Homework 1. PHY 564

Problem 1. 2 points. Lorentz transformations

Show explicitly that two successive Lorentz transformations in the same direction are equivalent to a single Lorentz transformation with $v = \dfrac{v_1 + v_2}{1 + (v_1 v_2 / c^2)}$.

Solution: Let's consider two successive Lorents transformation with $\beta_{1,2} = v_{1,2}/c$:

$$L_{1,2} = \gamma_{1.2}\begin{bmatrix} 1 & \beta_{1,2} \\ \beta_{1,2} & 1 \end{bmatrix}; L_{12} = L_2 L_1 = \gamma_1\gamma_2 \cdot \begin{bmatrix} 1 & \beta_2 \\ \beta_2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & \beta_1 \\ \beta_1 & 1 \end{bmatrix} = \gamma_1\gamma_2 \cdot \begin{bmatrix} 1+\beta_1\cdot\beta_2 & \beta_1+\beta_2 \\ \beta_1+\beta_2 & 1+\beta_1\cdot\beta_2 \end{bmatrix};$$

$$L_{12} = \gamma_{12}\begin{bmatrix} 1 & \beta_{12} \\ \beta_{12} & 1 \end{bmatrix}; \gamma_{12} = \gamma_1\gamma_2\left(1+\beta_1\cdot\beta_2\right); \beta_{12} = \frac{\beta_1+\beta_2}{1+\beta_1\cdot\beta_2} \to \mathrm{v}_{12} = \frac{\mathrm{v}_1+\mathrm{v}_2}{1+\dfrac{\mathrm{v}_1\cdot\mathrm{v}_2}{\mathrm{c}^2}}$$

where we paid attention only non-trivial portion of the Lorentz transformation for *(ct,x)* pair for 4-coordinates. The only one thing is required to check – even though is is automatically taken care of by structure of Lorentz transformation – the ratio between β_{12} and γ_{12}:

$$\gamma_{12} = \gamma_1\gamma_2\left(1+\beta_1\cdot\beta_2\right) = ? = \gamma_{12} = \frac{1}{\sqrt{1-\beta_{12}^2}}$$

$$\sqrt{1-\beta_{12}^2} = \sqrt{1-\left(\frac{\beta_1+\beta_2}{1+\beta_1\cdot\beta_2}\right)^2} = \frac{\sqrt{\left(1+\beta_1\cdot\beta_2\right)^2-\left(\beta_1+\beta_2\right)^2}}{1+\beta_1\cdot\beta_2}$$

$$\left(1+\beta_1\cdot\beta_2\right)^2-\left(\beta_1+\beta_2\right)^2 = 1-\beta_1^2 - \beta_2^2+\beta_1^2\cdot\beta_2 = \left(1-\beta_1^2\right)\left(1-\beta_2^2\right)$$

$$\frac{1}{\sqrt{1-\beta_{12}^2}} = \frac{1+\beta_1\cdot\beta_2}{\sqrt{1-\beta_1^2}\cdot\sqrt{1-\beta_2^2}} = \gamma_1\gamma_2\left(1+\beta_1\cdot\beta_2\right) = \gamma_{12}\#$$

Problem 2. 2 points. 4-invarints

Show that trace of a tensor is 4-invariant, i.e. $F_i^i \equiv \sum_{i=0}^{3} F_i^i = inv$.

Solution: this is easy one

$$Trace(F') = F'^i{}_i = \frac{\partial x'^i}{\partial x^k}\frac{\partial x^j}{\partial x'^i}F^k{}_j = \frac{\partial x^j}{\partial x^k}F^k{}_j = \delta^j_k F^k{}_j = F^k{}_k = Trace(F)\ \#$$

Problem 3. 6 points, Orbit of the antiparticle

Consider an accelerator with time independent (DC) magnetic elements, i.e. there is no electric field and know beam trajectory for a particle with charge *e*, energy γmc^2 and known trajectory $\vec{r}(t)$. Prove that placing at the same position antiparticle with opposite charge -*e*, the same energy and opposite direction of the momentum

$$\vec{r}_{e+} = \vec{r}_{e-}; \vec{p}_{e+} = - \vec{p}_{e-}$$

will result in the same trajectory traveled in the opposite direction.

Solution: Let's consider a particle with charge *e* traveling along trajectory $\vec{r}_e(t)$ in time independent magnetic field $\vec{B}(\vec{r})$:

$$\vec{v}_e(t) = \frac{d\vec{r}_e(t)}{dt}; \frac{d\vec{p}_e}{dt} = \frac{e}{c}\left[\vec{v}_e(t) \times B\left(r_e(t)\right)\right] \Rightarrow E_o = const, \ \gamma_o = const; \vec{p}_e = \gamma_o m\vec{v}_e$$

which means

$$\frac{d\vec{v}_e(t)}{dt} = \frac{e}{\gamma_o mc}\left[\vec{v}_e(t) \times B\left(r_e(t)\right)\right] \qquad (1)$$

Let's now reverse direction of motion: $\vec{r}_{-e}(t) = \vec{r}_e(\tau - t)$ with

$$\vec{v}_{-e}(t) = \frac{d\vec{r}_{-e}(t)}{dt} = -\frac{d\vec{r}_e(\tau - t)}{dt} = -\vec{v}_e(\tau - t)$$

which means

$$\frac{d\vec{v}_{-e}(t)}{dt} = -\frac{d\vec{v}_e(\tau - t)}{dt} = \frac{d\vec{v}_e(\tau - t)}{d(\tau - t)} = \frac{-e}{\gamma_o mc}\left[-\vec{v}_e(\tau - t) \times B\left(r_e(\tau - t)\right)\right];$$

$$\frac{d\vec{v}_e(\chi)}{d\chi} = \frac{e}{\gamma_o mc}\left[\vec{v}_e(\chi) \times B\left(r_e(\chi)\right)\right]$$

This equation is identical to (1) with substitution of χ->t. It means that return pass for a particle is identical to a pass for an anti-particle #