Due: Wednesday, September 7, 2016
Solutions:
HW 1 ( 5 point): Future Circular Collider (FCC, ) is under consideration by world physics community as a potentially next high energy collider.
(a) 1 point: The tunnel circumference would be 100 km https://en.wikipedia.org/wiki/Future_Circular_Collider. What average magnetic field is required to circulate 50 TeV proton beam?

The radius of curvature is defined by the particle momentum, charge and magnetic field (Lect 2, eq. (2.3)) and we defined an average "guiding" or "dipole" magnetic filed as defined by the ring circumference, C :

$$
\begin{gathered}
\rho=\frac{p c}{e B_{y}}=\frac{B \rho}{B_{y}} \rightarrow d \theta=\frac{d s}{\rho}=\frac{e B_{y}}{p c} d s \rightarrow \\
2 \pi=\oint d \theta=\frac{e}{p c} \oint_{C} B_{y} d s=\frac{e C}{p c}\left\langle B_{y}\right\rangle \\
\left\langle B_{y}\right\rangle=\frac{2 \pi \cdot p c}{e C}=\frac{2 \pi \cdot B \rho}{C}
\end{gathered}
$$

Now we can use eq. (2.4) and the fact that $p c=\sqrt{E^{2}-m_{p}{ }^{2} c^{4}} \cong E$. Indeed, $\mathrm{E}=50 \mathrm{TeV}$ and

$$
\begin{aligned}
& E=50,000 \mathrm{GeV} ; m_{p} c^{2}=938.272046 \mathrm{MeV} \approx 0.938 \mathrm{GeV} \\
& \gamma \cong 5.33 \cdot 10^{4} \Rightarrow \frac{E-p c}{E} \equiv \frac{c-\mathrm{v}}{c} \equiv 1-\beta \cong \frac{1}{2 \gamma^{2}}=1.76 \cdot 10^{-10}
\end{aligned}
$$

Then

$$
\begin{gathered}
B \rho[T \cdot k m] \cong \frac{p c[\mathrm{TeV}]}{0.299792458} \cong 167 T \cdot \mathrm{~km} \\
\left\langle B_{y}\right\rangle=\frac{2 \pi \cdot B \rho}{C}=10.48 T
\end{gathered}
$$

(b) 1 point: It is also considered for electron-positron collider with beam energy up to 175 GeV . What average magnetic field is required to circulate 175 GeV electron or positron beam?

Similarly, electrons are ultra relativistic:

$$
\begin{aligned}
& E=175 \mathrm{GeV} ; m_{e} c^{2}=0.511998910 \mathrm{MeV} \cong 0.511 \mathrm{MeV} \\
& \gamma \cong 3.43 \cdot 10^{5} \Rightarrow \frac{E-p c}{E} \equiv \frac{c-\mathrm{v}}{c} \equiv 1-\beta \cong \frac{1}{2 \gamma^{2}}=4.26 \cdot 10^{-12} \\
& B \rho[T \cdot \mathrm{~km}] \cong \frac{p c[\mathrm{TeV}]}{0.299792458} \cong 0.584 \mathrm{~T} \cdot \mathrm{~km} \\
& \left\langle B_{y}\right\rangle=\frac{2 \pi \cdot B \rho}{C}=0.0367 \mathrm{~T}=367 \mathrm{Gs}
\end{aligned}
$$

(c) 2 points: Show that the same ring (set of magnets) can be used to circulate electrons and positrons with the same energy but moving in opposite (colliding) directions. Specifically, write equation of motion for an electron and a positron and show that they can travel by the same trajectory but in opposite directions

Consider equation of motion on the same trajectory $\vec{r}_{o}(t)$ set by magnetic field $\vec{B}(\vec{r})$. Than equation of motion is:

$$
\frac{d \vec{p}}{d t}=\frac{e}{c}[\overrightarrow{\mathrm{v}} \times \vec{B}(\vec{r})] ; \overrightarrow{\mathrm{v}}(t) \equiv \frac{d \vec{r}_{o}(t)}{d t} ; \vec{p}(t)=\gamma m \overrightarrow{\mathrm{v}}(t)
$$

Since energy is preserved in magnetic field (neglecting radiation!) than $\gamma=$ const and velocity is a function of the trajectory

$$
\begin{equation*}
\frac{d^{2} \vec{r}_{o}(t)}{d t^{2}}=\frac{e}{\gamma m c}[\overrightarrow{\mathrm{v}} \times \vec{B}(\vec{r})]=\frac{e}{\gamma m c}\left[\frac{d \vec{r}_{o}(t)}{d t} \times \vec{B}\left(\vec{r}_{o}(t)\right)\right] ; \overrightarrow{\mathrm{v}}(t) \equiv \frac{d \vec{r}_{o}(t)}{d t}=\overrightarrow{\mathrm{v}}\left(\vec{r}_{o}(t)\right) \tag{I}
\end{equation*}
$$

Now let change signs of the particle charge $e \rightarrow-e$ and velocity $\overrightarrow{\mathrm{v}} \rightarrow-\overrightarrow{\mathrm{v}}$ to derive equation of motion for an antiparticle in the same $m$

$$
\frac{d^{2} \vec{r}_{a p}}{d t^{2}}=\frac{-e}{\gamma m c}\left[-\overrightarrow{\mathrm{v}} \times \vec{B}\left(\vec{r}_{a p}\right)\right]=\frac{-e}{\gamma m c}\left[\frac{d \vec{r}_{a p}}{d t} \times \vec{B}\left(\vec{r}_{a p}\right)\right] ;
$$

Now we can set antiparticle on the reverse trajectory:

$$
\vec{r}_{a p}(t)=\vec{r}_{o}\left(t_{o}-t\right)
$$

where $t_{o}$ is an arbitrary constant and check that its identical to equation of the particle (I):

$$
\begin{align*}
\frac{d^{2} \vec{r}_{o}\left(t_{o}-t\right)}{d t^{2}}= & \frac{-e}{\gamma m c}\left[\frac{d \vec{r}_{o}\left(t_{o}-t\right)}{d t} \times \vec{B}\left(\vec{r}_{o}\left(t_{o}-t\right)\right)\right] ; \tau=t_{o}-t ; d t=-d \tau  \tag{II}\\
& \frac{d^{2} \vec{r}_{o}(\tau)}{d \tau^{2}}=\frac{e}{\gamma m c}\left[\frac{d \vec{r}_{o}(\tau)}{d \tau} \times \vec{B}\left(\vec{r}_{o}(\tau)\right)\right]
\end{align*}
$$

In short, the changing signs of the particle charge and velocity does not change the value and the direction of the force and trajectory bends the same way.
(d) 1 point: Can be the same trick used to circulate and collide two proton beams?

No, the sign of the force changes for proton propagating in the opposite direction. It means that its trajectory will bend in opposite direction: dipole separator. Surprisingly, electric field would do the trick, but it is not useful for high energies: 10 T filed corresponds to electric field of $3,000 \mathrm{MV} / \mathrm{m}$ !

HW 2 (2 points): For a classical microtron having energy gain per pass of 1.022 MeV and operational RF frequency $3 \mathrm{GHz}\left(3 \times 10^{9} \mathrm{~Hz}\right)$ find required magnetic field (Hint: use $\mathrm{k}=1$ ). What will be radius of first orbit in this microtron?

Solution: Again, lets start from the radius of curvature

$$
\rho=\frac{p c}{e B_{y}}
$$

and calculate time of flight for a given energy

$$
T=\frac{2 \pi \rho}{\mathrm{v}}=\frac{2 \pi}{e B_{y}} \cdot \frac{p c}{\mathrm{v}}=\frac{2 \pi}{e B_{y}} \cdot \frac{E}{\mathrm{c}}
$$

The energy at n -turn is equal to the rest energy electron energy 0.511 MeV plus n -fold energy gain:

$$
\begin{aligned}
& E_{n}=m c^{2}+n \cdot \Delta E ; \quad \Delta E=2 m c^{2} ; \\
& E_{n}=(2 n+1) m c^{2}=(2 n+1) \cdot 0.511 \mathrm{MeV}
\end{aligned}
$$

Now we should use synchronization condition, e.g. that each turn should take an integer number of RF cycles

$$
\begin{aligned}
& T_{n}=N(n) \cdot T_{o} ; T_{o}=\frac{1}{f_{R F}} ; T_{n}=\frac{2 \pi}{e B_{y}} \cdot \frac{E_{n}}{\mathrm{c}}=(2 n+1) \frac{2 \pi m c}{e B_{y}} ; \\
& N(n)=k \cdot(2 n+1) ; \frac{2 \pi m c}{e B_{y}}=k T_{o}=\frac{k}{f_{R F}} \rightarrow B_{y}=\frac{1}{k} \frac{2 \pi m c^{2}}{e\left(c T_{o}\right)}
\end{aligned}
$$

where $k$ is a positive integer. Putting number together for $k=1$, we get for first pass

$$
\begin{gathered}
c T_{o}=\frac{c}{f_{o}}=9.993 \mathrm{~cm} \cong 10 \mathrm{~cm} ; \\
\frac{2 \pi m_{e} c^{2}}{e}=2 \pi \frac{0.511 \ldots}{0.29979 \ldots} \cong 2 \pi \cdot 1.705 \mathrm{kGs} \mathrm{~cm}=10.71 \mathrm{kGs} \mathrm{~cm} \\
B_{y} \cong 1.071 \mathrm{kGs} \\
\gamma=2 \mathrm{n}+1 \Rightarrow \gamma_{1}=3 ; \quad \beta_{1}=\sqrt{1-\gamma_{1}^{-2}} \cong 0.943 \\
p_{1} c=\gamma_{1} \beta_{1} m c^{2} \cong 1.445 \mathrm{MeV} \\
\rho_{1}[\mathrm{~cm}] \cong \frac{p_{1} c[\mathrm{MeV}]}{0.3 \cdot B_{y}}=4.5 \mathrm{~cm}
\end{gathered}
$$

The other way to find radius of first orbit: it takes 3 RF periods and orbit circumference is

$$
T_{1}=3 \cdot T_{o} ; \quad C_{1}=2 \pi \rho_{1}=\mathrm{v}_{1} T_{1}=\beta_{1} c T_{1}=26.26 \mathrm{~cm}
$$

Naturally, dividing the circumference by $2 \pi$ we get the same 4.5 cm radius of the first orbit.

HW 3 (3 points): Find available energy (so called C.M. energy) for a head-on collision of electrons and protons in two proposed electron-hadron eRHIC and LHeC:
(a) eRHIC plans to collide 20 GeV electrons with 250 GeV protons;
(b) LHeC plans to collide 60 GeV electrons with 7 TeV protons

$$
p_{p}^{\mu}=\left\{E_{p} / c, p_{p}, 0,0\right\} \quad p_{e}^{\mu}=\left\{E_{e} / c,-p_{e}, 0,0\right\}
$$



First let's find the c.m. energy using 4-momenta of both particles:

$$
\begin{gathered}
p_{e}^{\mu}=\left\{E_{e} / c,-p_{e}, 0,0\right\} ; p_{p}^{\mu}=\left\{E_{p} / c, p_{p}, 0,0\right\} ; \\
p_{c}^{\mu}=\left\{\frac{E_{p}+E_{e}}{c}, p_{p}-p_{e}, 0,0\right\} ; \frac{E_{c m}{ }^{2}}{c^{2}}=p_{c}^{\mu} p_{c}^{\mu}=\left(\frac{E_{p}+E_{e}}{c}\right)^{2}-\left(p_{p}-p_{e}\right)^{2} ; \\
E_{c m}{ }^{2}=\left(E_{p}+E_{e}\right)^{2}-\left(p_{p} c-p_{e} c\right)^{2}=E_{p}{ }^{2}-\left(p_{p} c\right)^{2}+E_{e}{ }^{2}-\left(p_{e} c\right)^{2}+2\left(E_{p} E_{e}+p_{p} p_{e} c^{2}\right) \\
E_{p}{ }^{2}-\left(p_{p} c\right)^{2}=\left(m_{p} c^{2}\right)^{2} ; E_{e}{ }^{2}-\left(p_{e} c\right)^{2}=\left(m_{e} c^{2}\right)^{2} ; \\
E_{c m}{ }^{2}=m_{p}{ }^{2} c^{4}+m_{e}{ }^{2} c^{4}+2 E_{p} E_{e}\left(1+\beta_{p} \beta_{e}\right) ; \\
E_{c m}=\sqrt{m_{p}{ }^{2} c^{4}+m_{e}{ }^{2} c^{4}+2 E_{p} E_{e}\left(1+\beta_{p} \beta_{e}\right)}
\end{gathered}
$$

For ultra-relativistic case $\left(\gamma_{p} \gg 1 ; \gamma_{e} \gg 1,1-\beta_{p} \beta_{e} \ll 1\right)$ we can approximately write

$$
E_{c m} \simeq 2 \sqrt{E_{p} E_{e}}
$$

(a) eRHIC with 20 GeV electrons and 250 GeV protons:

Exact $\mathrm{E}_{\mathrm{cm}}=141.4242197 \mathrm{GeV}$, approximate 141.4213562 is accurate in 5 digits.
(b) LHeC with 60 GeV electrons with 7 TeV protons: $\mathrm{E}_{\mathrm{cm}}=1.296 \mathrm{TeV} \mathrm{GeV}$. Difference between exact and proximate formulae $2.6 \mathrm{E}-7$ is negligible.

