Appendix B: Lorentz group http://en.wikipedia.org/wiki/Lorentz_group

Lorentz Group - Matrix representation

Jackson's Classical Electrodynamics, Section 11.7 [CED] has an excellent discussion of this topic. Here, we will review it briefly with some attention to the underlying mathematics. Generic Lorentz transformation involves a boost (a transformation from K to K' moving with some velocity $\vec{V}$) and an arbitrary rotation in 3D space. Matrix representation is well suited to describe 4-vectors transformations. The coordinate vector is defined as

$$X = \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}; \tag{B-29}$$

and standard scalar product of 4-vectors is defined by $(a,b) = \tilde{a}b$, where $\tilde{a}$ is the transposed vector. The 4-scalar product involves the metric tensor (matrix):

$$a \cdot b \equiv a^i \cdot b_i = (a, gb) = (ga, b) = \tilde{a}gb; \tag{B-30}$$

$$g = \tilde{g} = \{g^{ik}\} = \{g_{ik}\} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}. \tag{B-31}$$

Lorentz transformations A (or the group of Lorentz transformations[1]) are linear transformations that preserve the interval, or scalar product (B-30):

$$X' = AX;\ \tilde{X}'gX' = \tilde{X}\tilde{A}gAX = \tilde{X}gX; \Rightarrow \tilde{A}gA = g. \tag{B-33}$$

Using standard ratios for matrices

$$\det(\tilde{A}gA) = \det^2 A \det g = \det g \ \Rightarrow \det A = \pm 1; \tag{B-34}$$

we find that the matrices of Lorentz transformation have det=±1. We will consider only *proper Lorentz transformations* with unit determinants $\det A = +1$. Improper Lorentz transformations, like space- and time-inversions, should be considered as special transformations and added to the proper ones.

A 4x4 matrix has 16 elements. Equation (B-33) limited number of independent elements in matrix A of Lorentz transformations. Matrices on both sides are symmetric. Thus, there are 10 independent conditions on matrix A, leaving six independent elements there. This is unsurprising since

[1] Group G is defined as a set of elements , with a definition of a product of any two elements of the group; $P = A \bullet B \in G$; $A, B \in G$. The product must satisfy the associative law : $A \bullet (B \bullet C) = (A \bullet B) \bullet C$; there is an unit element in the group $E \in G; E \bullet A = A \bullet E = A: \forall A \in G;$ and inverse elements: $\forall A \in G; \exists B(called\ A^{-1}) \in G: A^{-1}A = AA^{-1} = E.$

Matrices NxN with non-zero determinants are examples of the group. Lorentz transformations are other examples : the product of two Lorentz is defined as two consequent Lorentz transformations. Therefore, the product also is a Lorentz transformation whose velocity is defined by rules discussed in previous lectures. The associative law is straightforward: unit Lorentz transformation is a transformation into the same system. Inverse Lorentz transformation is a transformation with reversed velocity. Add standard rotation s, to constitute the Lorentz Group rotation in 3D space is represented by 3 angles and a boost is represented by 3 components of velocity. Intuitively, then there are six independent rotations: (xy), (yz), (zx), (t, x), (t, y), and (t, z). No other combinations of 4D coordinates are possible: $C_4^2 = \frac{4!}{2!2!} = 6$.

We next consider the properties of A in standard way, representing A through a generator L:

$$A = e^L; \qquad \text{(B-35)}$$

where we use matrix exponent defined as the Taylor expansion:

$$e^L \downarrow_{def} \equiv \sum_{n=0}^{\infty} \frac{L^n}{n!}; L^0 = I; I = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}; \qquad \text{(B-36)}$$

where I is the unit matrix. Using (B-35) and $g^2 = I$ we find how to compose the inverse matrix for A:

$$\tilde{A}gA = g \Rightarrow A^{-1} = g\tilde{A}g; \qquad \text{(B-37)}$$

which, in combination with

$$\tilde{A} = transpose(e^L) = \sum_{n=0}^{\infty} \frac{\tilde{L}^n}{n!} = e^{\tilde{L}}; e^{gUg} = \sum_{n=0}^{\infty} \frac{(gUg)^n}{n!} = \sum_{n=0}^{\infty} g\frac{U^n}{n!}g; \qquad \text{(B-38)}$$

gives

$$A^{-1} = g\tilde{A}g = e^{g\tilde{L}g}. \qquad \text{(B-39)}$$

We can show that matrix exponent has similar properties as the regular exponent, i.e. $e^U e^{-U} = I$ by explicitly using Taylor expansion to collect the powers of U:

$$e^U e^{-U} = \left(\sum_{n=0}^{\infty} \frac{U^n}{n!}\right)\left(\sum_{k=0}^{\infty} (-1)^k \frac{U^k}{k!}\right) = \sum_{k=0,n=0}^{\infty} (-1)^k \frac{U^{n+k}}{n!k!} = I + \sum_{m=1}^{\infty} c_m U^m; \qquad \text{(B-40)}$$

and the well-known expansion of $(1-x)^m$. Our goal is to show that all c_m are zero:

$$(1-x)^m = \sum_{n=0}^{m} \frac{(-1)^n m!}{n!(m-n)!} x^n \Rightarrow m!c_m = \sum_{n=0}^{m} \frac{(-1)^n m!}{n!(m-n)!} = (1-1)^m = 0. \qquad \text{(B-41)}$$

Now (B-39) can be rewritten

$$A^{-1} = g\tilde{A}g = e^{g\tilde{L}g} = e^{-L} \Rightarrow g\tilde{L}g = -L; \Rightarrow \widetilde{gL} = -gL \qquad \text{(B-42)}$$

Hence, gL is an asymmetric matrix and has six independent elements as expected:

$$gL = \begin{bmatrix} 0 & L_{01} & L_{02} & L_{03} \\ -L_{01} & 0 & -L_{12} & -L_{13} \\ -L_{02} & L_{12} & 0 & -L_{23} \\ -L_{03} & L_{13} & L_{23} & 0 \end{bmatrix}; L = g(gL) = \begin{bmatrix} 0 & L_{01} & L_{02} & L_{03} \\ L_{01} & 0 & L_{12} & L_{13} \\ L_{02} & -L_{12} & 0 & L_{23} \\ L_{03} & -L_{13} & -L_{23} & 0 \end{bmatrix}. \qquad \text{(B-43)}$$

Each independent element represents an irreducible (fundamental) element of the Lorentz group or rotations and boosts, as discussed above. The six components of the L can be considered as six components of 3-vectors in the form ("-" is a convention):

$$L = -\vec{\omega}\vec{S} - \vec{\zeta}\vec{K}; A = e^{-\vec{\omega}\vec{S}-\vec{\zeta}\vec{K}}; \qquad \text{(B-44)}$$

with

$$\vec{S} = \hat{e}_x \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} + \hat{e}_y \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} + \hat{e}_z \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}; \qquad \text{(B-45)}$$

$$\vec{K} = \hat{e}_x \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \hat{e}_y \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \hat{e}_z \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}; \qquad \text{(B-46)}$$

where $\vec{\omega}\vec{S}$ represents the orthogonal group of rotations in 3D space (O_3^+), and $\vec{\zeta}\vec{K}$ represents the boosts caused by transformation into a moving system. It is easy to check that these matrices satisfy commutation rules of

$$[S_i, S_k] = e_{ikl}S_l; [S_i, K_k] = e_{ikl}K_l; [K_i, K_k] = -e_{ikl}S_l; \; [A,B] \equiv AB - BA; \qquad \text{(B-47)}$$

where e_{ikl} is the totally asymmetric 3D-tensor. You should be familiar with 3D rotation $e^{-\vec{\omega}\vec{S}}$ by $\vec{\omega}$: the direction of $\vec{\omega}$ is the axis of rotation and the value of $\vec{\omega}$ is the angle of rotation.

For the arbitrary unit vector $\hat{e}$

$$(\hat{e}\vec{S})^3 = -\hat{e}\vec{S}; \; (\hat{e}\vec{K})^3 = \hat{e}\vec{K}. \qquad \text{(B-48)}$$

Therefore, $\vec{S}$ "behaves" as an imaginary "i" and we should expect *sin* and *cos* to be generated by exp(..$\vec{S}$.); exp(..$\vec{K}$.) should generate hyperbolic functions *sinh* and *cosh.* It is left for your homework to show, in particular, that boost transformation is:

$$A(\vec{\beta} = \vec{V} / c) = e^{-\vec{\beta}\vec{K}\tanh^{-1}\beta}. \qquad \text{(B-49)}$$

Finally, all fully relativistic phenomena naturally have six independent parameters. For example, electromagnetic fields are described by two 3D vectors: the vector of the electric field and that of the magnetic field, or in equivalent form of an asymmetric 4-tensor of an electromagnetic field with six components. Furthermore, electric fields give charged particles energy boosts, while magnetic field rotates them without changing the energy….

Not surprisingly, the EM fields reflect the structure of the 4D space and its transformations.