Homework 21

1.

We are trying to find the approximate solution of the form

$$\lambda = a_0 + a_1\hat{C} + a_2\hat{C}^2 , \tag{1}$$

for the polynomial equation

$$\lambda^3 + 2i\hat{C}\lambda^2 - \hat{C}^2\lambda = i \ . \tag{2}$$

Inserting eq. (1) into eq. (2) yields

$$f\left(\hat{C}\right) \equiv \left(a_0 + a_1\hat{C} + a_2\hat{C}^2\right)^3 + 2i\hat{C}\left(a_0 + a_1\hat{C} + a_2\hat{C}^2\right)^2 - \hat{C}^2\left(a_0 + a_1\hat{C} + a_2\hat{C}^2\right) - i = 0 \ . \tag{3}$$

Requiring eq. (3) to be satisfied in the zeroth order in $\hat{C}$ leads to

$$f\left(0\right) \equiv a_0^{\ 3} - i = 0 \ ,$$

which leads to (we are searching for the growing mode, i.e. $\mathrm{Re}\left(a_0\right) > 0$)

$$a_0 = \frac{\sqrt{3}}{2} + i\frac{1}{2} \ . \tag{4}$$

Requiring eq. (3) to be satisfied in the first order in $\hat{C}$ leads to

$$\frac{d}{d\hat{C}} f\left(\hat{C}\right)\bigg|_{\hat{C}=0} = 0 \Rightarrow 3a_1 a_0^{\ 2} + 2ia_0^{\ 2} = 0 \Rightarrow a_1 = -i\frac{2}{3} \ . \tag{5}$$

Similarly, requiring eq. (3) to be satisfied in the second order in $\hat{C}$ leads to

$$\begin{aligned} \frac{d^2}{d\hat{C}^2} f\left(\hat{C}\right)\bigg|_{\hat{C}=0} = 0 \Rightarrow \frac{d}{d\hat{C}}\left[3\lambda^2\lambda' + 2i\lambda^2 + 4i\hat{C}\lambda\lambda' - 2\hat{C}\lambda - \hat{C}^2\lambda'\right]\bigg|_{\hat{C}=0} &= 0 \\ \Rightarrow 6a_0 a_1^{\ 2} + 3a_0^{\ 2} 2a_2 + 8ia_0 a_1 - 2a_0 &= 0 \ . \\ \Rightarrow a_2 = \frac{-3a_1^{\ 2} - 4ia_1 + 1}{3a_0} = -\frac{1}{9}\left(\frac{\sqrt{3}}{2} - i\frac{1}{2}\right) & \end{aligned} \tag{6}$$