Vector Calculus Refresher

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Gradient

$$grad(f) = \nabla f = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$$

$$\begin{aligned} \mathbf{A} &= (A_1, ..., A_n) \\ \mathbf{J_A} &= d\mathbf{A} = (\nabla \mathbf{A})^T = \left(\frac{\partial A_i}{\partial x_j}\right)_{ij} \end{aligned}$$

Divergence

$$\begin{aligned} \mathbf{F} &= F_x\mathbf{i} + F_y\mathbf{j} + F_z\mathbf{k} \\ div\mathbf{F} &= \nabla \cdot \mathbf{F} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \cdot (F_x, F_y, F_z) = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} \end{aligned}$$

Curl

$$
\begin{aligned}
\mathbf{F} &= F_x\mathbf{i} + F_y\mathbf{j} + F_z\mathbf{k} \\
curl\mathbf{F} &= \nabla \times \mathbf{F} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \times (F_x, F_y, F_z) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix} \\
&= \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right)\mathbf{i} + \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right)\mathbf{j} + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)\mathbf{k}
\end{aligned}
$$

Laplacian

$$
\begin{aligned}
\Delta f &= \nabla^2 f = \nabla \cdot (\nabla f) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \\
\Delta \mathbf{T} &= \nabla^2 \mathbf{T} = (\nabla \cdot \nabla)\mathbf{T}
\end{aligned}
$$

Properties

$$
\begin{aligned}
\nabla \cdot (\nabla \times \mathbf{A}) &= 0 \\
\nabla \cdot (\nabla \psi) &= \nabla^2 \psi = \Delta \psi \\
\nabla \cdot (\nabla \cdot \mathbf{A}) & \quad \text{is undefined} \\
\nabla \times (\nabla \varphi) &= \mathbf{0} \\
\nabla \times (\nabla \times \mathbf{A}) &= \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}
\end{aligned}
$$

Maxwell's equations

Permittivity of free space ε_0, permeability of free space μ_0, speed of light $c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}}$. In SI units

$$
\begin{aligned}
\nabla \cdot \mathbf{E} &= \frac{\rho}{\varepsilon_0} \\
\nabla \cdot \mathbf{B} &= 0 \\
\nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} \\
\nabla \times \mathbf{B} &= \mu_0 \left(\mathbf{J} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right)
\end{aligned}
$$

$\mathbf{D} = \varepsilon \mathbf{E}, \mathbf{H} = \frac{1}{\mu} \mathbf{B}$.

Vacuum with no charge ($\rho = 0$) and no currents ($\mathbf{J} = \mathbf{0}$)

$$\begin{aligned}
\nabla \cdot \mathbf{E} &= 0 \\
\nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} \\
\nabla \cdot \mathbf{B} &= 0 \\
\nabla \times \mathbf{B} &= \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} = \frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t}
\end{aligned}$$

$$\begin{aligned}
\frac{1}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} - \nabla^2 \mathbf{E} &= 0 \\
\frac{1}{c^2} \frac{\partial^2 \mathbf{B}}{\partial t^2} - \nabla^2 \mathbf{B} &= 0
\end{aligned}$$

$$\begin{aligned}
\mathbf{E} &= \mathbf{E}_0 \sin(-\omega t + \mathbf{k} \cdot \mathbf{r}) \\
\mathbf{B} &= \mathbf{B}_0 \sin(-\omega t + \mathbf{k} \cdot \mathbf{r}) \\
\mathbf{E}_0 \cdot \mathbf{B}_0 &= 0 = \mathbf{E}_0 \cdot \mathbf{k} = \mathbf{B}_0 \cdot \mathbf{k}
\end{aligned}$$

Refer to the reading material "Least Action Principle, Geometry of Special Relativity, Particles in E&M fields, by Prof. Litvinenko" for details.
4-potential $(\varphi, \vec{A})$, where φ is called the scalar potential and $\vec{A}$ is termed the vector potential of electromagnetic field.

$$\begin{aligned}\vec{E} &= -grad(\varphi) - \frac{1}{c}\frac{\partial \vec{A}}{\partial t} \Rightarrow curl\vec{E} = -curl(grad(\varphi)) - \frac{1}{c}curl\frac{\partial \vec{A}}{\partial t} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t} \\ \vec{B} &= curl\vec{A} \Rightarrow div\vec{B} = div(curl\vec{A}) = 0\end{aligned}$$