PHY554: HOMEWORK 2

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1. Problem 1

From the lecture notes, we have

$$\begin{pmatrix} X(s_2) \\ X'(s_2) \end{pmatrix} = M(s_2, s_1) \begin{pmatrix} X(s_1) \\ X'(s_1) \end{pmatrix}$$

where the transfer matrix $M(s_2, s_1)$ is

$$\begin{aligned} M(s_2, s_1) &= \begin{pmatrix} \sqrt{\frac{\beta_2}{\beta_1}}(\cos\mu + \alpha_1 \sin\mu) & \sqrt{\beta_1\beta_2}\sin\mu \\ -\frac{1+\alpha_1\alpha_2}{\sqrt{\beta_1\beta_2}}\sin\mu - \frac{\alpha_1-\alpha_2}{\sqrt{\beta_1\beta_2}}\cos\mu & \sqrt{\frac{\beta_2}{\beta_1}}(\cos\mu - \alpha_1\sin\mu) \end{pmatrix} \\ &= \begin{pmatrix} \sqrt{\beta_2} & 0 \\ -\frac{\alpha_2}{\sqrt{\beta_2}} & \frac{1}{\sqrt{\beta_1}} \end{pmatrix} \begin{pmatrix} \cos\mu & \sin\mu \\ -\sin\mu & \cos\mu \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{\beta_1}} & 0 \\ -\frac{\alpha_1}{\sqrt{\beta_1}} & \sqrt{\beta_1} \end{pmatrix} \end{aligned}$$

We can solve $X(s_2)$

$$X(s_2) = \sqrt{\frac{\beta_2}{\beta_1}}(\cos\mu + \alpha_1\sin\mu)\cdot X(s_1) + \sqrt{\beta_1\beta_2}\sin\mu\cdot X'(s_1)$$

If a particle is kicked at s_1 by angle θ, we have

$$\begin{pmatrix} X(s_2) + \Delta x_2 \\ X'(s_2) + \Delta x_2' \end{pmatrix} = M(s_2, s_1) \begin{pmatrix} X(s_1) \\ X'(s_1) + \theta \end{pmatrix}$$

We can solve $X(s_2) + \Delta x_2$

$$X(s_2) + \Delta x_2 = \sqrt{\frac{\beta_2}{\beta_1}}(\cos\mu + \alpha_1\sin\mu)\cdot X(s_1) + \sqrt{\beta_1\beta_2}\sin\mu\cdot (X'(s_1) + \theta)$$

Taking difference between $X(s_2)$ and $X(s_2) + \Delta x_2$, we have

$$\Delta x_2 = \theta\sqrt{\beta_1\beta_2}\sin\mu$$

Δx_2 is proportion to $\sqrt{\beta_1}$, and β_1 is the β function at the kicker location.
To obtain the maximum kicker strength, the kicker should be located in the position where β function reaches maximum. To obtain the minimum kicker strength, the kicker should be located in the position where β function reaches minimum.

2. Problem 2

2.1. Maximum. Maximum betatron functions are located at center of QFs, so a FODO cell is arranged as

$$QF/2 \Rightarrow B \Rightarrow QD \Rightarrow B \Rightarrow QF/2$$

Assuming the quadrupoles are thin lens, the corresponding transfer matrix is

$$\begin{aligned} M &= \begin{pmatrix} 1 & 0 \\ -\frac{1}{2f} & 1 \end{pmatrix} \begin{pmatrix} 1 & L_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{1}{f} & 1 \end{pmatrix} \begin{pmatrix} 1 & L_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{1}{2f} & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 - \frac{L_1^2}{2f^2} & 2L_1(1 + \frac{L_1}{2f}) \\ -\frac{L_1}{2f^2}(1 - \frac{L_1}{2f}) & 1 - \frac{L_1^2}{2f^2} \end{pmatrix} \end{aligned}$$

The transfer matrix can also be written as

$$M = \begin{pmatrix} \cos\Phi + \alpha \sin\Phi & \beta \sin\Phi \\ -\gamma \sin\Phi & \cos\Phi - \alpha \sin\Phi \end{pmatrix}$$

So we can solve that

$$\begin{aligned} \cos\Phi &= \frac{1}{2} Tr(M) \\ &= 1 - \frac{L_1^2}{2f^2} \end{aligned}$$

And with $\cos\Phi = 1 - \sin^2\frac{\Phi}{2}$, we can solve that $\sin\frac{\Phi}{2} = \frac{L_1}{2f}$, and we have

$$\begin{aligned} \alpha &= 0 \\ \beta &= \frac{2L_1(1 + \frac{L_1}{2f})}{\sin\Phi} \\ &= \frac{2L_1(1 + \sin\frac{\Phi}{2})}{\sin\Phi} \end{aligned}$$

In this problem, number of FODO cells is $n_{FODO} = 12$, circumference is $L = 180m$, so $L_1 = L/n_{FODO}/2 = 7.5m$.
The betatron tunes are $Q_x = 3.5, Q_y = 3.4$, so the phase advance for each FODO cell should be

$$\begin{aligned} \Phi_x &= \frac{2\pi Q_x}{n_{FODO}} \\ &= \frac{7\pi}{12} \\ \Phi_y &= \frac{2\pi Q_y}{n_{FODO}} \\ &= \frac{6.8\pi}{12} \end{aligned}$$

Therefore

$$\begin{aligned}\beta_{x,max} &= \frac{2L_1(1+\sin\frac{\Phi_x}{2})}{\sin\Phi_x}\\ &= 27.85m\\ \beta_{y,max} &= \frac{2L_1(1+\sin\frac{\Phi_y}{2})}{\sin\Phi_y}\\ &= 27.25m\end{aligned}$$

2.2. **Minimum.** Minimum betatron functions are located at center of QDs, so a FODO cell is arranged as

$$QD/2 \Rightarrow B \Rightarrow QF \Rightarrow B \Rightarrow QD/2$$

Assuming the quadrupoles are thin lens, the corresponding transfer matrix is

$$\begin{aligned}M &= \begin{pmatrix}1 & 0\\ \frac{1}{2f} & 1\end{pmatrix}\begin{pmatrix}1 & L_1\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ -\frac{1}{f} & 1\end{pmatrix}\begin{pmatrix}1 & L_1\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ \frac{1}{2f} & 1\end{pmatrix}\\ &= \begin{pmatrix}1-\frac{L_1^2}{2f^2} & 2L_1(1-\frac{L_1}{2f})\\ -\frac{L_1}{2f^2}(1+\frac{L_1}{2f}) & 1-\frac{L_1^2}{2f^2}\end{pmatrix}\end{aligned}$$

The transfer matrix can also be written as

$$M = \begin{pmatrix}\cos\Phi+\alpha\sin\Phi & \beta\sin\Phi\\ -\gamma\sin\Phi & \cos\Phi-\alpha\sin\Phi\end{pmatrix}$$

So we can solve that

$$\begin{aligned}\cos\Phi &= \frac{1}{2}Tr(M)\\ &= 1-\frac{L_1^2}{2f^2}\end{aligned}$$

And with $\cos\Phi = 1-\sin^2\frac{\Phi}{2}$, we can solve that $\sin\frac{\Phi}{2} = \frac{L_1}{2f}$, and we have

$$\begin{aligned}\alpha &= 0\\ \beta &= \frac{2L_1(1-\frac{L_1}{2f})}{\sin\Phi}\\ &= \frac{2L_1(1-\sin\frac{\Phi}{2})}{\sin\Phi}\end{aligned}$$

In this problem, number of FODO cells is $n_{FODO} = 12$, circumference is $L = 180m$, so $L_1 = L/n_{FODO}/2 = 7.5m$.
The betatron tunes are $Q_x = 3.5, Q_y = 3.4$, so the phase advance for each FODO cell should be

$$\begin{aligned}\Phi_x &= \frac{2\pi Q_x}{n_{FODO}}\\ &= \frac{7\pi}{12}\\ \Phi_y &= \frac{2\pi Q_y}{n_{FODO}}\\ &= \frac{6.8\pi}{12}\end{aligned}$$

Therefore

$$\begin{aligned}
\beta_{x,min} &= \frac{2L_1(1-\sin\frac{\Phi_x}{2})}{\sin\Phi_x} \\
&= 3.21m \\
\beta_{y,min} &= \frac{2L_1(1-\sin\frac{\Phi_y}{2})}{\sin\Phi_y} \\
&= 3.42m
\end{aligned}$$

2.3. **Chamber size.** Now we have maximum of betatron functions and RMS beam emittance, we can calculate the RMS beam size

$$\begin{aligned}
\sigma_x &= \sqrt{\beta_{x,max}\varepsilon} \\
&= \sqrt{27.85\cdot 1e-6} \\
&= 5.28e-3m \\
\sigma_y &= \sqrt{\beta_{y,max}\varepsilon} \\
&= \sqrt{27.25\cdot 1e-6} \\
&= 5.22e-3m
\end{aligned}$$

As we know, in a 1D normal distribution, integral of density function from -8σ to 8σ will be more than 99%. So if we take $8\cdot\max(\sigma_x,\sigma_y)$ as the chamber radius, the vacuum chamber will be large enough to house such beam. So the vacuum chamber size should be at least 4.224e-2m in radius or 8.448e-2m in diameter.

3. Problem 3

Given a distribution $\rho(X,X')$ with $\int\rho(X,X')dXdX'=1$, we have

$$\begin{aligned}
<X> &= \int X\rho(X,X')dXdX' \\
<X'> &= \int X'\rho(X,X')dXdX' \\
\sigma_X^2 &= \int (X-<X>)^2\rho(X,X')dXdX' \\
\sigma_{X'}^2 &= \int (X'-<X'>)^2\rho(X,X')dXdX' \\
\sigma_{XX'} &= \int (X-<X>)(X'-<X'>)\rho(X,X')dXdX'
\end{aligned}$$

And we can simplify $\sigma_X^2, \sigma_{X'}^2, \sigma_{XX'}$ as

$$\begin{aligned}
\sigma_X^2 &= <X^2>-<X>^2 \\
\sigma_{X'}^2 &= <X'^2>-<X'>^2 \\
\sigma_{XX'} &= <XX'>-<X><X'>
\end{aligned}$$

If we take derivative over s, we have

$$\begin{aligned}
\frac{d\sigma_X^2}{ds} &= 2 < XX' > -2 < X >< X' > \\
\frac{d\sigma_{X'}^2}{ds} &= 2 < X'X'' > -2 < X' >< X'' > \\
\frac{d\sigma_{XX'}}{ds} &= < X'^2 > - < X' >^2 - < X >< X'' > + < XX'' >
\end{aligned}$$

Note that X satisfies the function

$$X'' + KX = 0$$

So we can replace X'' by $-KX$.
And emittance is defined as

$$\varepsilon^2 = \sigma_X^2 \sigma_{X'}^2 - \sigma_{XX'}^2$$

Therefore

$$\begin{aligned}
\frac{d\varepsilon^2}{ds} &= \sigma_X^2 \frac{d\sigma_{X'}^2}{ds} + \sigma_{X'}^2 \frac{d\sigma_X^2}{ds} - 2\sigma_{XX'} \frac{d\sigma_{XX'}}{ds} \\
&= \left(< X^2 > - < X >^2\right)(2 < X'X'' > -2 < X' >< X'' >) \\
&\quad + \left(< X'^2 > - < X' >^2\right)(2 < XX' > -2 < X >< X' >) \\
&\quad -2(< XX' > - < X >< X' >)\left(< X'^2 > - < X' >^2 - < X >< X'' > + < XX'' >\right) \\
&= \left(< X^2 > - < X >^2\right)(-2K < XX' > +2K < X >< X' >) \\
&\quad + \left(< X'^2 > - < X' >^2\right)(2 < XX' > -2 < X >< X' >) \\
&\quad -2(< XX' > - < X >< X' >)\left(< X'^2 > - < X' >^2 +K < X >^2 -K < X^2 >\right) \\
&= 2K\left(< X^2 > - < X >^2\right)(< X >< X' > - < XX' >) \\
&\quad +2\left(< X'^2 > - < X' >^2\right)(< XX' > - < X >< X' >) \\
&\quad -2(< XX' > - < X >< X' >)\left(< X'^2 > - < X' >^2\right) \\
&\quad -2K(< XX' > - < X >< X' >)\left(< X >^2 - < X^2 >\right) \\
&= 0
\end{aligned}$$

Note that the two terms with K are canceled and the two terms without K are canceled. So we have

$$\frac{d\varepsilon^2}{ds} = 0$$