PHY 554. Homework 8.

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HW 1, 3 points. Using representation of transport matrices using β-functions from Lecture, and a weak quadrupole error:

$$M_o(s_1|s_2)=\begin{bmatrix} \sqrt{\frac{\beta_2}{\beta_1}}\left(\cos\Delta\psi_{12}+\alpha_1\sin\Delta\psi_{12}\right) & \sqrt{\beta_1\beta_2}\sin\Delta\psi_{12} \\ -\frac{1+\alpha_1\alpha_2}{\sqrt{\beta_1\beta_2}}\sin\Delta\psi_{12}-\frac{\alpha_1-\alpha_2}{\sqrt{\beta_1\beta_2}}\cos\Delta\psi_{12} & \sqrt{\frac{\beta_1}{\beta_2}}\left(\cos\Delta\psi_{12}-\alpha_2\sin\Delta\psi_{12}\right) \end{bmatrix};$$

$$\Delta\psi_{12}=\psi_2-\psi_1;\, M_\delta(s_1)=\begin{bmatrix} 1 & 0 \\ -k(s_1)ds & 1 \end{bmatrix};$$

$$M(s_2|s_2+C)=M_o(s_1|s_2+C)M_\delta(s_1)M_o(s_2|s_1);\beta_i\equiv\beta_o(s_i);\psi_i\equiv\psi_o(s_i)=\nu\phi_o(s_i);$$

$$\delta M_{12}(s_2|s_2+C)=M(s_2|s_2+C)-M_o(s_2|s_2+C);$$

prove the modification of the transport matrix element M_{12} is indeed what we used in Lecture 8:

$$\delta M_{12}(s_2|s_2+C)=-\beta_1\beta_2 k(s_1)ds\cdot\sin(\psi_1-\psi_2)\cdot\sin(\mu_o-\psi_1+\psi_2)$$
$$=\frac{1}{2}\beta_1\beta_2 k(s_1)ds\cdot\left[\cos\mu_o-\cos\left(\mu_o-2(\psi_1-\psi_2)\right)\right]$$

Solution:
First, to reduce amount of formulae lest do tin in matrix form:

$$\delta M(s_2|s_2+C)=T-T_o;\, T=M(s_2|s_2+C);T_o=M_o(s_2|s_2+C);A=M_o(s_1|s_2+C);B=M_o(s_2|s_1);$$

$$T=A\cdot M_\delta\cdot B;T_o=A\cdot B\equiv A\cdot I\cdot B;I=\begin{bmatrix}1&0\\0&1\end{bmatrix};A=\begin{bmatrix}a_{11}&a_{12}\\a_{21}&a_{22}\end{bmatrix};B=\begin{bmatrix}b_{11}&b_{12}\\b_{21}&b_{22}\end{bmatrix}$$

$$\delta M\equiv A\cdot M_\delta\cdot B-A\cdot I\cdot B\equiv A\cdot(M_\delta-I)\cdot B;M_\delta-I=\begin{bmatrix}0&0\\-kds&0\end{bmatrix}=-kds\begin{bmatrix}0&0\\1&0\end{bmatrix};$$

$$\delta M=-kds\begin{bmatrix}a_{11}&a_{12}\\a_{21}&a_{22}\end{bmatrix}\begin{bmatrix}0&0\\1&0\end{bmatrix}\begin{bmatrix}b_{11}&b_{12}\\b_{21}&b_{22}\end{bmatrix}=-kds\begin{bmatrix}a_{11}&a_{12}\\a_{21}&a_{22}\end{bmatrix}\begin{bmatrix}0&0\\b_{11}&b_{12}\end{bmatrix}=$$

$$-kds\begin{bmatrix}\sim&a_{12}b_{12}\\\sim&\sim\end{bmatrix}$$

where we are interested only in matrix elements 12:

$$\delta M_{12} = -kdsa_{12}b_{12};$$

And now we need to use expressions for matrix elemetns to find the exprestion:

$$a_{12} = M_o\left(s_1|s_2+C\right)_{12} = \sqrt{\beta_1\beta_2}\sin\left(\psi\left(s_2+C\right)-\psi\left(s_1\right)\right) = \sqrt{\beta_1\beta_2}\sin\left(\mu+\psi_2-\psi_1\right)$$

$$b_{12} = M_o\left(s_2|s_1\right)_{12} = \sqrt{\beta_1\beta_2}\sin\left(\psi_1-\psi_2\right); \beta_{1,2} = \beta\left(s_{1,2}\right); \psi_{1,2} = \psi\left(s_{1,2}\right)$$

$$\delta M_{12}\left(s_2|s_2+C\right) = -k\left(s_2\right)ds\cdot\beta_1\beta_2\cdot\sin\left(\mu+\psi_2-\psi_1\right)\cdot\sin\left(\psi_1-\psi_2\right)$$

With can be modified to its final form using

$$\cos\left(\theta+\varphi\right)-\cos\left(\theta-\varphi\right) = -2\sin\theta\sin\varphi; \quad \theta = \psi_1-\psi_2; \varphi = \mu_o-\theta.$$

$$\text{with } \theta+\varphi = \mu_o; \theta-\varphi = \mu_o-2\left(\psi_1-\psi_2\right).$$

HW2: 3 points. Prove that relative value of β-beat has the forced socillator equation with doubel betatron frequency:

$$f\left(s\right) = \frac{\delta\beta\left(s\right)}{\beta_o\left(s\right)} = -\frac{1}{2\sin\mu_o}\int_{\psi(s)}^{\psi(s)+\mu}\beta_o^2\left(z\right)k\left(z\right)\cdot\cos\left(\mu_o+2\left(\psi-\varphi\right)\right)d\varphi; d\varphi = \frac{ds}{\beta_o};$$

$$\frac{d^2}{d\psi^2}f\left(s\right)+4f\left(s\right) = -2\beta_o^2\left(s\right)k\left(s\right).$$

Solution: *just careful taking defirentials*

$$f\left(s\right) = \int_{\psi(s)}^{\psi(s)+\mu} g\left(z\right)\cdot\cos\left(\mu_o+2\left(\psi\left(s\right)-\varphi\right)\right)d\varphi; d\varphi = \frac{ds}{\beta_o}; g\left(z\right) = \frac{\beta_o^2\left(z\right)k\left(z\right)}{2\sin\mu_o}; g\left(z+C\right) = g\left(z\right);$$

$$\frac{d}{d\psi}f\left(s\right) = g\left(s\right)\cdot\cos\left(\mu_o+2\left(\psi-\varphi\right)\right)\Big|_{\varphi=\psi}^{\varphi=\psi+\mu} - 2\int_{\psi(s)}^{\psi(s)+\mu} g\left(z\right)\cdot\sin\left(\mu_o+2\left(\psi\left(s\right)-\varphi\right)\right)d\varphi;$$

$$\cos\left(\mu_o+2\left(\psi-\varphi\right)\right)\Big|_{\varphi=\psi}^{\varphi=\psi+\mu} = \cos\left(\mu_o\right)-\cos\left(-\mu_o\right)\Big| = 0;$$

$$\frac{d^2}{d\psi^2}f\left(s\right) = -2g\left(s\right)\cdot\sin\left(\mu_o+2\left(\psi-\varphi\right)\right)\Big|_{\varphi=\psi}^{\varphi=\psi+\mu} - 4\int_{\psi(s)}^{\psi(s)+\mu} g\left(z\right)\cdot\cos\left(\mu_o+2\left(\psi\left(s\right)-\varphi\right)\right)d\varphi;$$

$$\sin\left(\mu_o+2\left(\psi-\varphi\right)\right)\Big|_{\varphi=\psi}^{\varphi=\psi+\mu} = \sin\left(\mu_o\right)-\sin\left(-\mu_o\right)\Big| = 2\sin\mu_o;$$

$$\frac{d^2}{d\psi^2}f\left(s\right) = -4f\left(s\right)-4g\left(s\right)\sin\mu_o; \quad 2g\left(s\right)\sin\mu_o = \beta_o^2\left(z\right)k\left(z\right);$$

$$\frac{d^2}{d\psi^2}f+4f = -2\beta_o^2\left(z\right)k\left(z\right)\#$$

HW3, 4 points: Prove that it is impossible to compensate both horizontal and verticla chromaticity in a storage ring with uniform weak focusing.

Hints:

(a) Use the fact that β-functions are constants;
(b) Prove that both natural chromaticities are negative;
(c) Show hat dispersion fnction is constant and positive;
(d) Use this fact to show that sextupoles have equal opposite effect on tow chromaticities independently of locaion

Solution: First, let's find solutions for optics function for week focusing storage ring with field index 0<n<1:

$$x''+\frac{1-n}{\rho^2}x=0;\ \ y''+\frac{n}{\rho^2}y=0;\ \ D''+\frac{1-n}{\rho^2}D=\frac{1}{\rho};$$

It is easy to find constant solutions for D:

$$D=\frac{\rho}{1-n};$$

β-functions using Floquet theorem you learned about in lecture 3:

$$\beta_{x,y}={w_{x,y}}^2;w''_{x,y}+K_{x,y}w_{x,y}=\frac{1}{{w_{x,y}}^3};{\beta_{x,y}}^2={w_{x,y}}^4=\frac{1}{K_{x,y}};\alpha_{x,y}=0;$$

$$K_x=\frac{1-n}{\rho^2};K_x=\frac{1-n}{\rho^2}\Rightarrow\beta_x=\frac{\rho}{\sqrt{1-n}};\beta_y=\frac{\rho}{\sqrt{n}}.$$

More elaborate way is to use one-turn matrices, having the identical result:

$$M_{x,y}=\begin{bmatrix}\cos\mu_{x,y} & \dfrac{\sin\mu_{x,y}}{\sqrt{K_{x,y}}}\\ -\sqrt{K_{x,y}}\sin\mu_{x,y} & \cos\mu_{x,y}\end{bmatrix};\mu_{x,y}=2\pi\rho\sqrt{K_{x,y}}=2\pi\left\{\begin{matrix}\sqrt{1-n}\\ \sqrt{n}\end{matrix}\right\};$$

$$\alpha_{x,y}=0;\beta_{x,y}=\frac{1}{\sqrt{K_{x,y}}}=\rho\left\{\begin{matrix}1/\sqrt{1-n}\\ 1/\sqrt{n}\end{matrix}\right\}.$$

Now using lecture 8 formula

$$k_y=-K_y=-\frac{n}{\rho^2};k_x=-2K_o-k_y=-\frac{2-n}{\rho^2};$$

we can calculate natural chromaticities:

$$C=2\pi\rho;C_{x,y}\equiv\frac{1}{4\pi}\oint_C\beta_{x,y}(s)k_{x,y}(s)ds=\frac{\rho}{2}\beta_{x,y}k_{x,y};$$

$$C_y=-\frac{\sqrt{n}}{2};C_y=-\frac{2-n}{\sqrt{1-n}}.$$

If we attempt using sextupoles to compensate chromaticity we have:

$$C_x \equiv \frac{1}{4\pi}\oint_C \beta_x(s)\left\{D(s)K_2(s) - K_x(s)\right\}ds = -\frac{2-n}{\sqrt{1-n}} + \frac{D}{4\pi}\beta_x S;$$

$$C_y \equiv -\frac{1}{4\pi}\oint_C \beta_y(s)\left\{D(s)K_2(s) + K_y(s)\right\}ds = -\frac{\sqrt{n}}{2} - \frac{D}{4\pi}\beta_y S;$$

$$S = \oint_C K_2(s)\,ds$$

where we introduced the integral strength of the sextupoles: S. Since both $D\beta_{x,y} > 0$ are positive, it does not matter what sign of sextupoles we select – one of the chromaticities will remain negative.