Homework 6.

Problem 1. 10 points. Sylvester formula – dipole/quadrupole
For an uncoupled transverse motion with constant energy and Hamiltonian of a bending magnet with quadrupole term (e.g. field gradient):

$$\tilde{h}_n = \frac{p_x^2 + p_y^2}{2} + f\frac{x^2}{2} + g\frac{y^2}{2};$$

$$f = \left[K_o^2 - K_1\right]; g = -K_1;\; K_o = -\frac{e}{p_o c}B_y; K_1 = -\frac{e}{p_o c}\frac{\partial B_y}{\partial x}$$

(a) Define all cases for eigen values of D.
(b) Use Sylvester formula for one-dimensional motions (x and y) when $f \neq 0; g \neq 0;$ (non-degenerated cases) and write explicit form of the 2x2 transport matrices.
(c) Consider a case of pure quadrupole: $K_o = 0$, no bending
(d) Do the same as above using 4x4 matrix formulation (2D case) and show that results are identical

Problem 2. 10 points. Sylvester formula, SQ-quadrupole
For a coupled transverse motion with constant energy and Hamiltonian of a SQ-quadrupole:

$$\tilde{h}_n = \frac{p_x^2 + p_y^2}{2} + Nxy; \qquad N = \frac{e}{p_o c}\frac{\partial B_x}{\partial x}$$

(a) Use Sylvester formula and find matrix of SQ-quadrupole.
(b) Consider a "standard approach" – turn coordinates 45-degrees (use rotation matrix), to turn SQ-quad into a "normal". Then make the product of 45-degree turn, quad matrix, -45 degrees turn. Show that the matrix is the same as in case (a).
Note: for point (b), consider a rotation is around z-axis: it will rotate x,y and p_x,p_y:

$$\begin{pmatrix} x^1 \\ p_x^1 \\ y^1 \\ p_y^1 \end{pmatrix} = R\begin{pmatrix} x \\ p_x \\ y \\ p_y \end{pmatrix} = \begin{bmatrix} I\cdot\cos\varphi & I\cdot\sin\varphi \\ -I\cdot\sin\varphi & I\cdot\cos\varphi \end{bmatrix}\begin{pmatrix} x \\ p_x \\ y \\ p_y \end{pmatrix}; \; I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Thus
$$x^1 = x\cos\varphi + y\sin\varphi;\;\; y^1 = -x\sin\varphi + y\cos\varphi$$
$$p_x^1 = p_x\cos\varphi + p_y\sin\varphi; p_y^1 = -p_x\sin\varphi + p_y\cos\varphi$$

Do not forget that you need inverse matrix of R as well.
In rotated coordinates with $\varphi = \pi/2$ the Hamiltonian will have a decoupled form of one in quadrupole and you easily can calculate the matrix M_Q.. Finally, you need to rotate back to initial coordinates. $M_{SQ} = R^{-1}\cdot M_Q \cdot R$