

Periodic Beam Optics

EXERCISE: A basic brick of periodic optical systems, the “FODO CELL”

Consider the following optical sequence, along a straight axis:

a drift of length $l/2$; a focusing thin-lens with focal distance f ; a drift of length l ; a defocusing thin-lens with the same focal distance f ; a drift of length $l/2$.

Take f a positive quantity.

1. Theory

- 1.1/ Express the 2×2 transfer matrix T of this FODO cell, in terms of l and f .
- 1.2/ Verify that the determinant of T is 1.
- 1.3/ At what condition on f and l is this optical system periodically stable?
- 1.4/ Express the betatron phase advance μ over this cell, in terms of f and l .
- 1.5/ Express the periodic betatron function $\beta(s)$ at cell ends ($s=0$ and $s=2l$), in terms of μ , l and f .

2. Computer work: Simulate the above FODO cell in zgoubi.

Take $l=2\text{m}$, frozen. The strength of the lens, $1/f$, can be varied, we will start with $f=l$.

- 2.1/ Compare T from theory and from MATRIX or TWISS.
- 2.2/ What are the cell phase advance values, horizontal and vertical? The betatron function values at cell ends? Compare with theory.
- 2.3/ Check the stability limit $f > l/2$.
- 2.4/ Find the strengths ($K = G/B\rho$) of the F and D lenses to get phase advances of $0.27 \times 2\pi$ and $0.1 \times 2\pi$, respectively horizontal and vertical. FIT can be used for that. Theory can be used, as well.
- 2.5/ Launch 40 particles on a horizontal periodic invariant (use OBJET[KOBJ=8], with the previously found periodic $\alpha_{x,y}$, $\beta_{x,y}$). Transport that bunch through the cell:
 - plot initial and final phase spaces, on a common graph (use FAISTORE);
 - graphically, verify that the ellipse parameters coincide with the periodic $\alpha_{x,y}$, $\beta_{x,y}$;
 - plot the trajectories $Y(s)$ through the cell (use DRIFT[split] for fine resolution; use IL=2 to log particle data);
 - from this graph data, get an estimate of Y_{\min} and Y_{\max} . Give their exact locations $s(Y_{\min})$ and $s(Y_{\max})$.
 - plot the phase space ellipses, right upstream and right downstream of the F lens, on the same graph. Same for the D lens.

ANSWER

$$1.1/ \quad T = \begin{pmatrix} 1 & l/2 \\ 0 & 1 \end{pmatrix} \times \begin{pmatrix} 1 & 0 \\ 1/f & 1 \end{pmatrix} \times \begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix} \times \begin{pmatrix} 1 & 0 \\ -1/f & 1 \end{pmatrix} \times \begin{pmatrix} 1 & l/2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 - \frac{l}{f}(1 + \frac{l}{2f}) & l(2 - \frac{l^2}{4f^2}) \\ -\frac{l}{f^2} & 1 + \frac{l}{f}(1 - \frac{l}{2f}) \end{pmatrix}$$

1.2/ Development of $\det = ad-bc$ shows that the matrix T above does feature $\det(T)=1$.

1-3/ Periodic stability requires $|\text{trace}(T)/2| < 1$. This yields $|1 - \frac{l^2}{2f^2}| < 1$, which happens iff $\frac{l^2}{2f^2} < 2$, i.e. $f > l/2$.