**Homework 5.**

**Problem 1. 3 x 5 points.** Function of a Jordan block

(a) Show that powers of m x m Jordan block

$$
\mathbf{G} = \begin{bmatrix}
\lambda & 1 & 0 & 0 \\
0 & \lambda & 1 & 0 \\
0 & 0 & \dots & 1 \\
0 & 0 & 0 & \lambda
\end{bmatrix}
$$

are

$$
\mathbf{G}^n = \begin{bmatrix}
\lambda^n & C_1^n\lambda^{n-1} & C_2^n\lambda^{n-2} & \dots & C_k^n\lambda^{n-k} & C_{k+1}^n\lambda^{n-k-1} & \dots \\
0 & \lambda^n & C_1^n\lambda^{n-1} & \dots & C_{k-1}^n\lambda^{n+1-k} & C_k^n\lambda^{n-k} & \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots \\
0 & 0 & 0 & 0 & 0 & 0 & \lambda^n
\end{bmatrix}; C_k^n = \frac{n!}{(n-k)!k!} \quad (1)
$$

Suggestion: use $\mathbf{G}^0 = \mathbf{I}; \mathbf{G}^1 = \mathbf{G}$ - as first step, they satisfy 1. Than use induction assuming that (1) is correct for $n$ and show that $\mathbf{G}^{n+1} = \mathbf{G} \cdot \mathbf{G}^n$ satisfy (1) for **n+1**. Use a well know ratio $C_k^{n+1} = C_k^n + C_{k-1}^n$.

(b) For a polynomial function $f(x) = \sum\limits_{n=0}^{N} f_n x^n$ show that

$$
f(\mathbf{G}) = \sum\limits_{n=0}^{N} f_n \mathbf{G}^n = \begin{bmatrix}
\sum\limits_{n=0}^{N} f_n \lambda^n & \dots & \sum\limits_{n=0}^{N} f_n C_k^n \lambda^{n-k} & \dots \\
0 & \dots & \dots & \dots \\
0 & 0 & & \sum\limits_{n=0}^{N} f_n \lambda^n
\end{bmatrix}
$$

and $\sum\limits_{n=0}^{N} f_n C_k^n \lambda^{n-k} = \dfrac{1}{k!} \dfrac{d^k f}{d\lambda^k}$  #

(c) Prove that for an arbitrary (well behaved function!) $f(x) = \sum\limits_{n=0}^{\infty} f_n x^n$

(d) $f(\mathbf{G}) = \begin{bmatrix}
f(\lambda) & f'(\lambda)/1! & \dots f^{(k)}(\lambda)/k! & f^{(n-1)}(\lambda)/(n-1)! \\
0 & f(\lambda) & \dots & f^{(n-2)}(\lambda)/(n-2)! \\
\dots & \dots & \dots & \dots \\
0 & 0 & \dots & f'(\lambda)/1! \\
0 & 0 & \dots & f(\lambda)
\end{bmatrix}$