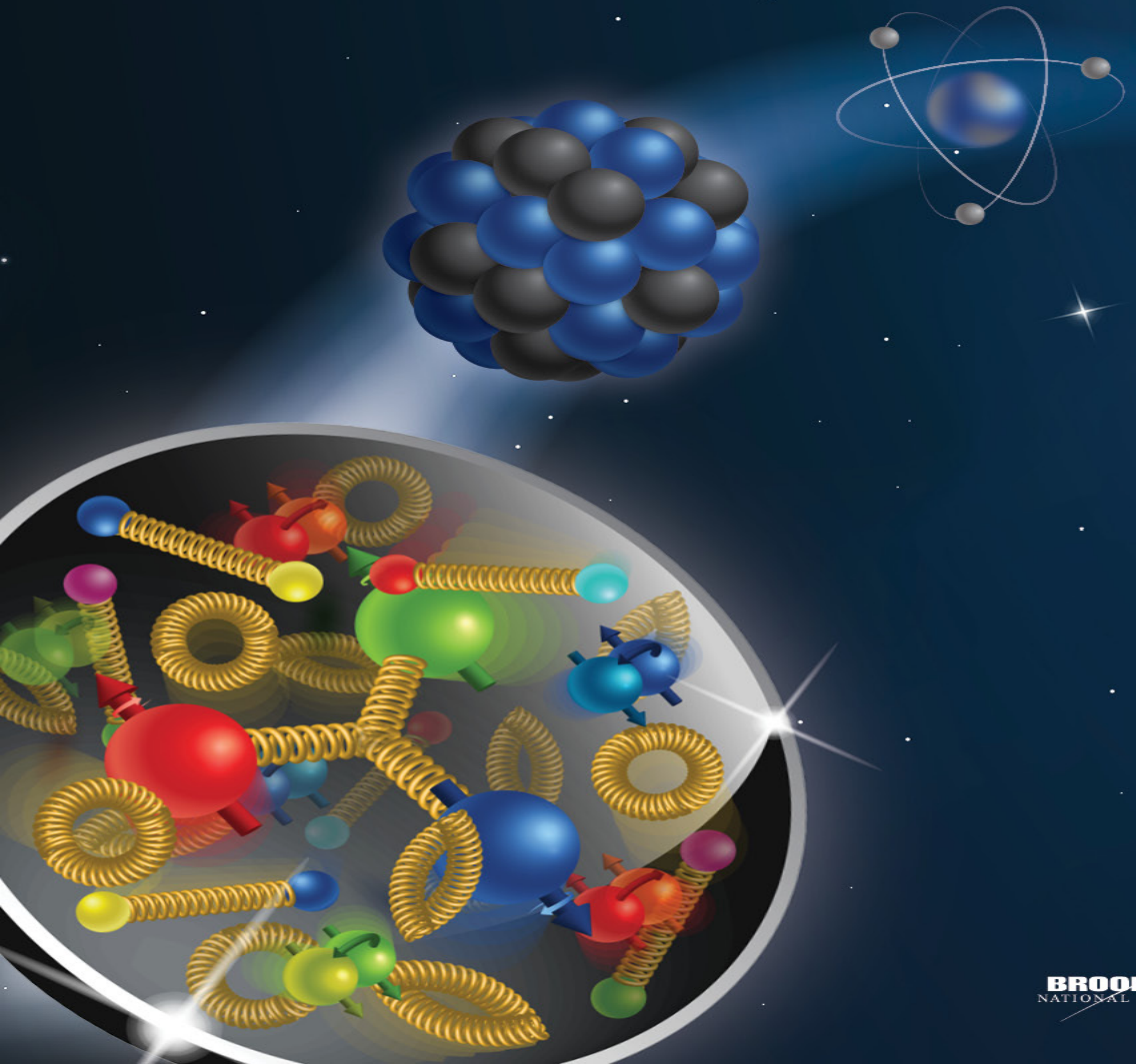


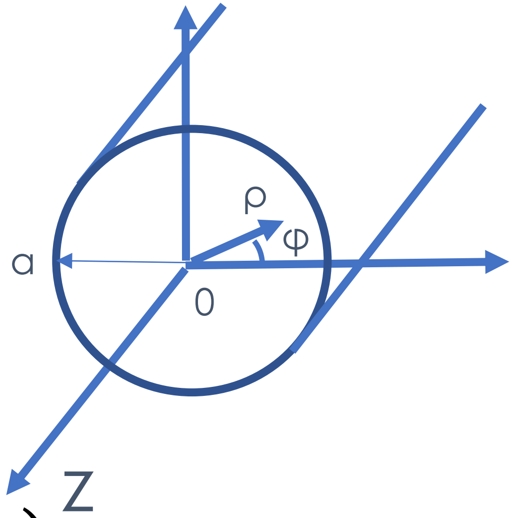
High Power RF Engineering -Waveguide (2)

Binping Xiao

Electron-Ion Collider

BROOKHAVEN NATIONAL LABORATORY
Jefferson Lab
U.S. DEPARTMENT OF ENERGY | Office of Science

Cylindrical coordinate



∇ x **E** = -jωμ**H** & ∇ x **H** = jωε**E**

$$\nabla \times \mathbf{E} = \boldsymbol{\rho}\left(\frac{1}{\rho}\frac{\partial E_z}{\partial \varphi} - \frac{\partial E_\varphi}{\partial z}\right) + \boldsymbol{\varphi}\left(\frac{\partial E_\rho}{\partial z} - \frac{\partial E_z}{\partial \rho}\right) + \mathbf{z}\frac{1}{\rho}\left(\frac{\partial(\rho E_\varphi)}{\partial \rho} - \frac{\partial E_\rho}{\partial \varphi}\right)$$

$$= \boldsymbol{\rho}\left(\frac{1}{\rho}\frac{\partial E_z}{\partial \varphi} + j\beta E_\varphi\right) + \boldsymbol{\varphi}\left(-j\beta E_\rho - \frac{\partial E_z}{\partial \rho}\right) + \mathbf{z}\frac{1}{\rho}\left(\frac{\partial(\rho E_\varphi)}{\partial \rho} - \frac{\partial E_\rho}{\partial \varphi}\right)$$

$$= \boldsymbol{\rho}(-j\omega\mu H_\rho) + \boldsymbol{\varphi}(-j\omega\mu H_\varphi) + \mathbf{z}(-j\omega\mu H_z)$$

Similarly

$$\boldsymbol{\rho}\left(\frac{1}{\rho}\frac{\partial H_z}{\partial \varphi} + j\beta H_\varphi\right) + \boldsymbol{\varphi}\left(-j\beta H_\rho - \frac{\partial H_z}{\partial \rho}\right) + \mathbf{z}\frac{1}{\rho}\left(\frac{\partial(\rho H_\varphi)}{\partial \rho} - \frac{\partial H_\rho}{\partial \varphi}\right)$$

$$= \boldsymbol{\rho}(j\omega\varepsilon E_\rho) + \boldsymbol{\varphi}(j\omega\varepsilon E_\varphi) + \mathbf{z}(j\omega\varepsilon E_z)$$

Field Distribution

$$\frac{1}{\rho}\frac{\partial E_z}{\partial \varphi} + j\beta E_\varphi = -j\omega\mu H_\rho$$

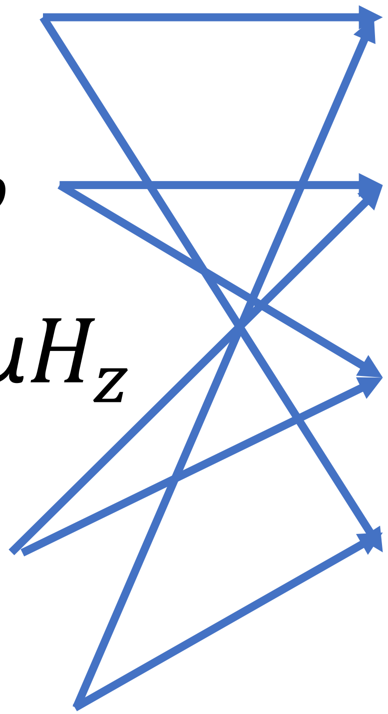
$$-j\beta E_\rho - \frac{\partial E_z}{\partial \rho} = -j\omega\mu H_\varphi$$

$$\frac{1}{\rho}\left(\frac{\partial(\rho E_\varphi)}{\partial \rho} - \frac{\partial E_\rho}{\partial \varphi}\right) = -j\omega\mu H_z$$

$$\frac{1}{\rho}\frac{\partial H_z}{\partial \varphi} + j\beta H_\varphi = j\omega\varepsilon E_\rho$$

$$-j\beta H_\rho - \frac{\partial H_z}{\partial \rho} = j\omega\varepsilon E_\varphi$$

$$\frac{1}{\rho}\left(\frac{\partial(\rho H_\varphi)}{\partial \rho} - \frac{\partial H_\rho}{\partial \varphi}\right) = j\omega\varepsilon E_z$$



$$k_c^2 H_\rho = j\left(\frac{\omega\varepsilon}{\rho}\frac{\partial E_z}{\partial \varphi} - \beta\frac{\partial H_z}{\partial \rho}\right)$$

$$k_c^2 H_\varphi = -j\left(\omega\varepsilon\frac{\partial E_z}{\partial \rho} + \frac{\beta}{\rho}\frac{\partial H_z}{\partial \varphi}\right)$$

$$k_c^2 E_\rho = -j\left(\beta\frac{\partial E_z}{\partial \rho} + \frac{\omega\mu}{\rho}\frac{\partial H_z}{\partial \varphi}\right)$$

$$k_c^2 E_\varphi = j\left(-\frac{\beta}{\rho}\frac{\partial E_z}{\partial \varphi} + \omega\mu\frac{\partial H_z}{\partial \rho}\right)$$

$$k_c^2 = k^2 - \beta^2 \; \& \; k = \omega\sqrt{\mu\varepsilon}$$

Note: there are two equations that have not been used yet.

Transverse Electric (TE)

- $E_z = 0$ & $H_z \neq 0$.
- Wave impedance

$Z_{TE} = E_\rho / H_\varphi = -E_\varphi / H_\rho = \omega\mu/\beta = k\eta/\beta$

$$H_\rho = \frac{-j\beta}{k_c^2}\frac{\partial H_z}{\partial \rho}$$

$$H_\varphi = \frac{-j\beta}{\rho k_c^2}\frac{\partial H_z}{\partial \varphi}$$

$$E_\rho = \frac{-j\omega\mu}{\rho k_c^2}\frac{\partial H_z}{\partial \varphi}$$

$$E_\varphi = \frac{j\omega\mu}{k_c^2}\frac{\partial H_z}{\partial \rho}$$

$$\frac{\partial(\rho H_\varphi)}{\partial \rho} = \frac{\partial(H_\rho)}{\partial \varphi}$$

Transverse Magnetic (TM)

- $E_z \neq 0$ & $H_z = 0$.
- Wave impedance

$Z_{TM} = E_\rho / H_\varphi = -E_\varphi / H_\rho = \beta/\omega\varepsilon = \beta\eta/k$

$$H_\rho = \frac{j\omega\varepsilon}{\rho k_c^2}\frac{\partial E_z}{\partial \varphi}$$

$$H_\varphi = \frac{-j\omega\varepsilon}{k_c^2}\frac{\partial E_z}{\partial \rho}$$

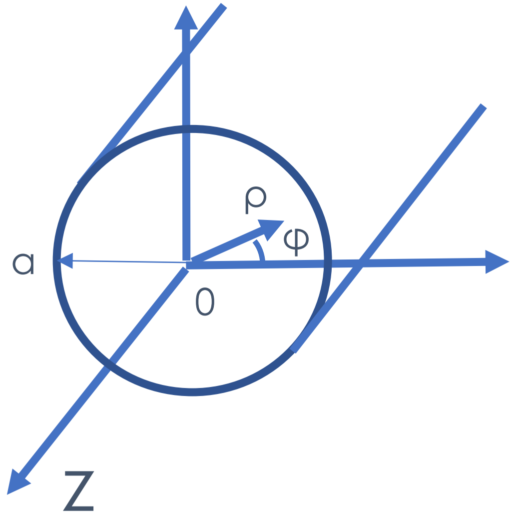
$$E_\rho = \frac{-j\beta}{k_c^2}\frac{\partial E_z}{\partial \rho}$$

$$E_\varphi = -\frac{j\beta}{\rho k_c^2}\frac{\partial E_z}{\partial \varphi}$$

$$\frac{\partial(\rho E_\varphi)}{\partial \rho} = \frac{\partial E_\rho}{\partial \varphi}$$

Circular Waveguide

TE (1)



$$\frac{1}{\rho}\left(\frac{\partial(\rho E_\varphi)}{\partial\rho}-\frac{\partial E_\rho}{\partial\varphi}\right)=-j\omega\mu H_z$$

& $E_\varphi = \frac{j\omega\mu}{k_c^2}\frac{\partial H_z}{\partial\rho}$ & $E_\rho = \frac{-j\omega\mu}{\rho k_c^2}\frac{\partial H_z}{\partial\varphi}$

$$\rightarrow \left(\frac{\partial^2}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial}{\partial\rho}+\frac{1}{\rho^2}\frac{\partial^2}{\partial\varphi^2}+k_c^2\right)H_z = 0$$

and then use " separation of variables" $H_z(\rho,\varphi) = P(\rho)\Phi(\varphi)e^{-j\beta z}$

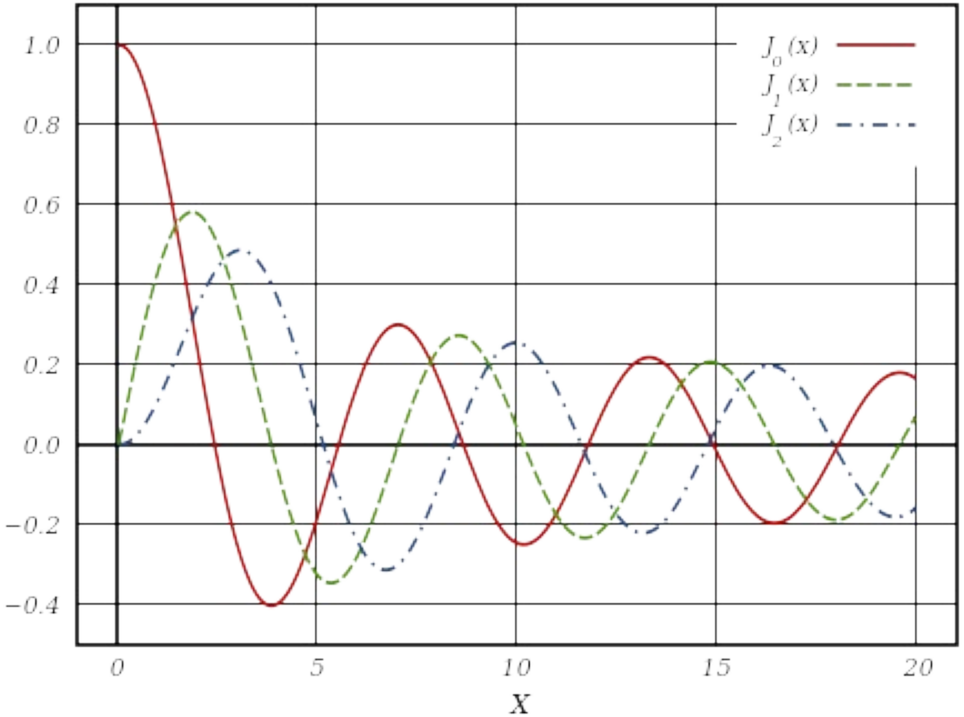
$$\rho^2\frac{d^2P}{d\rho^2}+\rho\frac{dP}{d\rho}+\left(k_c^2\rho^2-k_\varphi^2\right)P = 0 \ \& \ \frac{d^2\Phi}{d\varphi^2}+k_\varphi^2\Phi = 0$$

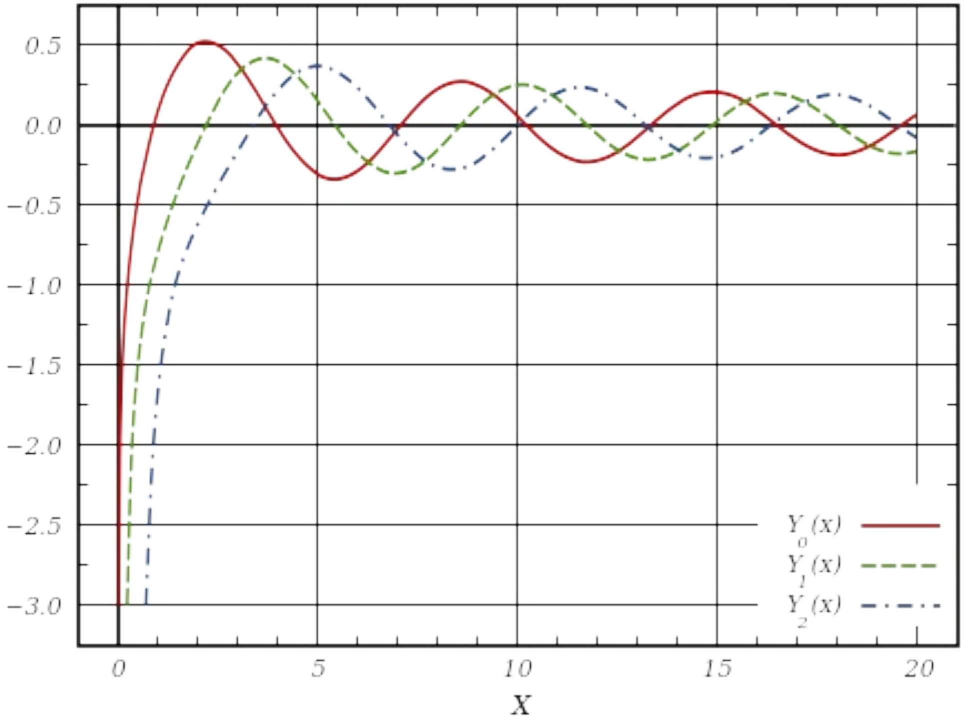
For $\frac{d^2\Phi}{d\varphi^2}+k_\varphi^2\Phi = 0$, $\Phi(\varphi) = A\sin k_\varphi\varphi + B\cos k_\varphi\varphi$, and Φ should be periodic every 2π, thus $k_\varphi = n = 0,1,2,3\ldots$

Bessel functions

Bessel's differential equation $x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+(x^2-\alpha^2)y=0$, α can be a complex number, in our application α = 0,1,2,3…

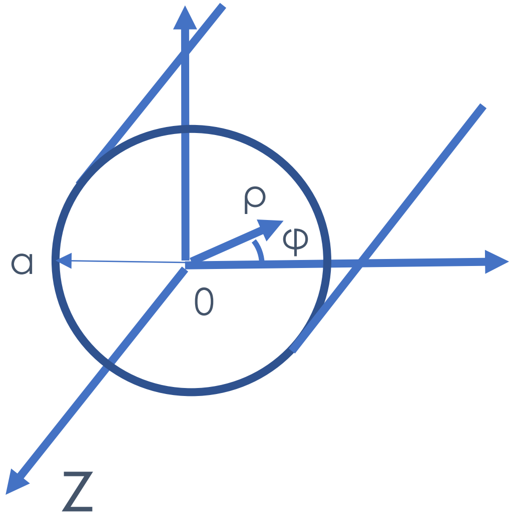
Bessel functions of the 1st kind (left) and the 2nd kind (right)





From: Wikipedia https://en.wikipedia.org/wiki/Bessel_function

TE (2)



$\rho^2 \frac{d^2P}{d\rho^2} + \rho \frac{dP}{d\rho} + (k_c^2\rho^2 - n^2)P = 0$ is Bessel equation

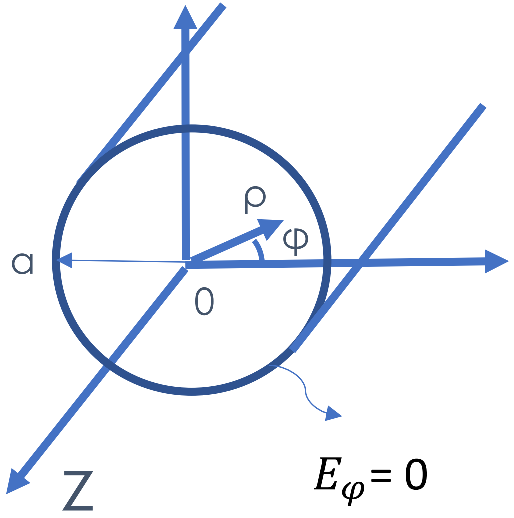
The solution is $P(\rho) = CJ_n(k_c\rho) + DY_n(k_c\rho)$, $J_n(x)$ and $Y_n(x)$ are Bessel functions of the 1st and 2nd kind. $Y_n(0) = -\infty$ is not acceptable since the field at ρ=0 cannot be infinite, thus D=0 and $P(\rho) = CJ_n(k_c\rho)$. We have

$$H_z(\rho,\varphi) = k_c^2(A\sin n\varphi + B\cos n\varphi)J_n(k_c\rho)e^{-j\beta z}$$

and $E_\varphi = \frac{j\omega\mu}{k_c^2}\frac{\partial H_z}{\partial \rho}$, thus

$$E_\varphi = j\omega\mu k_c(A\sin n\varphi + B\cos n\varphi)J_n'(k_c\rho)e^{-j\beta z}$$

TE (3)



Boundary conditions:

E field should be perpendicular to the metal walls, thus $E_\varphi|_{\rho=a} = 0$

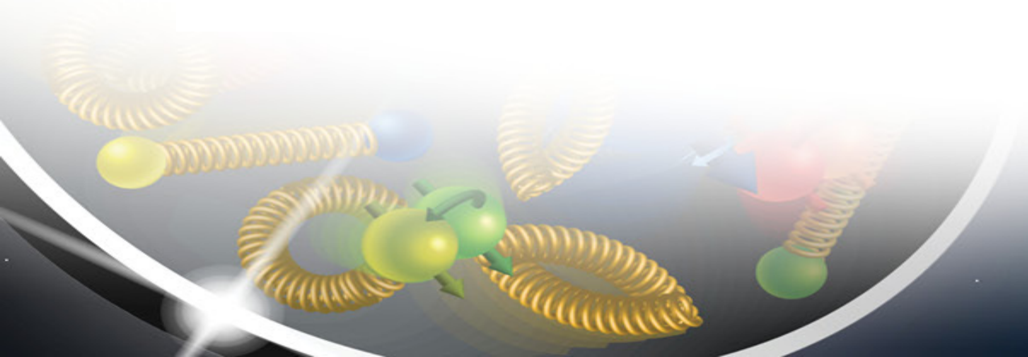
$J'_n(k_c a) = 0$

We define P'_{nm} the m^th root of J'_n, with m = 1,2,3…

we have $k_{c\_nm} = P'_{nm}/a$.

Values of P'_{nm}

n	P'_{n1}	P'_{n2}	P'_{n3}
0	3.832	7.016	10.174
1	1.841	5.331	8.536
2	3.054	6.706	9.970



TE (4)

$$E_\rho = \frac{-j\omega\mu n}{\rho}(A\cos n\varphi - B\sin n\varphi)J_n\left(\frac{P'_{nm}}{a}\rho\right)e^{-j\beta z}$$

$$E_\varphi = j\omega\mu\frac{P'_{nm}}{a}(A\sin n\varphi + B\cos n\varphi)J'_n\left(\frac{P'_{nm}}{a}\rho\right)e^{-j\beta z}$$

$$E_z = 0$$

$$H_\rho = -j\beta\frac{P'_{nm}}{a}(A\sin n\varphi + B\cos n\varphi)J'_n\left(\frac{P'_{nm}}{a}\rho\right)e^{-j\beta z}$$

$$H_\varphi = \frac{-j\beta n}{\rho}(A\cos n\varphi - B\sin n\varphi)J_n\left(\frac{P'_{nm}}{a}\rho\right)e^{-j\beta z}$$

$$H_z = \left(\frac{P'_{nm}}{a}\right)^2(A\sin n\varphi + B\cos n\varphi)J_n\left(\frac{P'_{nm}}{a}\rho\right)e^{-j\beta z}$$

with $k_{c\_nm} = P'_{nm}/a$

Terms containing A can be get by rotating terms containing B by 90° (orthogonal, while n ≠ 0, polarization degeneracy). One can choose either the term that containing A or the term that containing B, for example:

$$E_\rho = \frac{j\omega\mu n}{\rho}B\sin n\varphi J_n\left(\frac{P'_{nm}}{a}\rho\right)e^{-j\beta z}$$

$$E_\varphi = j\omega\mu\frac{P'_{nm}}{a}B\cos n\varphi J'_n\left(\frac{P'_{nm}}{a}\rho\right)e^{-j\beta z}$$

$$E_z = 0$$

$$H_\rho = -j\beta\frac{P'_{nm}}{a}B\cos n\varphi J'_n\left(\frac{P'_{nm}}{a}\rho\right)e^{-j\beta z}$$

$$H_\varphi = \frac{j\beta n}{\rho}B\sin n\varphi J_n\left(\frac{P'_{nm}}{a}\rho\right)e^{-j\beta z}$$

$$H_z = \left(\frac{P'_{nm}}{a}\right)^2B\cos n\varphi J_n\left(\frac{P'_{nm}}{a}\rho\right)e^{-j\beta z}$$

$Asinn\varphi + Bcosn\varphi$ and degeneracy

- Degenerate modes: modes that have the same cutoff frequency in a waveguide or have the same frequency in a cavity.
- Mathematically $Asinn\varphi + Bcosn\varphi \neq Bcosn\varphi$, we choose cos so that when $n = 0$, it is non-zero.
- For a cylindrical symmetric structure, however, If you rotate $90^\circ/n$ of the field pattern that containing term *A*, you will get the field pattern that containing term *B* (while $n \neq 0$, polarization degeneracy). One can choose either the term that containing *A* or the term that containing *B*.
- The mode pattern inside is determined by the input signal.
- We will show an example about polarization degeneracy soon.

TE (5)

n = 0,1,2,3… and m = 1,2,3…

TE_{nm} with n for φ and m for ρ.

There is no TE_{n0} mode in a circular waveguide.

There are TE_{0m} modes in it.

The mode with lowest cutoff frequency for TE modes is TE_{11}, with $f_{c\_TE_{11}} = \frac{1.841c}{2\pi a}$

$f_{c\_TE_{21}} = \frac{3.054c}{2\pi a}$ & $f_{c\_TE_{01}} = \frac{3.832c}{2\pi a}$

TE_{11} is the dominant mode.

Values of P'_{nm}

n	P'_{n1}	P'_{n2}	P'_{n3}
0	3.832	7.016	10.174
1	1.841	5.331	8.536
2	3.054	6.706	9.970

Meaning of *n*

- $E_\varphi = j\omega\mu \frac{P'_{nm}}{a} B cos n\varphi J'_n\left(\frac{P'_{nm}}{a}\rho\right) e^{-j\beta z}$
- *n* = 0 means the fields do not change along φ (recall that TEM mode in coax line do not change along φ as well), it is called monopole.
- *n* = 1 means the fields change 1 cycle ($cos\varphi$) along φ, it is called dipole.
- *n* = 2 means the fields change 2 cycles ($cos2\varphi$) along φ, it is called quadrupole.
- *n* = 3 sextupole/hexapole, *n* = 4 octupole…

Meaning of *m*

- The mth root for $J_n'\left(\frac{P_{nm}'}{a}\rho\right)$
- There is no zeroth root thus m = 1,2,3…

TE_{11}

$$E_\rho = \frac{-j\omega\mu}{\rho} A\cos\varphi J_1\left(\frac{P'_{11}}{a}\rho\right) e^{-j\beta z}$$

$$E_\varphi = j\omega\mu\frac{P'_{11}}{a} A\sin\varphi J'_1\left(\frac{P'_{11}}{a}\rho\right) e^{-j\beta z}$$

$$E_z = 0$$

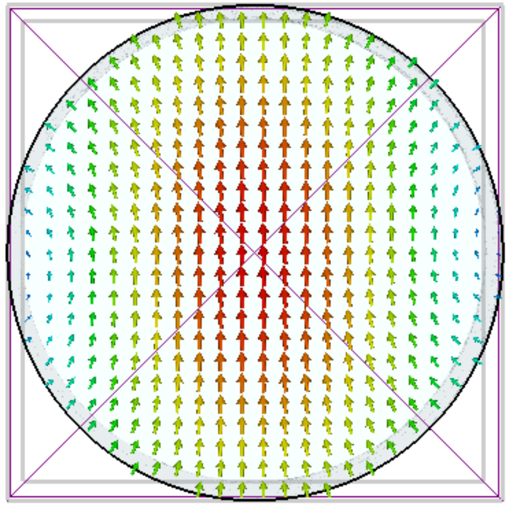
$$H_\rho = -j\beta\frac{P'_{11}}{a} A\sin\varphi J'_1\left(\frac{P'_{11}}{a}\rho\right) e^{-j\beta z}$$

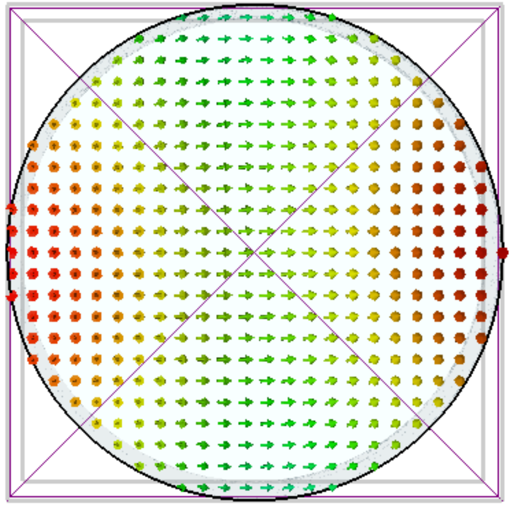
$$H_\varphi = \frac{-j\beta}{\rho} A\cos\varphi J_1\left(\frac{P'_{11}}{a}\rho\right) e^{-j\beta z}$$

$$H_z = \left(\frac{P'_{11}}{a}\right)^2 A\sin\varphi J_1\left(\frac{P'_{11}}{a}\rho\right) e^{-j\beta z}$$

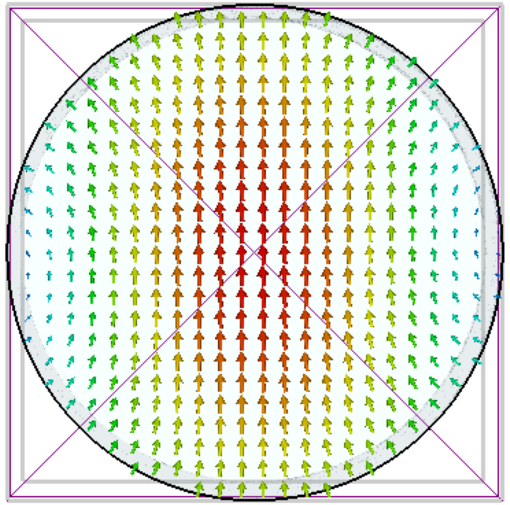
with $k_{c\_11} = \frac{P'_{11}}{a} = \frac{1.841}{a}$

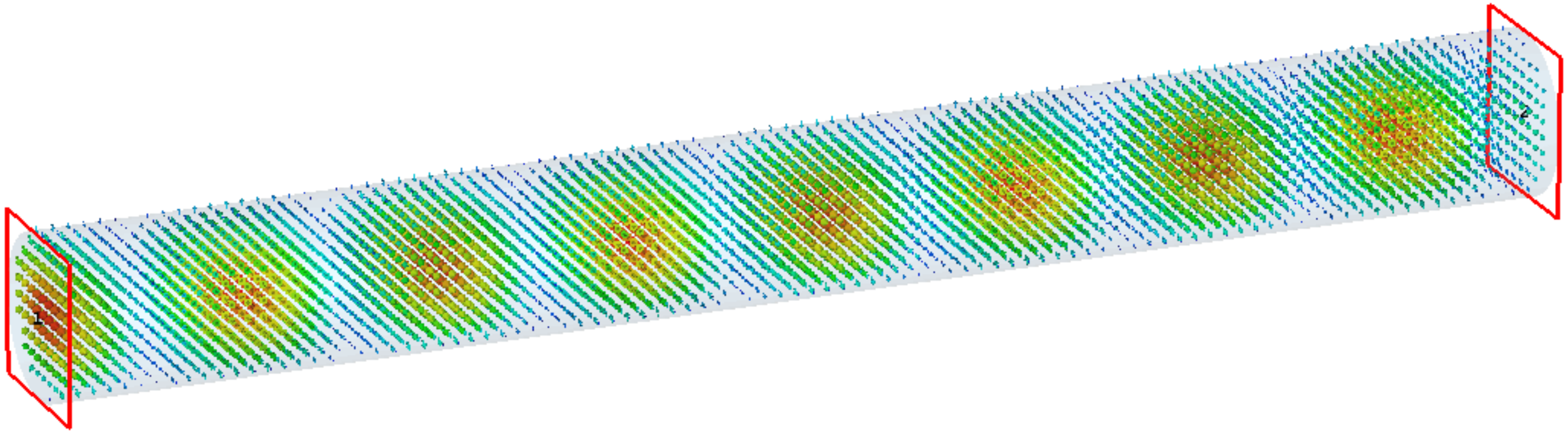
and another TE_{11} that is orthogonal to it.

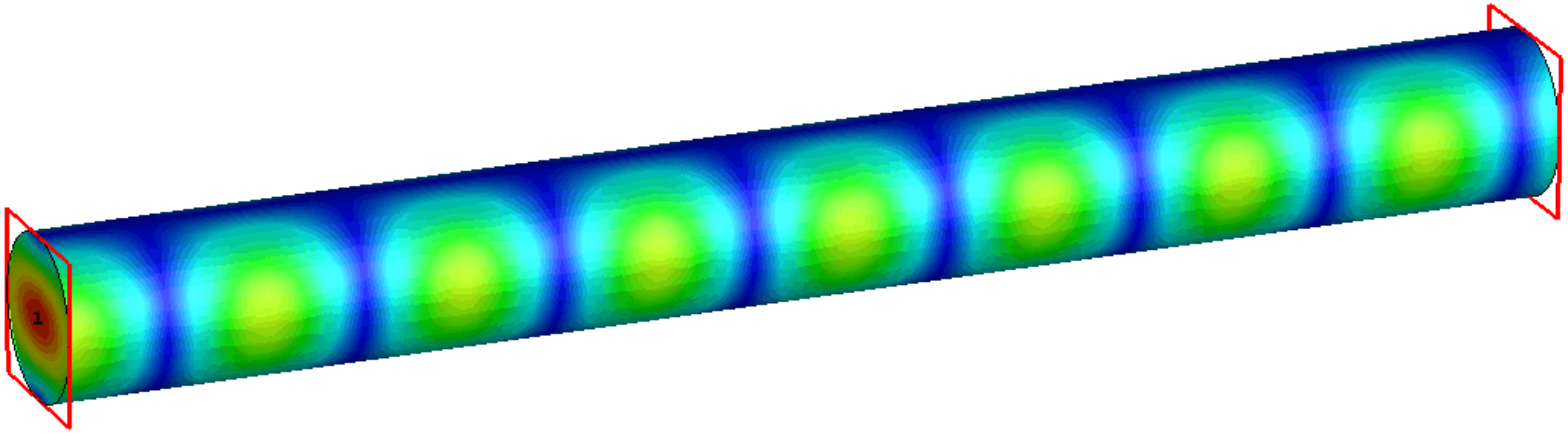




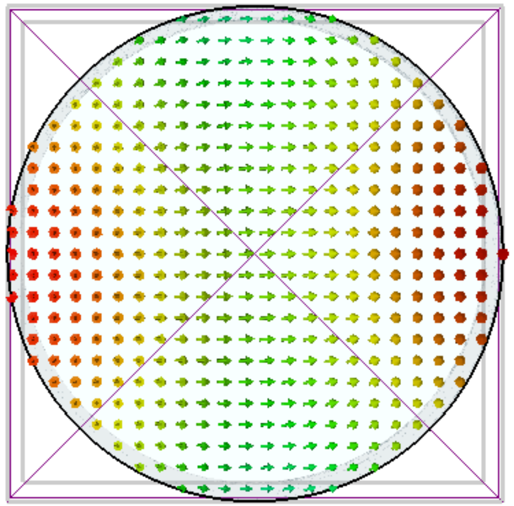
TE_{11} E

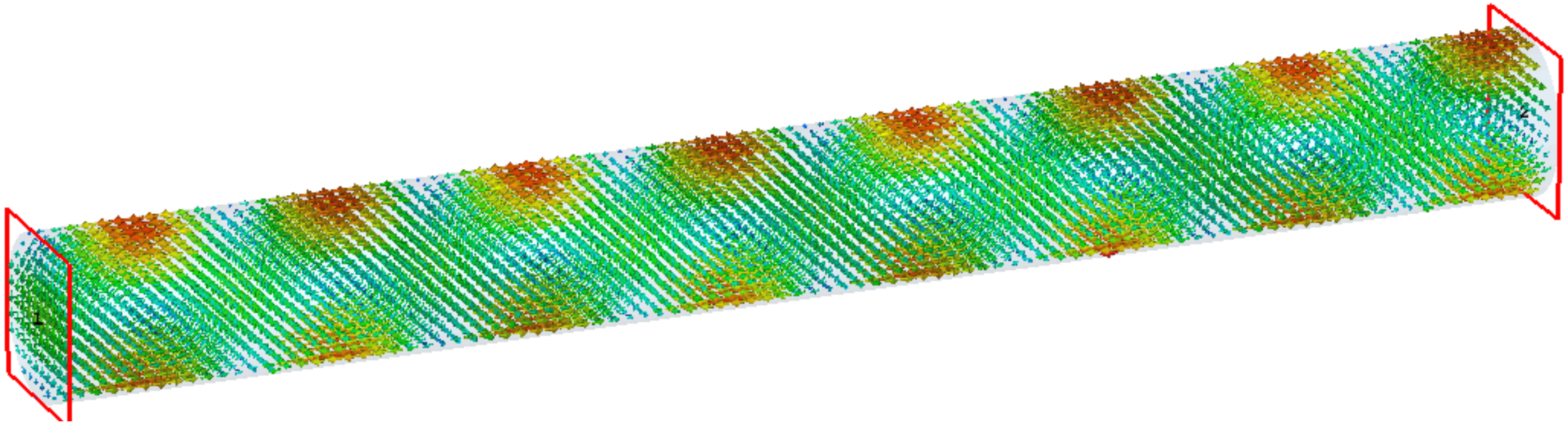


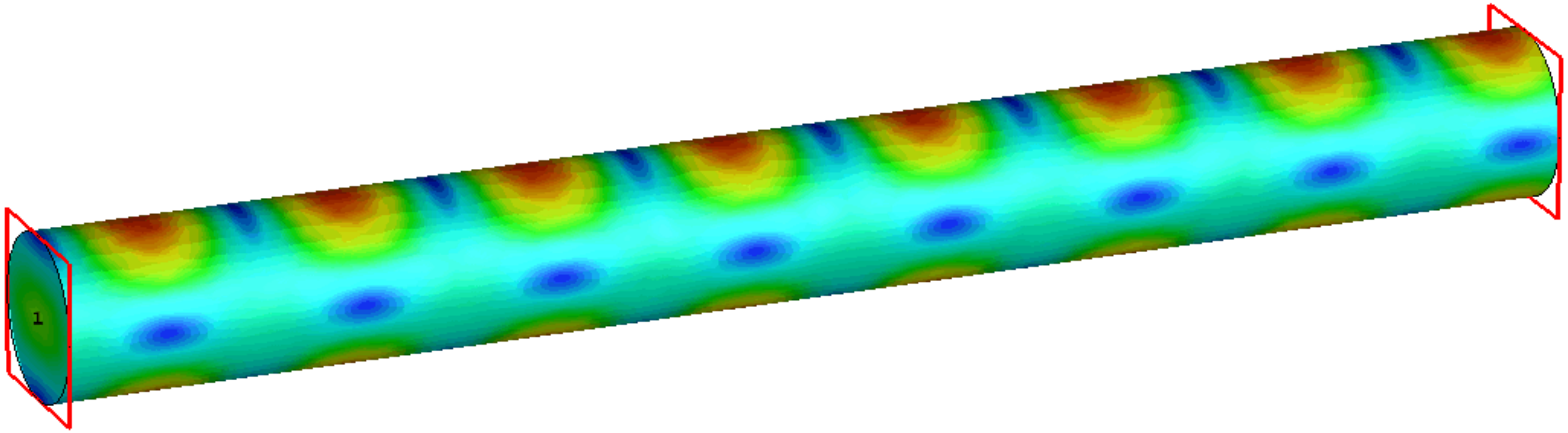




TE_{11} H

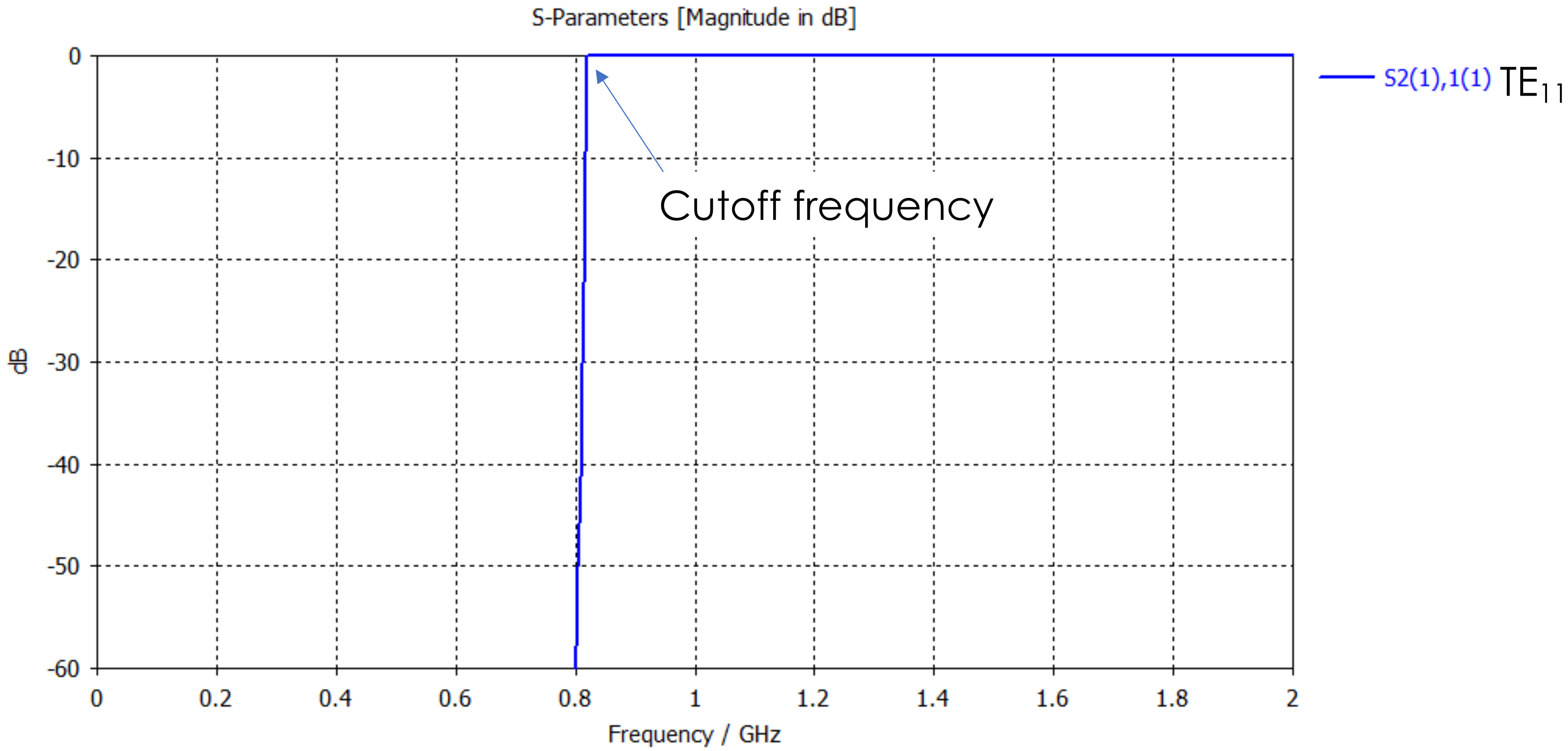






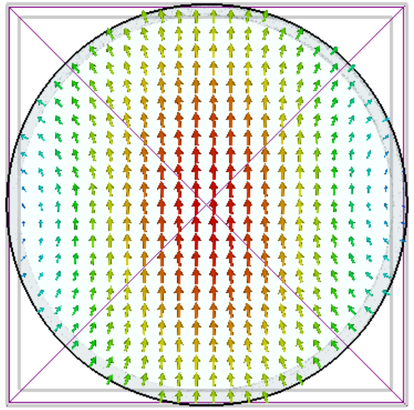
TE_{11} – Cutoff

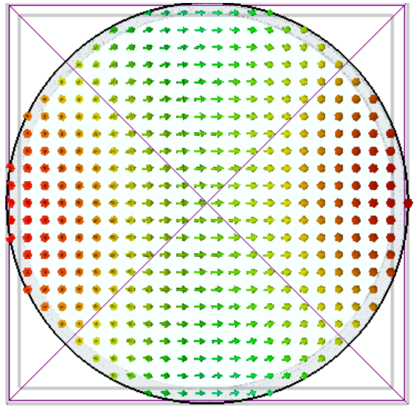
- Loss of TE_{11} versus frequency (assuming 2m long waveguide with Cu wall).

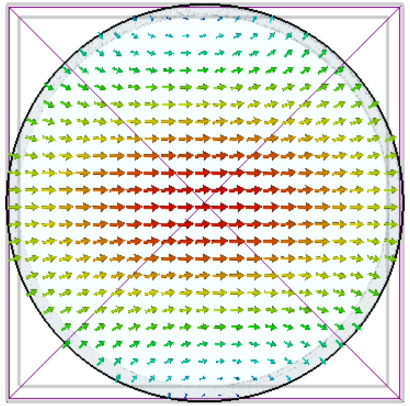


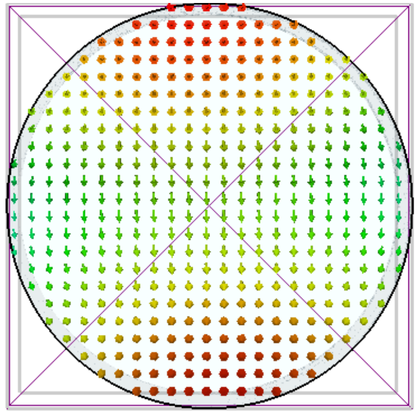
Polarization Degeneracy in TE_{11}

n = 1, dipole, fields change 1 cycle along φ. Another mode that has the same field pattern but rotated 90^{o}.



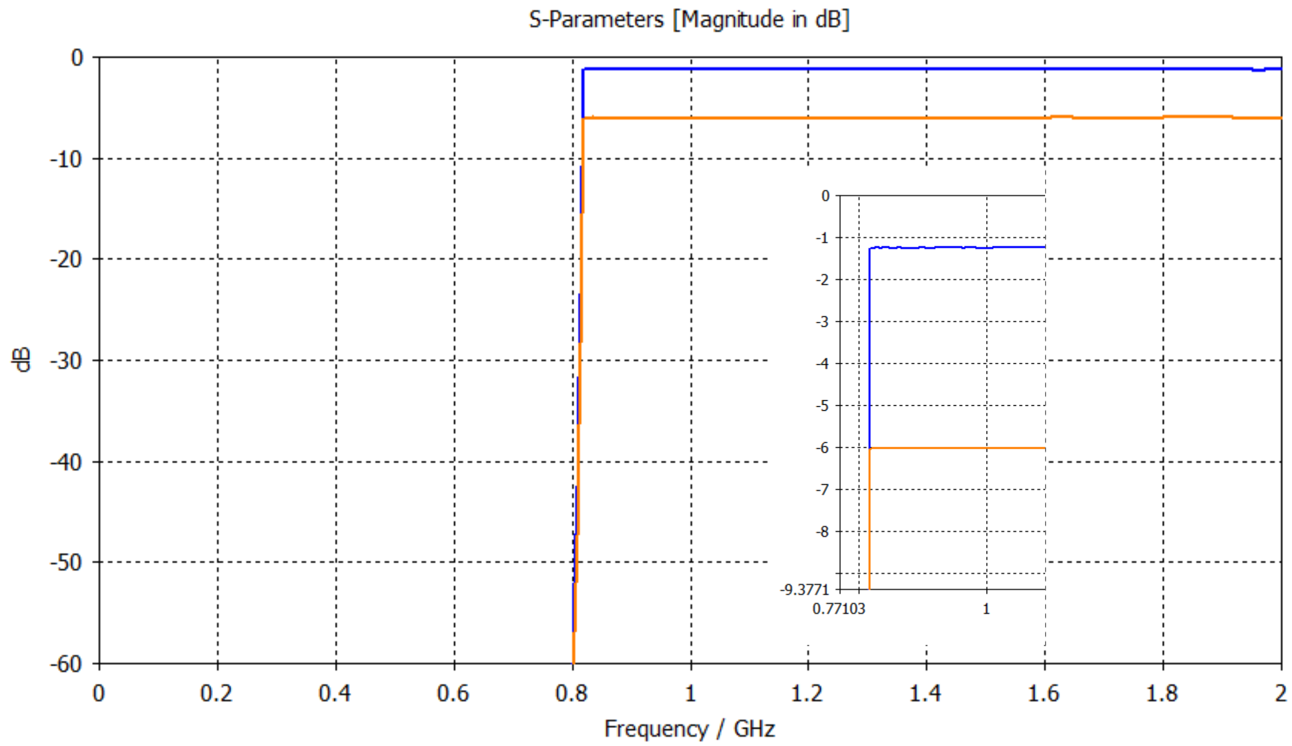






Polarization Degeneracy in TE_{11}

- Loss of TE_{11} (degenerated) versus frequency (assuming 2m long cable with Cu walls).

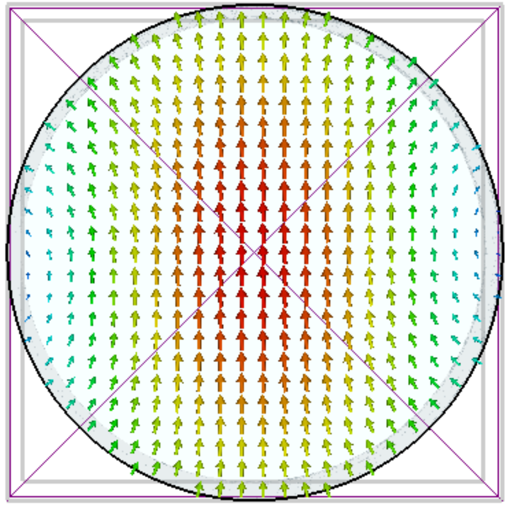


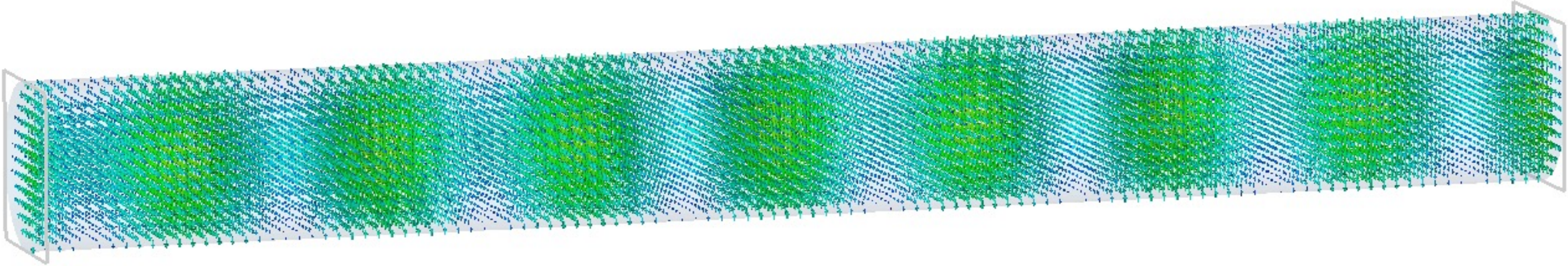
S2(1),1(1) TE_{11} 30° between 2 ports
S2(2),1(1) TE_{11} 60° between 2 ports

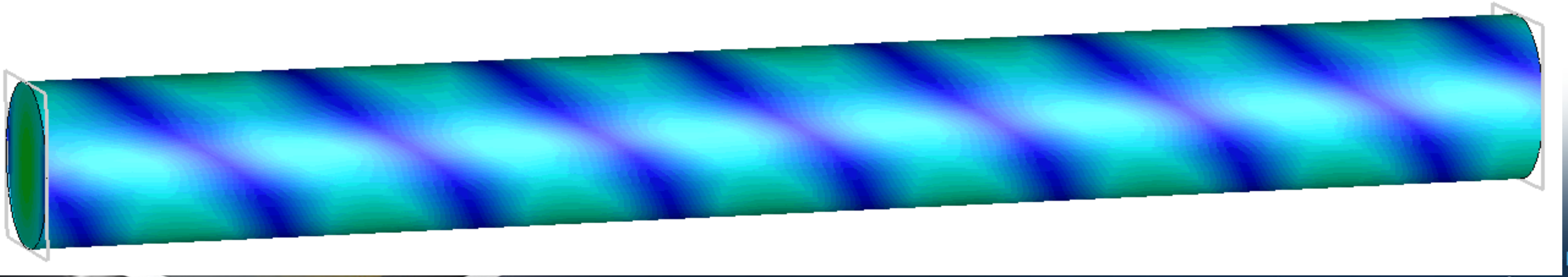
The attenuation depends on the polarization of the port modes on the input and output ports, if they are not aligned, the attenuation will be higher, but it is not real, the total power going to two degenerated modes on the output should be ~100% (minus wall loss), 25% attenuation (-1.25dB) for the blue curve and 75% attenuation (-6dB) for the orange curve in the plot.

TE_{11} E with a circular polarization input signal

rotating

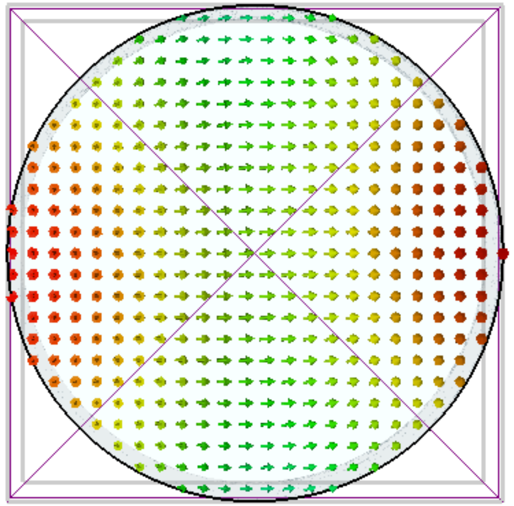


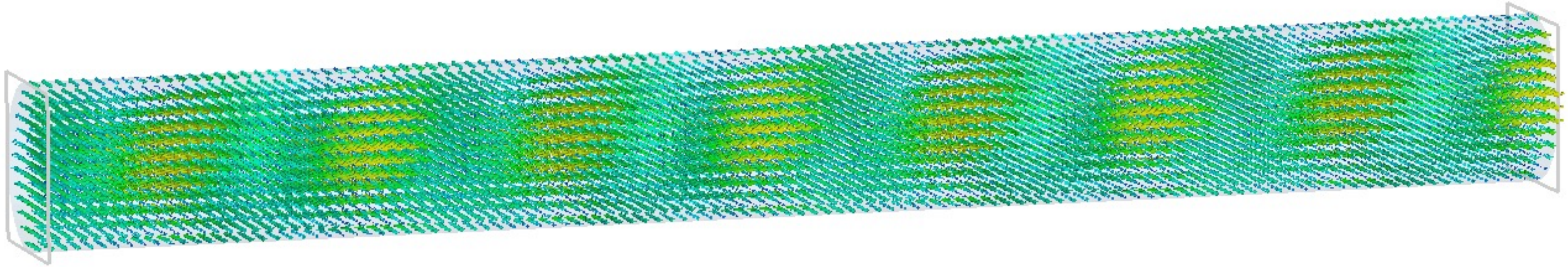


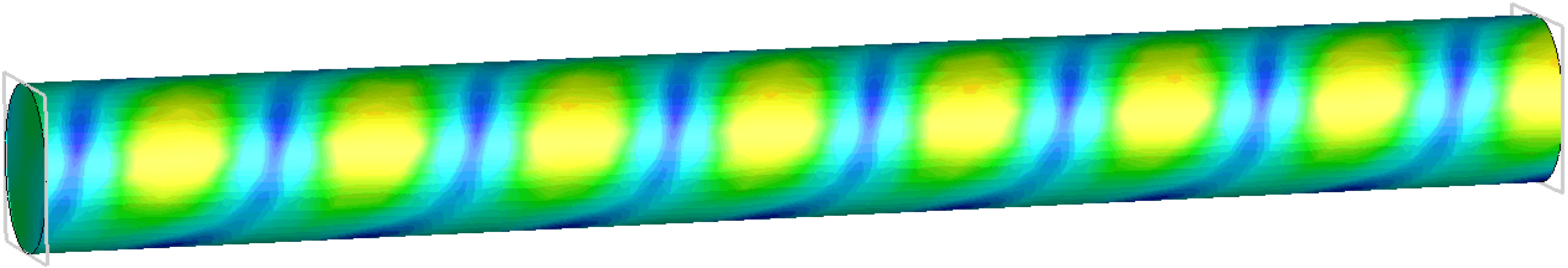


TE_{11} H with a circular polarization input signal

rotating







Circular Waveguide – TE_{01}

$E_\rho = 0$

$E_\varphi = j\omega\mu \frac{P'_{01}}{a} A J'_0\left(\frac{P'_{01}}{a}\rho\right) e^{-j\beta z}$

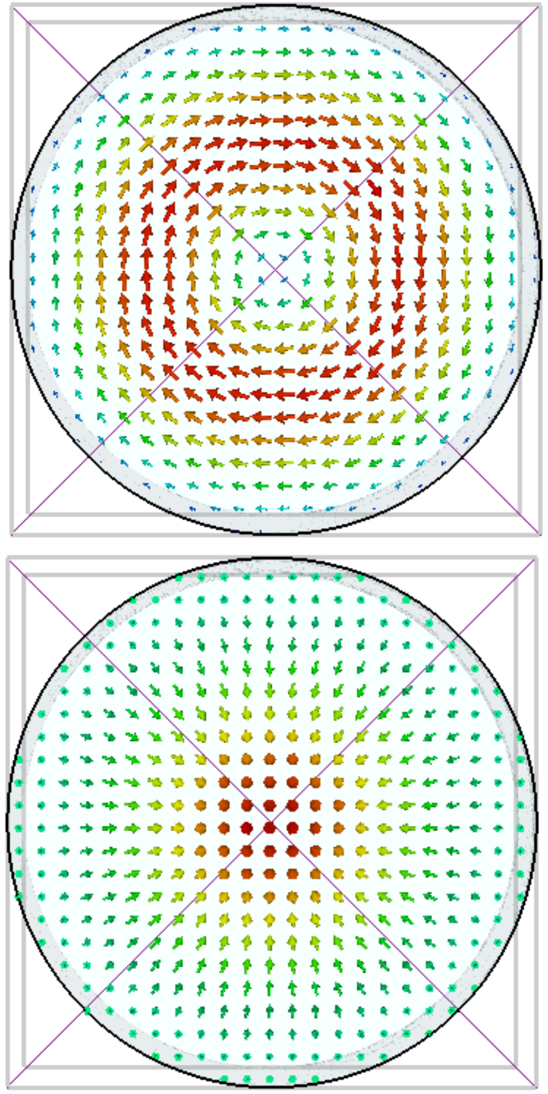
$E_z = 0$

$H_\rho = -j\beta \frac{P'_{01}}{a} A J'_0\left(\frac{P'_{01}}{a}\rho\right) e^{-j\beta z}$

$H_\varphi = 0$

$H_z = (\frac{P'_{01}}{a})^2 A J_0\left(\frac{P'_{01}}{a}\rho\right) e^{-j\beta z}$

with $k_{c\_01} = \frac{P'_{01}}{a} = \frac{3.832}{a}$

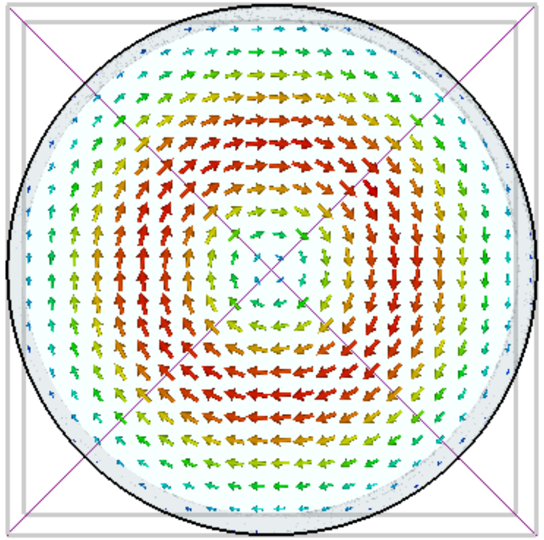


There is no orthogonal TE_{01} since it is not φ dependent.

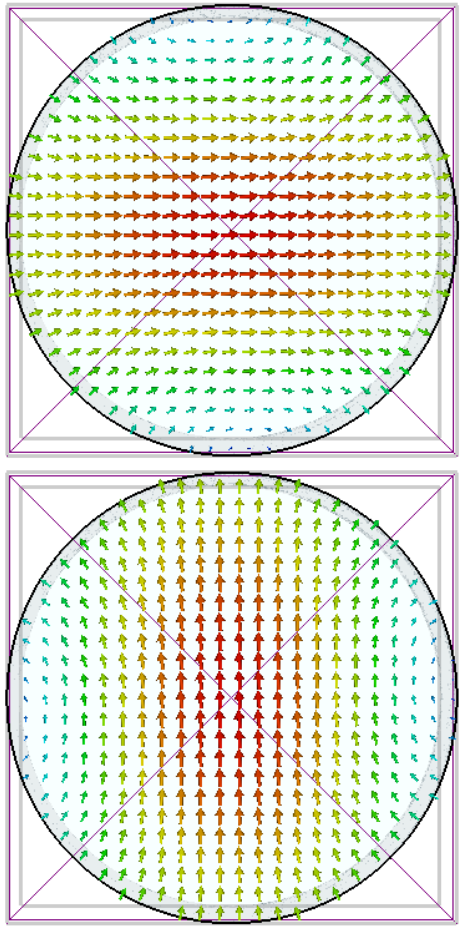
E fields

TE_{n1}

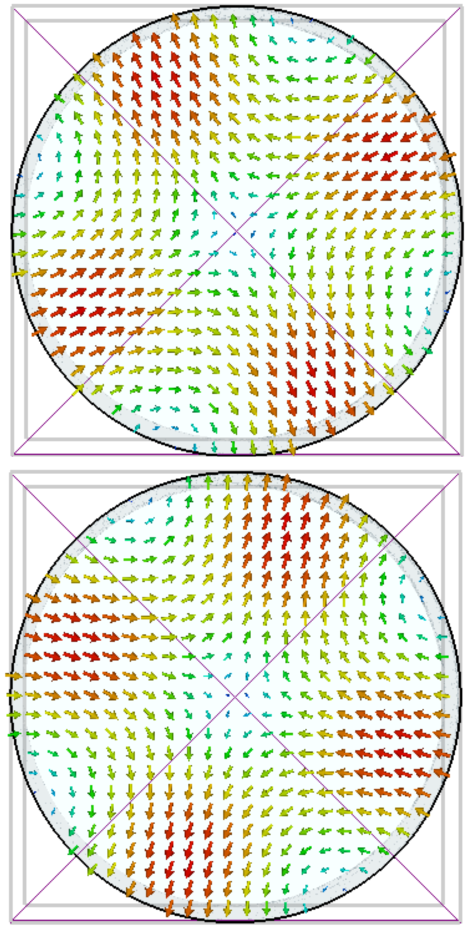
TE_{01} $n = 0$
monopole



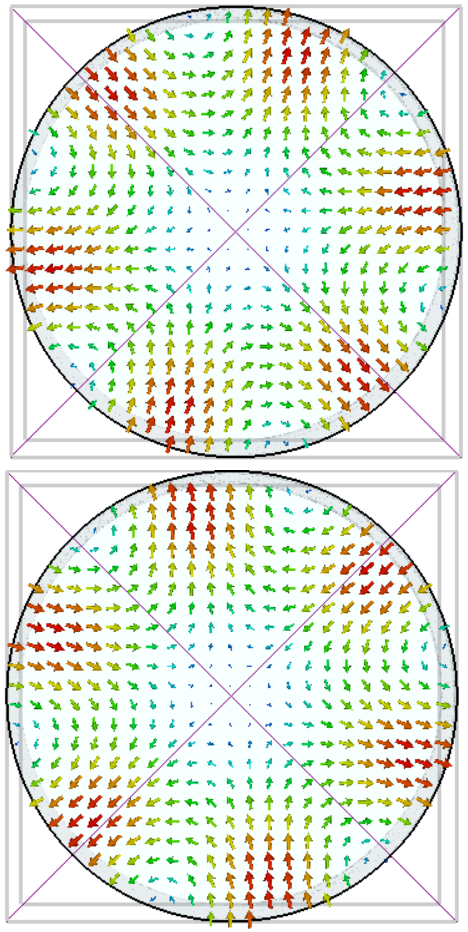
TE_{11} $n = 1$
dipole



TE_{21} $n = 2$
quadrupole

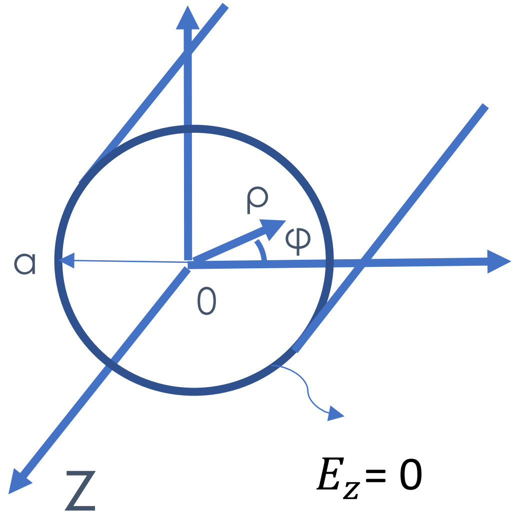


TE_{31} $n = 3$
sextupole/hexapole



Modes are degenerated, 90°/n rotation

Circular Waveguide – TM (1)



$$\frac{1}{\rho}\left(\frac{\partial(\rho H_\varphi)}{\partial\rho} - \frac{\partial H_\rho}{\partial\varphi}\right) = j\omega\varepsilon E_z$$

$$\& \; H_\rho = \frac{j\omega\varepsilon}{\rho k_c^2}\frac{\partial E_z}{\partial\varphi} \; \& \; H_\varphi = \frac{-j\omega\varepsilon}{k_c^2}\frac{\partial E_z}{\partial\rho}$$

$$\rightarrow \left(\frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\varphi^2} + k_c^2\right) E_z = 0$$

thus $E_z(\rho,\varphi) = k_c^2(A sin n\varphi + B cos n\varphi) J_n(k_c\rho) e^{-j\beta z}$

and boundary condition $E_z|_{\rho=a} = 0$, so $J_n(k_c a) = 0$

We define P_{nm} the m[th] root of $J_n(P_{nm})$, with m = 1,2,3…

we have $k_{c\_nm} = P_{nm}/a$.

Values of P_{nm}

n	P_{n1}	P_{n2}	P_{n3}
0	2.405	5.520	8.654
1	3.832	7.016	10.174
2	5.135	8.417	11.620

Bessel functions

Bessel functions of the 1st kind

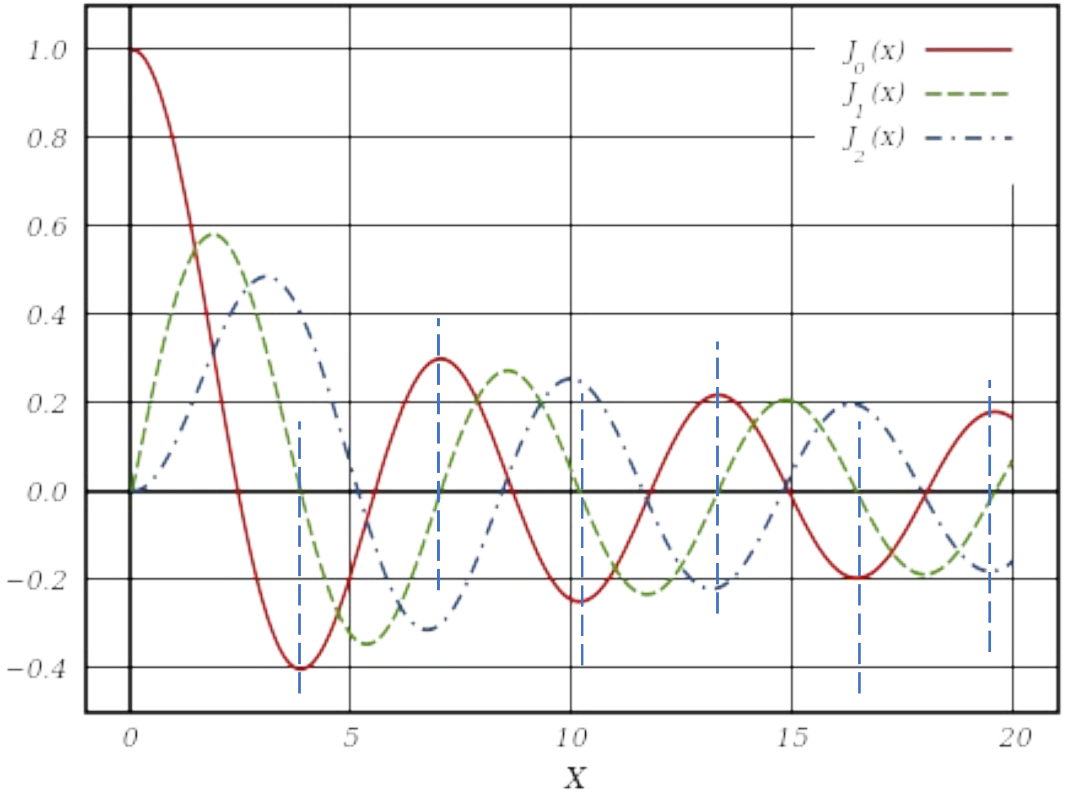
We define P'_{nm} the m^th root of $J'_n(P'_{nm})$, with m = 1,2,3…

We define P_{nm} the m^th root of $J_n(P_{nm})$, with m = 1,2,3…

$P_{1m} = P'_{0m}$

In fact $J'_0(x) = -J_1(x)$

TM_{1n} & TE_{0n} have the same cutoff frequency, TE-TM degeneration.



From: Wikipedia https://en.wikipedia.org/wiki/Bessel_function

Circular Waveguide – TM (2)

$$E_\rho = -j\beta\frac{P_{nm}}{a}(A\sin n\varphi + B\cos n\varphi)J_n'\left(\frac{P_{nm}}{a}\rho\right)e^{-j\beta z}$$

$$E_\varphi = \frac{-j\beta n}{\rho}(A\cos n\varphi - B\sin n\varphi)J_n\left(\frac{P_{nm}}{a}\rho\right)e^{-j\beta z}$$

$$E_z = \left(\frac{P_{nm}}{a}\right)^2(A\sin n\varphi + B\cos n\varphi)J_n\left(\frac{P_{nm}}{a}\rho\right)e^{-j\beta z}$$

$$H_\rho = \frac{j\omega\varepsilon n}{\rho}(A\cos n\varphi - B\sin n\varphi)J_n\left(\frac{P_{nm}}{a}\rho\right)e^{-j\beta z}$$

$$H_\varphi = -j\omega\varepsilon\frac{P_{nm}}{a}(A\sin n\varphi + B\cos n\varphi)J_n'\left(\frac{P_{nm}}{a}\rho\right)e^{-j\beta z}$$

$$H_z = 0$$

Terms containing A can be get by rotating terms containing B by 90° (orthogonal, while n ≠ 0). One can choose either the term that containing A or the term that containing B, for example:

$$E_\rho = -j\beta\frac{P_{nm}}{a}A\sin n\varphi J_n'\left(\frac{P_{nm}}{a}\rho\right)e^{-j\beta z}$$

$$E_\varphi = \frac{-j\beta n}{\rho}A\cos n\varphi J_n\left(\frac{P_{nm}}{a}\rho\right)e^{-j\beta z}$$

$$E_z = \left(\frac{P_{nm}}{a}\right)^2 A\sin n\varphi J_n\left(\frac{P_{nm}}{a}\rho\right)e^{-j\beta z}$$

$$H_\rho = \frac{j\omega\varepsilon n}{\rho}A\cos n\varphi J_n\left(\frac{P_{nm}}{a}\rho\right)e^{-j\beta z}$$

$$H_\varphi = -j\omega\varepsilon\frac{P_{nm}}{a}A\sin n\varphi J_n'\left(\frac{P_{nm}}{a}\rho\right)e^{-j\beta z}$$

$$H_z = 0$$

Circular Waveguide – TM (3)

$n = 0,1,2,3\ldots$ and $m = 1,2,3\ldots$

There is no TM_{n0} mode in a circular waveguide.

There are TM_{0m} modes in it.

The mode with lowest cutoff frequency for TM modes is TM_{01}, with $f_{c\_TM_{01}} = \frac{2.405c}{2\pi a}$

$f_{c\_TM_{01}}$ is between $f_{c\_TE_{11}}$ & $f_{c\_TE_{21}}$

Values of P_{nm}

n	P_{n1}	P_{n2}	P_{n3}
0	2.405	5.520	8.654
1	3.832	7.016	10.174
2	5.135	8.417	11.620

Circular Waveguide – TM_{01}

$$E_\rho = -j\beta\frac{P_{01}}{a}AJ_0'\left(\frac{P_{01}}{a}\rho\right)e^{-j\beta z}$$

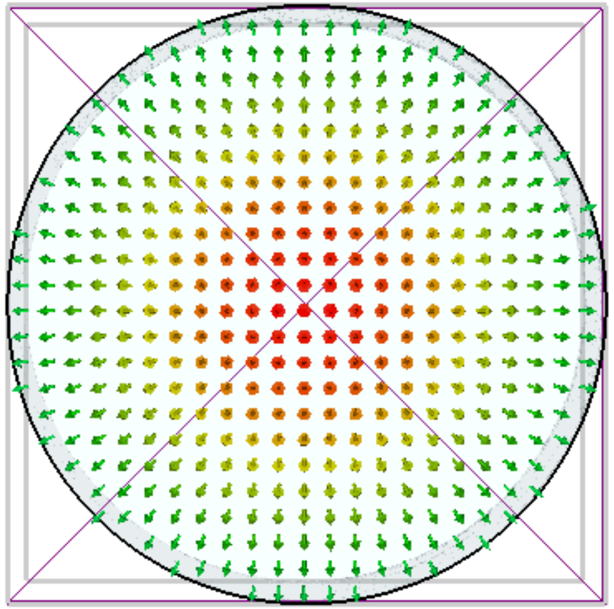
$$E_\varphi = 0$$

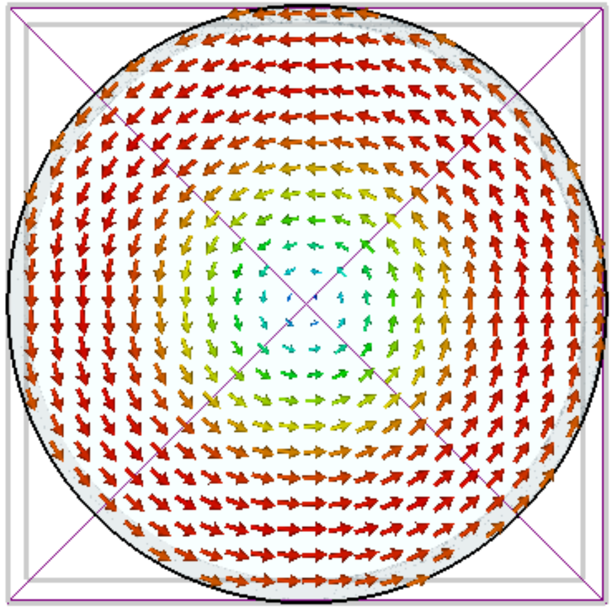
$$E_z(\rho,\varphi) = \left(\frac{P_{01}}{a}\right)^2 AJ_0\left(\frac{P_{01}}{a}\rho\right)e^{-j\beta z}$$

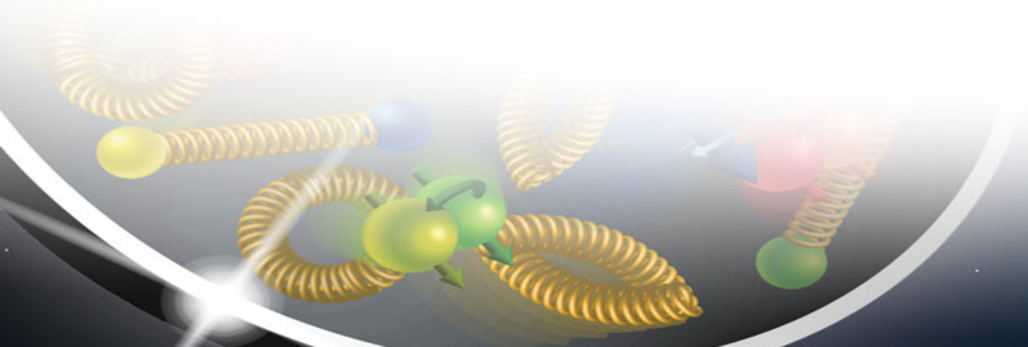
$$H_\rho = 0$$

$$H_\varphi = -j\omega\varepsilon\frac{P_{01}}{a}AJ_0'\left(\frac{P_{01}}{a}\rho\right)e^{-j\beta z}$$

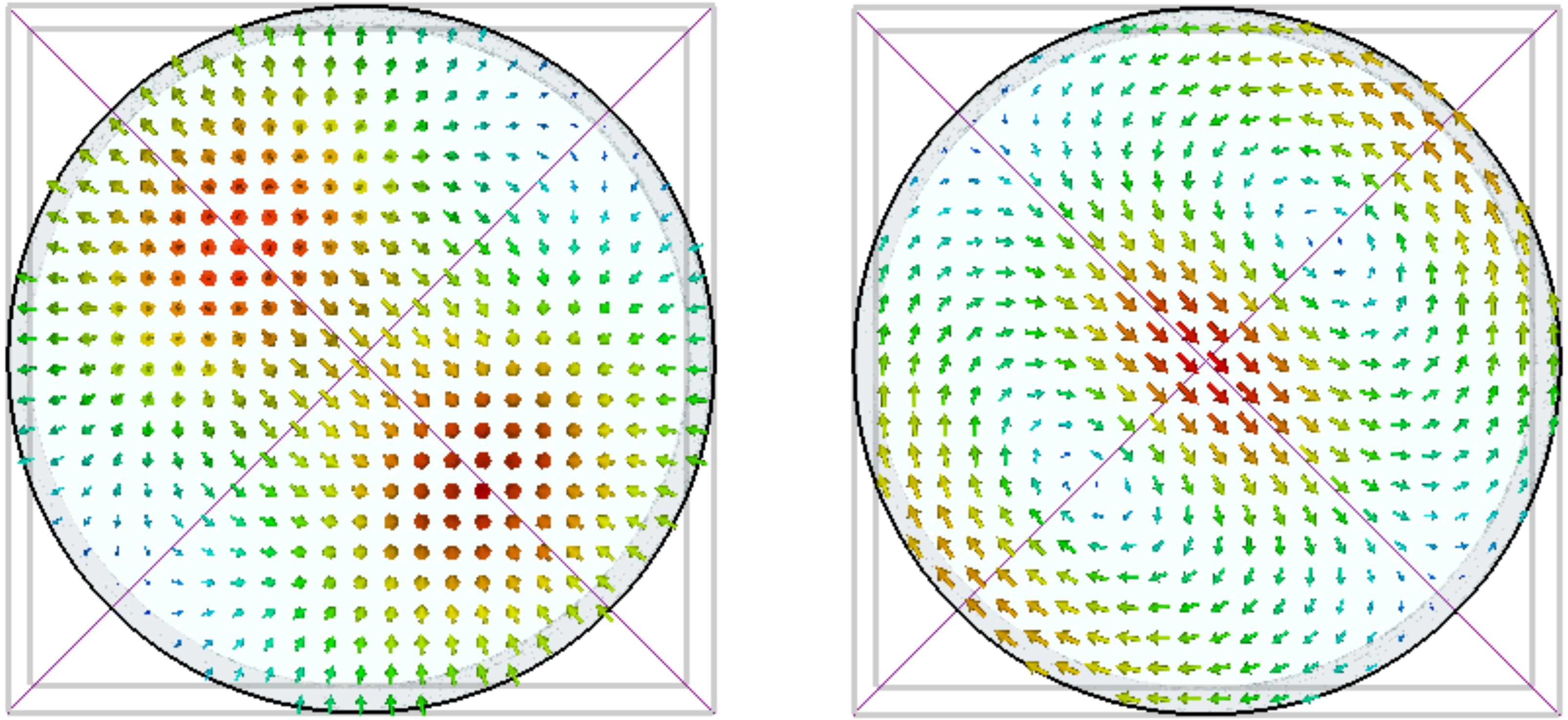
$$H_z = 0$$







Circular Waveguide – TM_{11}



TE_{01} & TM_{11} degeneracy

- Two modes with the same cutoff, with $k_c b$ =3.832 → TE-TM degeneracy. BTW, it is also possible to have TE-TE or TM-TM degeneracy.