Homework 13 solution:

1. Following the definition of the loss factor and that of the impedance, we obtain

$$
\begin{aligned}
k_{//} &= \int_{-\infty}^{\infty} V_{//}(z)\lambda(z)\,dz \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \lambda(z_1)\,w_{//}(z_1-z)\lambda(z)\,dz_1 dz \\
&= \frac{1}{(2\pi)^3}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} Z_{//}(\omega_2)e^{i(\omega_2-\omega)z/c}\tilde{\lambda}(\omega)\tilde{\lambda}(\omega_1)e^{-i(\omega_1+\omega_2)z_1/c}d\omega_2 d\omega_1 d\omega dz_1 dz \\
&= \frac{c^2}{(2\pi)}\int_{-\infty}^{\infty} Z_{//}(\omega)\tilde{\lambda}(\omega)\tilde{\lambda}(-\omega)\,d\omega \\
&= \frac{c^2}{2\pi}\int_{-\infty}^{\infty} Z_{//}(\omega)\tilde{\lambda}(\omega)\tilde{\lambda}^*(\omega)\,d\omega \\
&= \frac{c^2}{2\pi}\int_{-\infty}^{\infty} Z_{//}(\omega)\left|\tilde{\lambda}(\omega)\right|^2 d\omega
\end{aligned}
\quad , (1)
$$

where I used $\tilde{\lambda}^*(\omega)=\tilde{\lambda}(-\omega)$ since $\lambda(z)$ is real.

2.

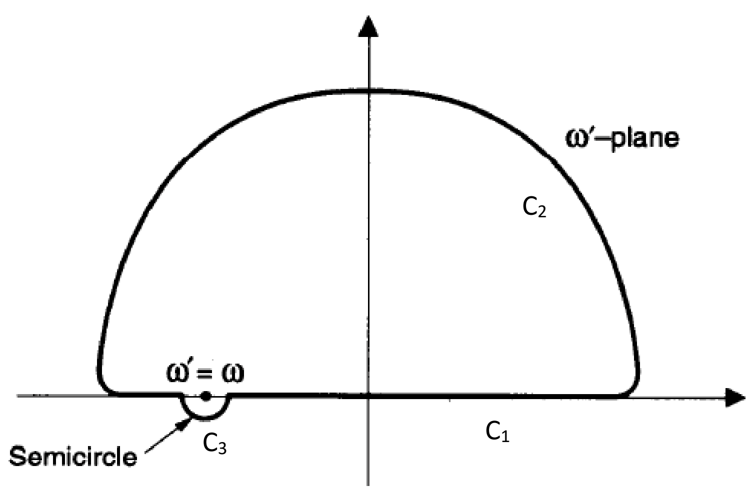

Figure 1: Integration contour in complex ω' plane.

From Cauchy residue theorem, the contour integral can be calculated as

$$\int_C \frac{Z_{//}(\omega')}{\omega'-\omega}d\omega' = 2\pi i Z_{//}(\omega)\ . \qquad (1)$$

The LHS of (1) can be split into the following form

$$\begin{aligned}\int_C \frac{Z_{//}(\omega')}{\omega'-\omega}d\omega' &= \int_{C_1} \frac{Z_{//}(\omega')}{\omega'-\omega}d\omega' + \int_{C_2} \frac{Z_{//}(\omega')}{\omega'-\omega}d\omega' + \int_{C_3} \frac{Z_{//}(\omega')}{\omega'-\omega}d\omega' \\ &= P.V.\int_{-\infty}^{\infty} \frac{Z_{//}(\omega')}{\omega'-\omega}d\omega' + 0 + \int_{e^{i\pi}}^{e^{i2\pi}} \frac{Z_{//}(\omega)}{e^{i\theta}}de^{i\theta} \quad , \\ &= P.V.\int_{-\infty}^{\infty} \frac{Z_{//}(\omega')}{\omega'-\omega}d\omega' + 0 + i\pi Z_{//}(\omega)\end{aligned} \tag{2}$$

where the integral along C_2 vanishes since we assume $Z_{//}(\omega')$ is well behaved at large $|\omega'|$. From eq. (1) and (2), it follows

$$Z_{//}(\omega) = -\frac{i}{\pi} P.V.\int_{-\infty}^{\infty} \frac{Z_{//}(\omega')}{\omega'-\omega}d\omega' \ . \tag{3}$$

Splitting eq. (3) into the real and imaginary part leads to

$$\mathrm{Re}\left[Z_{//}(\omega)\right] = \frac{1}{\pi} P.V.\int_{-\infty}^{\infty} \frac{\mathrm{Im}\left[Z_{//}(\omega')\right]}{\omega'-\omega}d\omega' \ , \tag{4}$$

and

$$\mathrm{Im}\left[Z_{//}(\omega)\right] = -\frac{1}{\pi} P.V.\int_{-\infty}^{\infty} \frac{\mathrm{Re}\left[Z_{//}(\omega')\right]}{\omega'-\omega}d\omega' \ . \tag{5}$$