Homework 20

1. A point particle with charge Q moves with velocity $\vec{v} = (0, 0, v_z)$ in the longituinal direction. The particle has a horizontal offset, $x = a$. Show that, in the cylindrical system, the charge distribution due to the particle can be written as:

$$\rho(r, \theta, z, t) = \sum_{m=0}^{\infty} \frac{Q_m \cos(m\theta)}{\pi a^{m+1}(1 + \delta_{m,0})} \delta(z - v_z t) \delta(r - a)$$

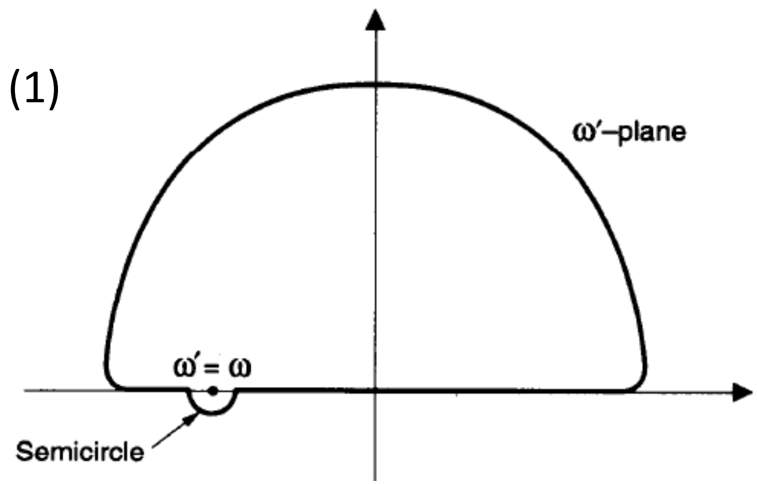
with $Q_m = Qa^m$ and $\delta_{m,0} = \begin{cases} 1 & \text{if } m = 0 \\ 0 & \text{otherwise} \end{cases}$

Hint 1. prove that the delta function in cylindrical coordinate is $\rho(x, y, z) = \frac{1}{r}\delta(r - a)\delta_p(\theta)\delta(z)$
with $\delta_p(\theta) = \sum_{k=-\infty}^{\infty} \delta(\theta - 2\pi k)$ (Jackson page 120 footnote)
Hint 2. perform Fourier transformation to $\delta_p(\theta)$

Homeworks
(Taken from excercise 2.16 from [4] by A. Chao)

2. Perform a contour integral of $\frac{Z_{//}(\omega')}{\omega'-\omega}$ in the complex ω'-plane over the upper half plane along the contour shown in the figure. Show that if $Z_{//}(\omega')$ converges sufficiently fast as $|\omega'| \to \infty$

$$Z_{//}(\omega) = -\frac{i}{\pi} P.V. \int_{-\infty}^{\infty} \frac{Z_{//}(\omega')}{\omega'-\omega} d\omega' \quad (1)$$



Show that eq. (1) leads to Kramers-Kronig relations

$$\mathrm{Re}\left[Z_{//}(\omega)\right] = \frac{1}{\pi} P.V. \int_{-\infty}^{\infty} \frac{\mathrm{Im}\left[Z_{//}(\omega')\right]}{\omega'-\omega} d\omega'$$

$$\mathrm{Im}\left[Z_{//}(\omega)\right] = -\frac{1}{\pi} P.V. \int_{-\infty}^{\infty} \frac{\mathrm{Re}\left[Z_{//}(\omega')\right]}{\omega'-\omega} d\omega'$$

About Principal Value Integral:

The trick of P.V. is to utilize the property that the divergences on the side $\omega' < \omega$ and the side $\omega' > \omega$ are of opposite signs and, if the integration is taken *symmetrically* about the singularity so that the divergences on the two sides cancel each other, the integral is actually well defined. Algebraically, this leads to

$$\mathrm{P.V.} \int_{-\infty}^{\infty} dx \frac{f(x)}{x-a} = \int_{0}^{\infty} du \frac{f(a+u) - f(a-u)}{u}, \quad (2.95)$$

where the expression on the right is well behaved at $u = 0$.