PHY 554
Fundamentals of Accelerator Physics

Mon, Wed 6:30-7:50PM Physics D103

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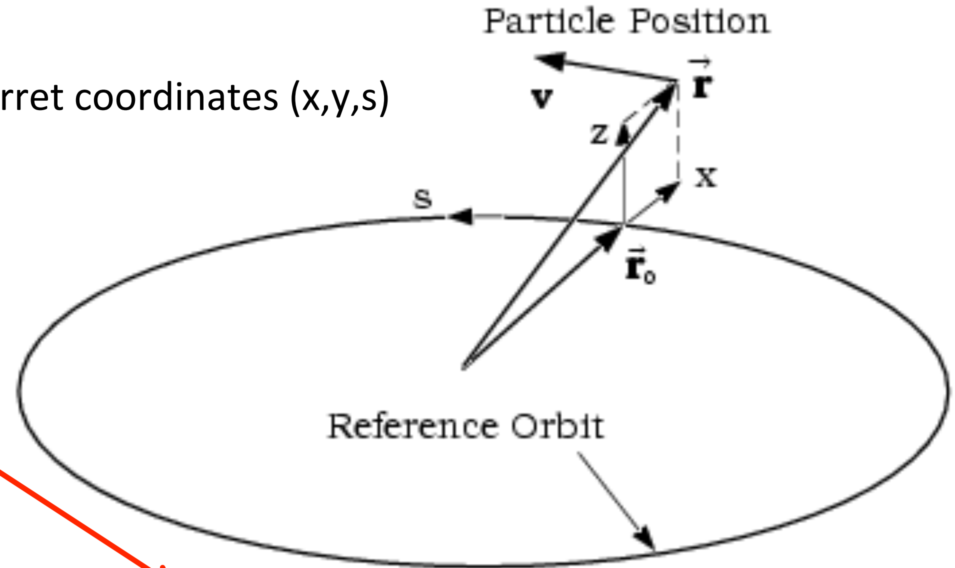
Transverse (Betatron) Motion

- Linear betatron motion
- Dispersion function of off momentum particle
- Simple Lattice design considerations
- Nonlinearities

What we learned:

Frenet-Serret coordinates (x,y,s)



Hill's equations (derivatives w.r.t. s)

$$x'' + K_x(s)x = \pm\frac{\Delta B_z}{B\rho}, \quad y'' + K_y(s)y = \mp\frac{\Delta B_x}{B\rho}$$

$$K_x(s) = \frac{1}{\rho^2} \mp \frac{B_1}{B\rho}, \qquad K_y(s) = \pm\frac{B_1}{B\rho}$$

Natural focusing from dipoles (curvature)

Focusing from quadrupoles

Higher order magnet, usually field errors

$$\theta = \frac{s}{R} = \frac{\beta ct}{R}$$

Solution of Hill's equations X(s), X'(s) form a coordinate set and can be transformed thru matrix representation

$$\begin{pmatrix} X(s) \\ X'(s) \end{pmatrix} = M(s,s_0)\begin{pmatrix} X(s_0) \\ X'(s_0) \end{pmatrix}$$

X can be x or y

$$|M(s,s_0)| = 1 \qquad |Trace(M(s,s_0))| \leq 2$$

Stable solution conditions

Courant-Snyder parameterization

$$\mathrm{M}(s) = \begin{pmatrix} \cos\Phi + \alpha\sin\Phi & \beta\sin\Phi \\ -\gamma\sin\Phi & \cos\Phi - \alpha\sin\Phi \end{pmatrix} = I\cos\Phi + J\sin\Phi$$

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad J = \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix}, \quad J^2 = -I, \quad \text{or} \quad \beta\gamma = 1 + \alpha^2$$

Where α,β,γ,ϕ are functions of s and describes position dependent beam properties.

Focusing quadrupole:

$$M(s,s_0) = \begin{pmatrix} \cos\sqrt{K}\ell & \frac{1}{\sqrt{K}}\sin\sqrt{K}\ell \\ -\sqrt{K}\sin\sqrt{K}\ell & \cos\sqrt{K}\ell \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 \\ -1/f & 1 \end{pmatrix}$$

Defocusing quadrupole:

$$M(s,s_0) = \begin{pmatrix} \cosh\sqrt{|K|}\ell & \frac{1}{\sqrt{|K|}}\sinh\sqrt{|K|}\ell \\ \sqrt{|K|}\sinh\sqrt{|K|}\ell & \cosh\sqrt{|K|}\ell \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 \\ 1/f & 1 \end{pmatrix}$$

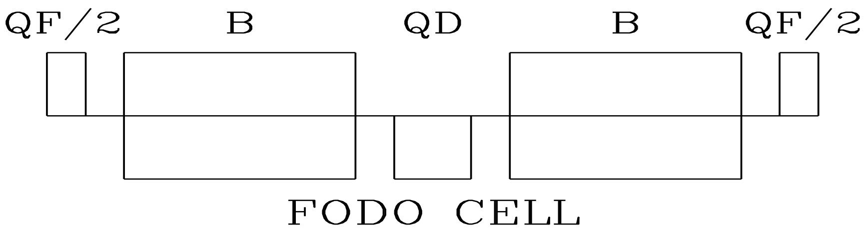
Dipole: $K=1/\rho^2$

$$M(s,s_0) = \begin{pmatrix} \cos\frac{\ell}{\rho} & \rho\sin\frac{\ell}{\rho} \\ -\frac{1}{\rho}\sin\frac{\ell}{\rho} & \cos\frac{\ell}{\rho} \end{pmatrix} \rightarrow \begin{pmatrix} 1 & \ell \\ 0 & 1 \end{pmatrix}$$

Drift space: $K=0$

$$M(s,s_0) = \begin{pmatrix} 1 & \ell \\ 0 & 1 \end{pmatrix}$$

Example: FODO cell



A FODO cell is a basic block in beam transport, where the transfer matrices for dipoles (B) can be approximated by drift spaces, and QF and QD are the focusing and defocusing quadrupoles.

$$\mathbf{M} = \begin{pmatrix} 1 & 0 \\ -\frac{1}{2f} & 1 \end{pmatrix} \begin{pmatrix} 1 & L_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{1}{f} & 1 \end{pmatrix} \begin{pmatrix} 1 & L_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{1}{2f} & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 - \frac{L_1^2}{2f^2} & 2L_1(1 + \frac{L_1}{2f}) \\ -\frac{L_1}{2f^2}(1 - \frac{L_1}{2f}) & 1 - \frac{L_1^2}{2f^2} \end{pmatrix}$$

$$\mathrm{M}(s) = \begin{pmatrix} \cos\Phi + \alpha\sin\Phi & \beta\sin\Phi \\ -\gamma\sin\Phi & \cos\Phi - \alpha\sin\Phi \end{pmatrix}$$

$$\cos\Phi = \tfrac{1}{2}\mathrm{Tr(M)}$$

$$\cos\Phi = 1 - \frac{L_1^2}{2f^2}, \quad \sin\frac{\Phi}{2} = \frac{L_1}{2f}$$

$$\beta = \frac{2L_1(1 + \frac{L_1}{2f})}{\sin\Phi} = \frac{2L_1(1 + \sin\frac{\Phi}{2})}{\sin\Phi}$$

$$\alpha = 0$$

How good is a fodo cell by thin lens approximation?

Qx= 4.793020 Qy =4.787549

betax(max) = 33.816143 betax(min)=5.034 m

$\Phi x==2\pi Qx/18=1.673$ rad

Thin lens approximation:

Focal length = 1/(0.150*1)=6.67m

$$\sin\frac{\Phi}{2}=\frac{L_1}{2f}$$

$\sin(\Phi x/2)=0.742329$ $L_1/2f=0.75$

$$\beta=\frac{2L_1(1\pm\sin\frac{\Phi}{2})}{\sin\Phi}$$

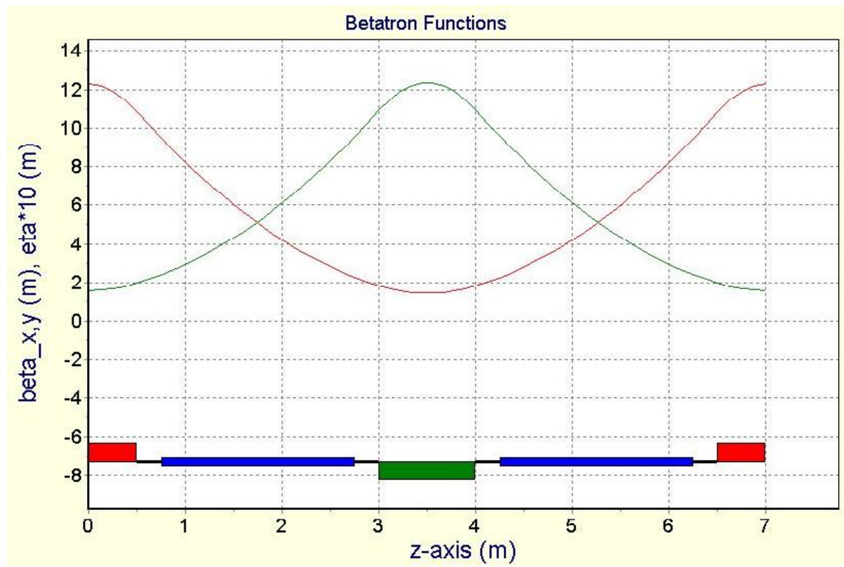
Betax(max)=35.03 betax(min)=5.18

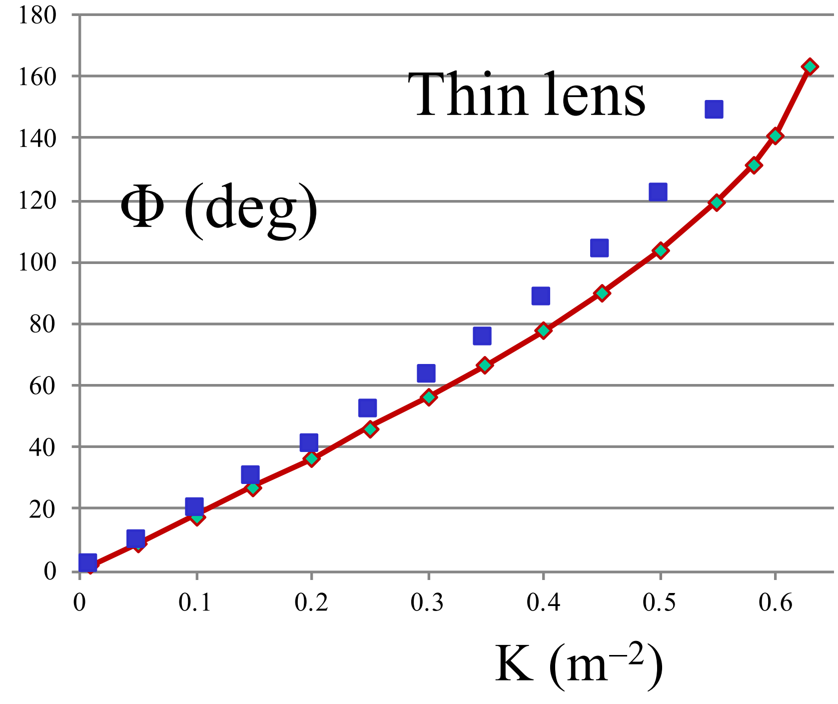
Thin lens – use with care

$$\cos\Phi = 1 - \frac{L_1^2}{2f^2}, \quad \sin\frac{\Phi}{2} = \frac{L_1}{2f}$$

In this FODO cell with Lq=1.0 m, L_dipole=2.0 m, drift length of 0.25 m, and thus L1=3.5 m. Thin lens approximation is good except when the focusing strength is high. The percentage error at high focusing gradient can be larger than 11%.

L_q=1m, K_q=1m^{-2}, L_1=3.5m





Floquet Theorem

We consider the linear Hill's equation of motion **X'' + K(s)X = 0**,where X(s) and X'(s) are conjugate coordinates, K(s) is the focusing function, and the prime is the derivative with respect to the independent variable s. In many accelerator applications, K(s) is a periodic function of s with period L, i.e. **K(s+L)=K(s)**. Floquet theorem states we can express the solution in amplitude and phase functions which satisfy a periodic boundary condition similar to that of the potential function K(s), i.e.

$$X(s) = aw(s)e^{j\psi(s)}, \quad w(s) = w(s+L), \quad \psi(s+L) - \psi(s) = 2\pi\mu$$

where the phase advance μ in one period is independent of s. Using the Floquet transformation on Hill's equation, we get the differential equation

$$2w'\psi' + w\psi'' = 0, \quad w'' + K(s)w - w\psi'^2 = 0$$

$$\psi' = \frac{1}{w^2}, \quad \psi = \int_{s_0}^{s} \frac{ds}{w^2}, \quad w'' + K(s)w - \frac{1}{w^3} = 0$$

Floquet transformation: $$X'' + K(s)X = 0$$

$$X(s) = aw(s)e^{j\psi(s)} \qquad w'' + K(s)w - \frac{1}{w^3} = 0 \qquad \psi' = \frac{1}{w^2}$$

What is the transfer matrix $M(s_2,s_1)$?
$\psi_2=\psi_1+\mu$, where μ is the phase advance.

$$\begin{pmatrix} X(s_2) \\ X'(s_2) \end{pmatrix} = M(s_2,s_1)\begin{pmatrix} X(s_1) \\ X'(s_1) \end{pmatrix}$$

$$M(s_2,s_1) = \begin{pmatrix} \frac{w_2}{w_1}\cos\mu - w_2 w_1' \sin\mu & w_1 w_2 \sin\mu \\ -\frac{1+w_1 w_1' w_2 w_2'}{w_1 w_2}\sin\mu - (\frac{w_1'}{w_2} - \frac{w_2'}{w_1})\cos\mu & \frac{w_1}{w_2}\cos\mu + w_1 w_2' \sin\mu \end{pmatrix}$$

$$w_1 = w_2, \quad w_1' = w_2', \quad \psi_2 - \psi_1 = \mu$$

$$M(s) = \begin{pmatrix} \cos\mu - ww'\sin\mu & w^2\sin\mu \\ -(1+w'^2)\sin\mu & \cos\mu + ww'\sin\mu \end{pmatrix} \Leftrightarrow \begin{pmatrix} \cos\Phi + \alpha\sin\Phi & \beta\sin\Phi \\ -\gamma\sin\Phi & \cos\Phi - \alpha\sin\Phi \end{pmatrix}$$

$$\beta(s) = w^2, \quad \alpha = -\frac{1}{2}\beta', \quad \gamma = \frac{1+\alpha^2}{\beta}, \qquad w(s) = \sqrt{\beta(s)}, \quad \psi(s) = \int_{s_0}^{s}\frac{1}{\beta}ds$$

$$\frac{1}{2}\beta'' + K\beta - \frac{1}{\beta}[1 + (\frac{\beta'}{2})^2] = 0, \quad \text{or} \quad \alpha' = K\beta - \frac{1}{\beta}[1+\alpha^2]$$

$$M(s_2,s_1)=\begin{pmatrix} \frac{w_2}{w_1}\cos\psi - w_2 w_1' \sin\psi & w_1 w_2 \sin\psi \\ -\frac{1+w_1w_1'w_2w_2'}{w_1w_2}\sin\psi - \left(\frac{w_1'}{w_2}-\frac{w_2'}{w_1}\right)\cos\psi & \frac{w_1}{w_2}\cos\psi + w_1 w_2' \sin\psi \end{pmatrix}$$

$$w_1 = w(s_1), \quad w_2 = w(s_2), \quad w_1' = w'(s_1), \quad w_2' = w'(s_2)$$

The transfer matrix from s_1 to s_2 in any beam transport line becomes

$$M(s_2,s_1)=\begin{pmatrix} \sqrt{\frac{\beta_2}{\beta_1}}(\cos\mu+\alpha_1\sin\mu) & \sqrt{\beta_1\beta_2}\sin\mu \\ -\frac{1+\alpha_1\alpha_2}{\sqrt{\beta_1\beta_2}}\sin\mu - \frac{\alpha_1-\alpha_2}{\sqrt{\beta_1\beta_2}}\cos\mu & \sqrt{\frac{\beta_2}{\beta_1}}(\cos\mu-\alpha_1\sin\mu) \end{pmatrix}$$

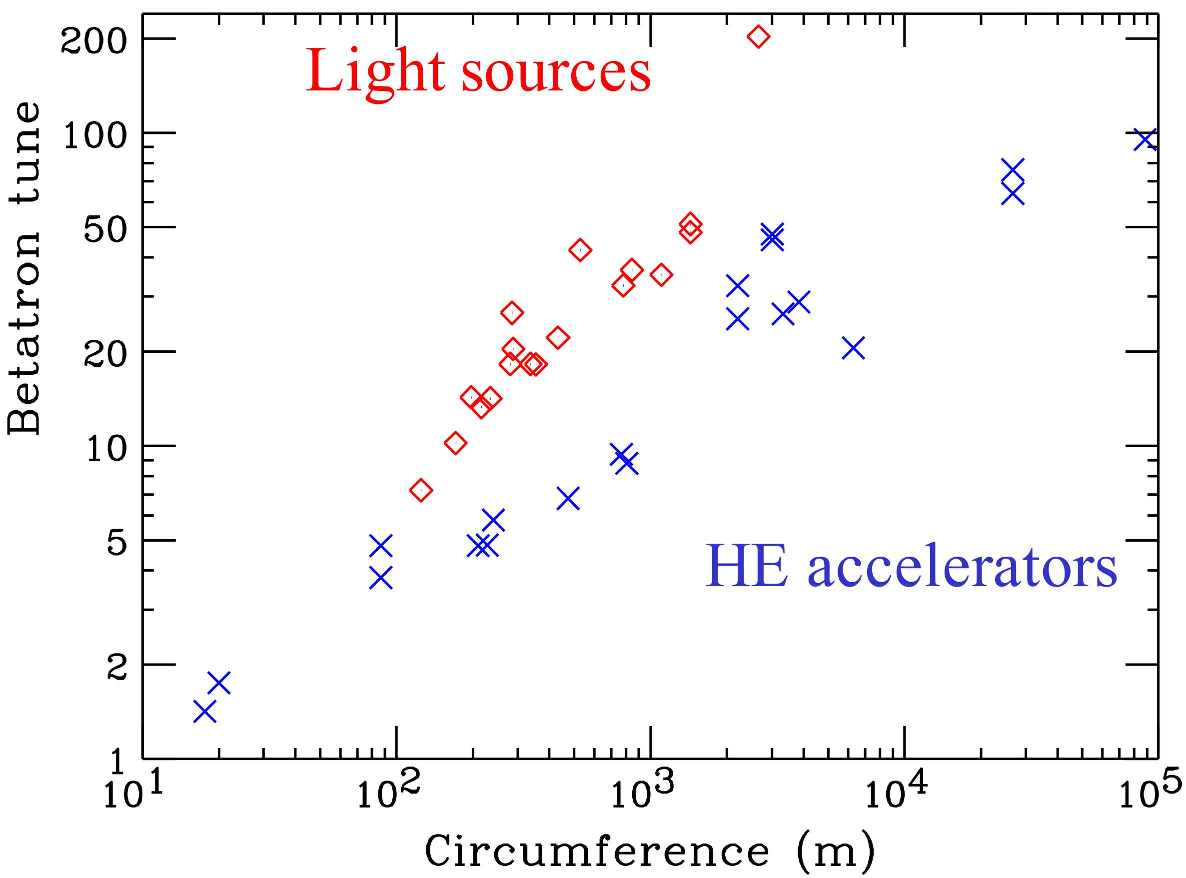
$$=\begin{pmatrix} \sqrt{\beta_2} & 0 \\ -\frac{\alpha_2}{\sqrt{\beta_2}} & \frac{1}{\sqrt{\beta_2}} \end{pmatrix}\begin{pmatrix} \cos\mu & \sin\mu \\ -\sin\mu & \cos\mu \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{\beta_1}} & 0 \\ -\frac{\alpha_1}{\sqrt{\beta_1}} & \sqrt{\beta_1} \end{pmatrix}$$

The amplitude function is $w(s) = \sqrt{\beta(s)}$

$$y(s) = a\sqrt{\beta_y(s)}\cos[\psi_y(s) + \xi_y] \quad \text{with} \quad \psi_y(s) = \int_0^s \frac{ds}{\beta_y(s)}$$

Betatron tune (ν_y, or Q_y.): The **number of betatron oscillations in one revolution**.

$$Q_y \equiv \nu_y = \frac{1}{2\pi}\oint \frac{ds}{\beta_y}$$



Floquet theorem: Many accelerator components obey the periodic condition: $K(s+L)=K(s)$. The solution of Hill's equation is periodic. In matrix representation, we obtain

$M_1M_2\ldots\ldots\ldots\ldots M_n \quad M_1M_2\ldots\ldots\ldots\ldots M_n$

$M(s_1+L|s_1)=M_nM_{n-1}M_{n-2}\ldots M_2M_1=\mathbf{M}(s_1)$
$M(s_2+L|s_2)=M_1M_nM_{n-1}M_{n-2}\ldots M_2=\mathbf{M}(s_2)=M_1\mathbf{M}(s_1)M_1^{-1}$

Each matrix is a product of identical number of matrices. They are related by **similarity** transformation. The eigen-values of the periodic matrix $\mathbf{M}(s_i)$ are identical.

$$M(s_2) = M(s_2 \mid s_1)M(s_1)[M(s_2 \mid s_1)]^{-1}$$

With the similarity transformation of the transfer matrix,

$$M(s_2) = M(s_2 \mid s_1) M(s_1) [M(s_2 \mid s_1)]^{-1}$$

the values of the Courant–Snyder parameters α_2, β_2, γ_2 at s_2 are related to α_1, β_1, γ_1 at s_1 by

$$\mathrm{M}(s_2) = I\cos\Phi + \begin{pmatrix} \alpha_2 & \beta_2 \\ -\gamma_2 & -\alpha_2 \end{pmatrix} \sin\Phi = I\cos\Phi + J_2 \sin\Phi$$

$$\mathrm{M}(s_1) = I\cos\Phi + \begin{pmatrix} \alpha_1 & \beta_1 \\ -\gamma_1 & -\alpha_1 \end{pmatrix} \sin\Phi = I\cos\Phi + J_1 \sin\Phi$$

$$J_2 = M(s_2 \mid s_1) J_1 [M(s_2 \mid s_1)]^{-1}$$

$$\begin{pmatrix} \alpha_2 & \beta_2 \\ -\gamma_2 & -\alpha_2 \end{pmatrix} = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \begin{pmatrix} \alpha_1 & \beta_1 \\ -\gamma_1 & -\alpha_1 \end{pmatrix} \begin{pmatrix} M_{22} & -M_{12} \\ -M_{21} & M_{11} \end{pmatrix}$$

$$\begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_2 = \begin{pmatrix} M_{11}^2 & -2M_{11}M_{12} & M_{12}^2 \\ -M_{11}M_{21} & M_{11}M_{22} + M_{12}M_{21} & -M_{12}M_{22} \\ M_{21}^2 & -2M_{21}M_{22} & M_{22}^2 \end{pmatrix} \begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_1$$

M_{ij} is the *ij*-th component of the matrix $M(s_2,s_1)$

$$\begin{pmatrix}\beta\\ \alpha\\ \gamma\end{pmatrix}_2 = \begin{pmatrix} M_{11}^2 & -2M_{11}M_{12} & M_{12}^2 \\ -M_{11}M_{21} & M_{11}M_{22}+M_{12}M_{21} & -M_{12}M_{22} \\ M_{21}^2 & -2M_{21}M_{22} & M_{22}^2 \end{pmatrix}\begin{pmatrix}\beta\\ \alpha\\ \gamma\end{pmatrix}_1$$

1. The evolution of the betatron amplitude function in a drift space is

$$\beta_2 = \frac{1}{\gamma_1} + \gamma_1 (s - \frac{\alpha_1}{\gamma_1})^2 = \beta^* + \frac{(s-s^*)^2}{\beta^*},$$

$$\alpha_2 = \alpha_1 - \gamma_1 s = -\frac{(s-s^*)}{\beta^*}, \quad \gamma_2 = \gamma_1 = \frac{1}{\beta^*}$$

Note that γ is constant in a drift space, and $\mathbf{s^*=\alpha_1/\gamma_1}$ is the location for an extremum of the betatron amplitude function with $\alpha(s^*) = 0$.

2. Passing through a thin-lens quadrupole, the evolution of betatron function is

$$\beta_2 = \beta_1, \quad \alpha_2 = \alpha_1 + \frac{\beta_1}{f}, \quad \gamma_2 = \gamma_1 + \frac{2\alpha_1}{f} + \frac{\beta_1}{f^2}$$

where *f* is the focal length of the quadrupole. Thus a thin-lens quadrupole gives rise to an angular kick to the betatron amplitude function without changing its magnitude.

Floquet transformation:

$$y'' + Ky = \frac{\Delta B}{B\rho} \qquad x'' + K_x(s)x = \pm\frac{\Delta B_z}{B\rho}, \qquad z'' + K_z(s)z = \mp\frac{\Delta B_x}{B\rho}$$

Define the normalized coordinate $\eta = \frac{y}{\sqrt{\beta}}, \qquad \varphi = \frac{1}{\nu}\int^{s}\frac{1}{\beta}ds$

$$\frac{d^2}{d\varphi^2}\eta = \nu\beta\frac{d}{ds}\nu\beta\frac{d}{ds}\frac{y}{\sqrt{\beta}} = \nu^2\beta^{3/2}y'' + \nu^2\left\{-\frac{1}{2}\beta\beta'' + \frac{\beta'^2}{4}\right\}\frac{y}{\sqrt{\beta}}$$

$$= \nu^2\beta^{3/2}\frac{\Delta B}{B\rho} - \nu^2\frac{y}{\sqrt{\beta}}$$

$$\frac{d^2}{d\varphi^2}\eta + \nu^2\eta = \nu^2\beta^{3/2}\frac{\Delta B}{B\rho}$$

If ΔB=0, we find $$\frac{d^2}{d\varphi^2}\eta + \nu^2\eta = 0$$

A simple harmonic oscillator

$$X'' + K(s)X = 0$$

Since $$X(s) = a\sqrt{\beta(s)}\cos(\psi(s) + \psi_0) \quad \textit{with} \quad \psi(s) = \int_0^s \frac{ds}{\beta(s)}$$

Thus $$X' = -\frac{X}{\beta}(\tan\psi - \frac{\beta'}{2})$$

$$\frac{1}{2\beta}[X^2 + (\beta X' + \alpha X)^2] = \frac{X^2}{2\beta}\sec^2\psi = \frac{a^2}{2} \equiv J$$

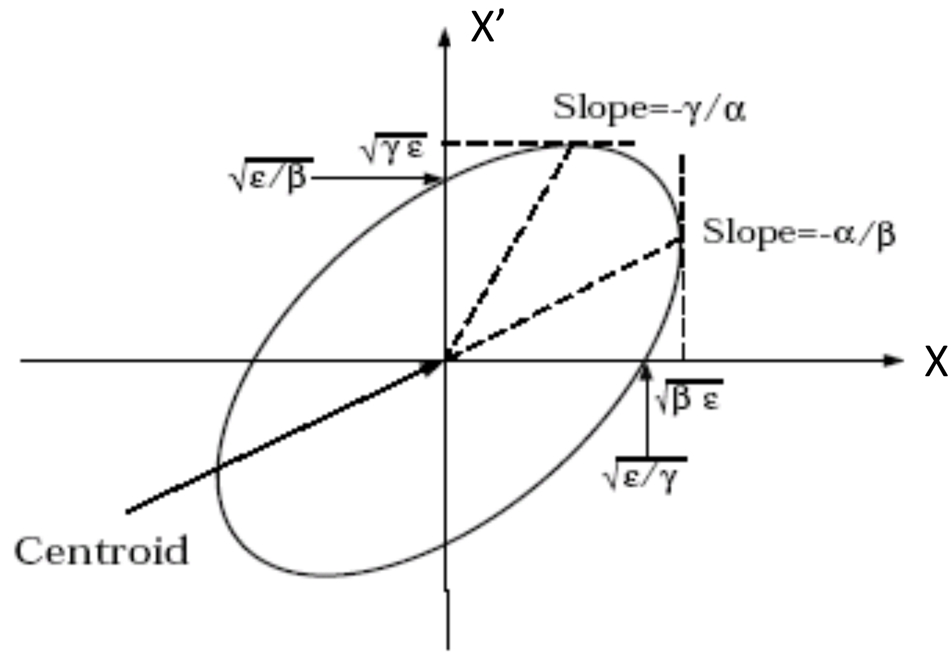
$$X = \sqrt{2\beta J}\cos\psi, \quad X' = -\sqrt{\frac{2J}{\beta}}(\sin\psi + \alpha\cos\psi)$$

Define: $$P_X = \beta X' + \alpha X = -\sqrt{2\beta J}\sin\psi$$

(X, P_X) form a normalized phase space coordinates with $X^2 + P_X^2 = 2\beta J$, here J is called **action**.

Courant-Snyder Invariant

$$\gamma X^2 + 2\alpha X X' + \beta X'^2 = \frac{1}{\beta}\left[X^2 + (\alpha X + \beta X')^2\right] = 2J \equiv \varepsilon$$



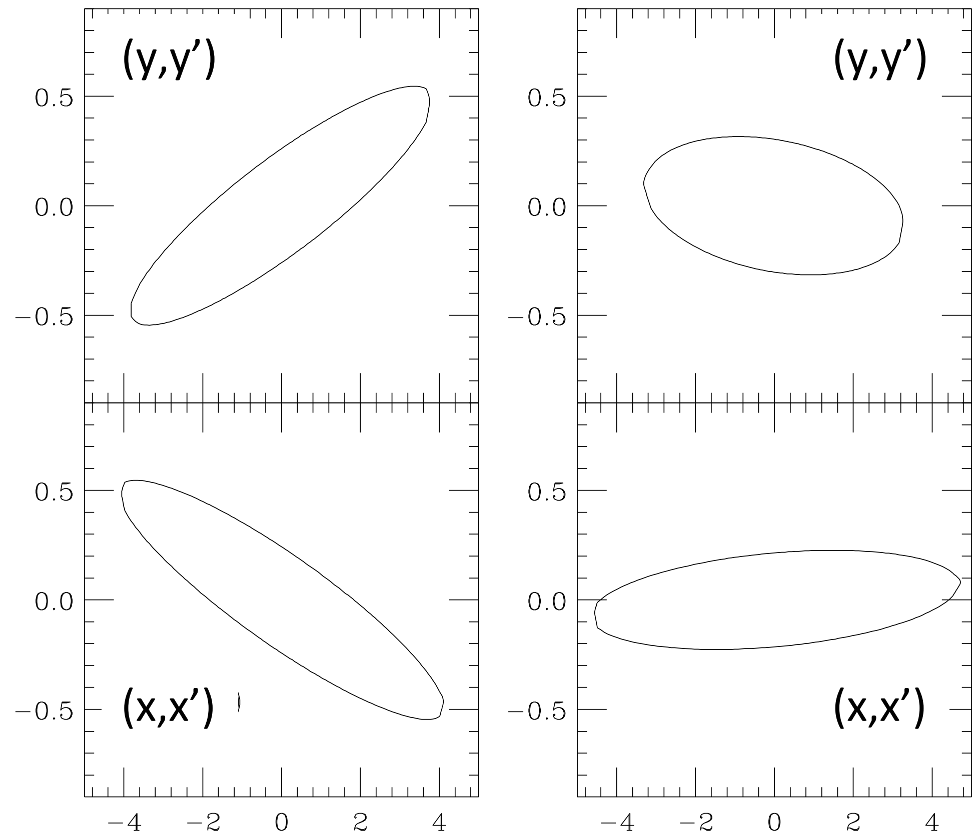
Questions:

1) When we have two particles with different action J_1 and $J_2 = 2J_1$, what will their ellipses look like?

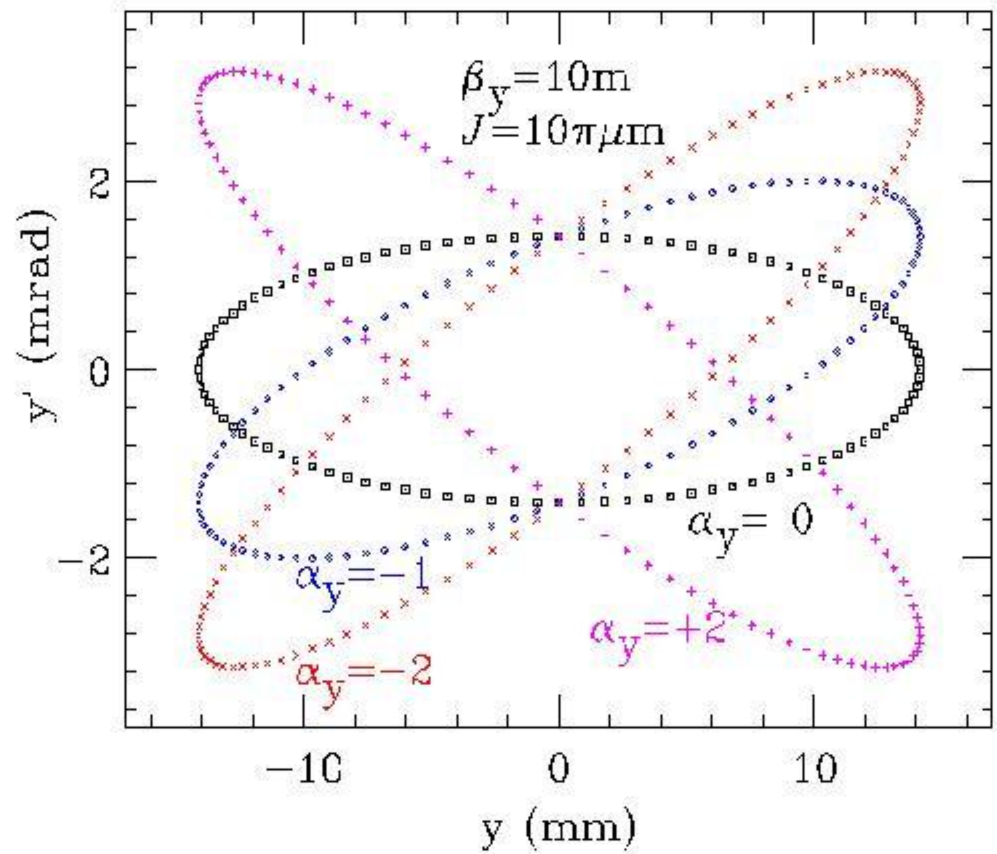
2) Will the ellipses intersect with each other?

$$\begin{pmatrix} X(s_0) \\ X'(s_0) \end{pmatrix}_{n+1} = M_X \begin{pmatrix} X(s_0) \\ X'(s_0) \end{pmatrix}_n \qquad M_X = \begin{pmatrix} \cos\Phi + \alpha\sin\Phi & \beta\sin\Phi \\ -\gamma\sin\Phi & \cos\Phi - \alpha\sin\Phi \end{pmatrix}$$

The horizontal and vertical betatron ellipses for a particle with actions J_x=J_y=0.5π mm-mrad at the end of the first dipole (left plots) and the end of the fourth dipole of the AGS lattice. The scale for the ordinate x or y is in mm, and that for the coordinate x' or y' is in mrad. Left plots: β_x=17.0 m, α_x=2.02, β_y=14.7 m, and α_y=−1.84. Right plots: β_x=21.7 m, α_x=−0.33, β_y=10.9 m, and α_y=0.29.



Example: Ellipses (vertical) with different optical parameters



The betatron phase space ellipses of a particle with actions J = 10π mm-mrad. The btatron parameters are β_y = 10m, and α_y shown by each curve. The scale for the ordinate y is mm, and y' in mrad. The betatron parameters for each ellipse are marked on the graph. All ellipses has the maximum y coordinate at $(2\beta_y J)^{1/2}$. The maximum anglular coordiante y' is $(2(1 + \alpha_y^2)J/\beta_y)^{1/2}$. All ellipses have the same phase space area of 2J.