Homework 14.

Problem 1 – 15 points

Prove Lie operators identity:

$$:f:[g,h]=[:f:g,h]+[g,:f:h]$$

Solution: It is straightforward but multi-term derivation:

$$\begin{aligned}
&:f:[g,h]=[:f:g,h]+[g,:f:h]\Leftrightarrow[f,[g,h]]=[[f,g],h]+[g,[f,h]]\\
&\partial_{qi}f\cdot\partial_{pi}(\partial_{qj}g\,\partial_{pj}h-\partial_{pj}g\,\partial_{qj}h)-\partial_{pi}f\cdot\partial_{qi}(\partial_{qj}g\,\partial_{pj}h-\partial_{pj}g\,\partial_{qj}h)=\\
&-\partial_{qi}h\cdot\partial_{pi}(\partial_{qj}f\,\partial_{pj}g-\partial_{pj}f\,\partial_{qj}g)+\partial_{pi}h\cdot\partial_{qi}(\partial_{qj}f\,\partial_{pj}g-\partial_{pj}f\,\partial_{qj}g)+\\
&+\partial_{qi}g\cdot\partial_{pi}(\partial_{qj}f\,\partial_{pj}h-\partial_{pj}f\,\partial_{qj}h)+\partial_{pi}g\cdot\partial_{qi}(\partial_{qj}f\,\partial_{pj}h-\partial_{pj}f\,\partial_{qj}h)
\end{aligned}$$

It takes a good 2-3 pages to collect all terms and find that *rhs* and *lhs* are indeed identical. More compact prove uses symplectic generation matrix $S=[s_{ij}]$:

$$\begin{aligned}
&[g,h]\equiv s_{ij}\cdot\partial_i g\cdot\partial_j h\Rightarrow\\
&[f,[g,h]]=s_{kl}\cdot\partial_k f\cdot\partial_l(s_{ij}\cdot\partial_i g\cdot\partial_j h)=s_{kl}\cdot s_{ij}\cdot\partial_k f\cdot(\partial_{il}g\cdot\partial_j h+\partial_i g\cdot\partial_{jl}h)\\
&[[f,g],h]=s_{kl}\cdot\partial_k(s_{ij}\cdot\partial_i f\,\partial_j g)\cdot\partial_l h=\ s_{kl}\cdot s_{ij}\cdot\partial_l h\cdot(\partial_{ik}f\cdot\partial_j g+\partial_i f\cdot\partial_{jk}g)\\
&[g,[f,h]]=s_{kl}\cdot\partial_k g\cdot\partial_l(s_{ij}\cdot\partial_i f\,\partial_j h)=s_{kl}\cdot s_{ij}\cdot\partial_k g\cdot(\partial_{il}f\cdot\partial_j h+\partial_i f\cdot\partial_{jl}h)
\end{aligned}$$

where we assume silent summation over all for indexes from 1 to 2n. It worth noting that each term contains second order derivative of *f, g or h*. Collecting term using asymmetry of *S* solve the puzzle. The easiest is

$$\begin{aligned}
&s_{kl}\cdot s_{ij}\cdot\partial_l h\cdot(\partial_{ik}f\cdot\partial_j g+\partial_i f\cdot\partial_{jk}g)\equiv-s_{lk}\cdot s_{ij}\cdot\partial_k h\cdot(\partial_{il}f\cdot\partial_j g+\partial_i f\cdot\partial_{jl}g)\\
&[[f,g],h]+[g,[f,h]]\Rightarrow\partial_{il}f\cdot\ s_{kl}\cdot s_{ij}\cdot(\partial_k g\cdot\partial_j h-\partial_k h\,\partial_j g);\\
&\sum_{k,l,i,j}\partial_{il}f\cdot\ s_{kl}\cdot s_{ij}\cdot\partial_k g\cdot\partial_j h=\sum_{k,l,i,j}\partial_{il}f\cdot\ s_{jl}\cdot s_{ik}\cdot\partial_l g\cdot\partial_k h\ \#
\end{aligned}$$

which proves that terms with second derivative of *f* vanish. Similarly, we can prove that second order derivatives of *g* and h are equal on both sides of equation. But the easiest is to notice that our problem is reduced to Jacoby identity

$$[f,[g,h]]-[[f,g],h]-[g,[f,h]]=[f,[g,h]]+[g,[h,f]]+[h,[f,g]]$$

where all three functions *f, g, h* appear identically. It means that their second order derivatives show up in same sequence and vanish. But you can easily check it.

Problem 2 – 25 points

Prove for 1D case that $\exp(:f:) = \sum_{n=0}^{\infty} \frac{:f:^n}{n!}$ with $f = -\left(\frac{p^2}{2} + K_1 \frac{x^2}{2}\right) \cdot s$ generates matrices for focusing and defocusing quadrupole. Consider also case of drift with $K_1=0$.

Solution. Let's start from reminder of Lie operator

$$\exp(:f:) = \sum_{n=0}^{\infty} \frac{:f:^n}{n!}; \ \exp(:f)\ g = g + [f,g] \frac{+[f,[f,g]]}{2!} + \ldots$$

and apply it to x and p:

$$\exp(:f)\ x = x - s \cdot \left[\left(\frac{p^2}{2} + K_1 \frac{x^2}{2}\right), x\right] + \frac{s^2}{2!} \cdot \left[\left(\frac{p^2}{2} + K_1 \frac{x^2}{2}\right), \left[\left(\frac{p^2}{2} + K_1 \frac{x^2}{2}\right), x\right]\right] + \ldots$$

$$\exp(:f)\ p = p - s \cdot \left[\left(\frac{p^2}{2} + K_1 \frac{x^2}{2}\right), p\right] + \frac{s^2}{2!} \cdot \left[\left(\frac{p^2}{2} + K_1 \frac{x^2}{2}\right), \left[\left(\frac{p^2}{2} + K_1 \frac{x^2}{2}\right), p\right]\right] + \ldots$$

$$\left[\left(\frac{p^2}{2} + K_1 \frac{x^2}{2}\right), x\right] = -p; \left[\left(\frac{p^2}{2} + K_1 \frac{x^2}{2}\right), p\right] = K_1 x;$$

$$\exp(:f)\ x = x + s \cdot p - \frac{s^2}{2!} \cdot K_1 \cdot x - \frac{s^3}{2!} \cdot K_1 \cdot x + \frac{s^4}{4!} \cdot K_1{}^2 \cdot x .. = x \cdot \sum_{n=0}^{\infty} \frac{\left(-K_1 s^2\right)^n}{2n!} + p \cdot \sum_{n=0}^{\infty} \frac{s^{2n-1} \cdot K_1{}^{n-1}}{(2n+1)!};$$

$$\exp(:f)\ p = p - K_1 s \cdot x - \frac{s^2}{2!} \cdot K_1 \cdot p + \frac{s^3}{2!} \cdot K_1{}^2 \cdot x + \frac{s^4}{4!} \cdot K_1{}^2 \cdot p .. = p \cdot \sum_{n=0}^{\infty} \frac{\left(-K_1 s^2\right)^n}{2n!} - x \cdot \sum_{n=0}^{\infty} \frac{s^{2n-1} \cdot K_1{}^{n}}{(2n+1)!}.$$

The only remaining task is to compare it with 3 cases:

Focusing: $K_1>0$

$$x \to x \cdot \cos\left(\sqrt{K_1} s\right) + \frac{\sin\left(\sqrt{K_1} s\right)}{\sqrt{K_1}} \cdot p = x \cdot \sum_{n=0}^{\infty} \frac{\left(-\sqrt{K_1} s\right)^{2n}}{2n!} - \frac{p}{\sqrt{K_1}} \cdot \sum_{n=0}^{\infty} \frac{\left(-s \cdot \sqrt{K_1}\right)^{2n-1}}{(2n+1)!};$$

$$p \to p \cdot \cos\left(\sqrt{K_1} s\right) - \sqrt{K_1} \cdot \sin\left(\sqrt{K_1} s\right) \cdot x = p \cdot \sum_{n=0}^{\infty} \frac{\left(-\sqrt{K_1} s\right)^{2n}}{2n!} + x \cdot \sqrt{K_1} \cdot \sum_{n=0}^{\infty} \frac{\left(-s \cdot \sqrt{K_1}\right)^{2n-1}}{(2n+1)!}.$$

Defocusing: $K_1<0$

$$x \to x \cdot \cosh\left(\sqrt{-K_1} s\right) + \frac{\sinh\left(\sqrt{-K_1} s\right)}{\sqrt{-K_1}} \cdot p = x \cdot \sum_{n=0}^{\infty} \frac{\left(-\sqrt{K_1} s\right)^{2n}}{2n!} + \frac{p}{\sqrt{-K_1}} \cdot \sum_{n=0}^{\infty} \frac{\left(s \cdot \sqrt{-K_1}\right)^{2n-1}}{(2n+1)!};$$

$$p \to p \cdot \cosh\left(\sqrt{-K_1} s\right) + \sqrt{-K_1} \cdot \sinh\left(\sqrt{-K_1} s\right) \cdot x = p \cdot \sum_{n=0}^{\infty} \frac{\left(-\sqrt{K_1} s\right)^{2n}}{2n!} - x \cdot \sqrt{-K_1} \cdot \sum_{n=0}^{\infty} \frac{\left(s \cdot \sqrt{-K_1}\right)^{2n-1}}{(2n+1)!}.$$

Drift: $x \to x + s \cdot p;\ p \to p.$

You can compare terms to see that they are identical for all three cases. ###