PHY 554. Homework 1.

HW 1.1 (3 points): Find available energy (so called C.M. energy) for a head-on collision in these scenarios:

(a) In CERN, SPS produced 160 GeV muons collide with protons at rest (the rest energy of proton is 0.938257 GeV, and rest energy of muons is 0.1057 GeV);
Note: there was an unintentional typo in the HW posted at the web with 1 missing in 0.1057 GeV and turning 0.057 GeV – those of you who corrected it will have extra points!!!

(b) Super-KEKB collides 7 GeV electrons with 4 GeV positions (the rest energy of electrons and positrons is 0.511 MeV);

Solution: we should use formula for available c.m. energy:

$$E_{cm} \equiv Mc^2 = c\sqrt{P_i P^i} = \sqrt{E^2 - (c \cdot \vec{p})^2}; E = E_1 + E_2; \vec{p} = \vec{p}_1 + \vec{p}_2 \quad (1)$$

For those of you who are most curious, this energy to create new particles with mass M is available in frame

$$\vec{v}_{cm} = c^2 \cdot \frac{\vec{p}_1 + \vec{p}_2}{E_1 + E_2}$$

Calculations are simple if you do not forget that positrons and electrons are colliding head-on, i.e. their momenta have opposite signs:

	E_1, GeV	m_1c^2, GeV	p_1c, GeV	E_2, GeV	m_2c^2, GeV	p_2c, GeV	Mc^2, GeV
CERN muons	160	0.1057	159.999965	0.938257	0.938257	0	17.353
Super-KEKB	7	5.11E-05	7	4	5.11E-05	-4	10.583

HW 1.2 (2 points): Future circular collider at CERN plans to initially collide 180 GeV electron and position beam and later 50 TeV protons beam circulating in storage ring with 100 km circumference.

(a) 1 point: Assuming that bending magnets fill 70% of the ring circumference, what will be bending radius in the magnets? What magnetic field is required to circulate 50 TeV proton beam?

(b) 1 point: What magnetic field is required to turn 180 GeV electrons and positrons with the same radius?

Solution: we should use formula for available c.m. energy:

$$B\rho = \frac{pc}{e} \Leftrightarrow \left\{ \begin{array}{c} B\rho\left[kGs\cdot cm\right] = \dfrac{pc[MeV]}{0.299792458} \cong \dfrac{pc[MeV]}{0.3} \\ B\rho\left[T\cdot m\right] = \dfrac{pc[GeV]}{0.299792458} \cong \dfrac{pc[GeV]}{0.3} \\ B\rho\left[T\cdot km\right] = \dfrac{pc[TeV]}{0.299792458} \cong \dfrac{pc[TeV]}{0.3} \end{array} \right\}$$

Again, this is just a simple arithmetic:

C, km	<R>, km	fill factor	R_{magnet}, km
100	15.92	0.7	11.14

	E_1, TeV	m_1c^2, TeV	p_1c, GeV	Bρ, T m	B, T
50 TeV protons	50	9.38E-04	50.0000	166.78	14.97
0.18 TeV e+e-	0.18	5.11E-05	0.180000	0.60	0.0539

HW 1.3 (2 points): For a classical microtron with orbit factor k=1 and energy gain per pass of 0.511 MeV and operational RF frequency 3 GHz (3 x 10^9 Hz) find required magnetic field. What will be radius of first orbit in this microtron?

Hint: Note that rest energy of electron with γ=1 is 0.511 MeV. This is energy gain per pass will define available n numbers in eq. (2.6)

Solution: By design, the election's energy gain on the RF cavity is equal to its rest energy – it means that electron at the first orbit has *γ=2* and energy of 1.533 MeV. It also means that on the second orbit electrons will have *γ=3, γ=4* at the third orbit, etc... With k=1 and integer *γ* and *n*, the resonance conditions (2.6)

$$\frac{2\pi mc\cdot f_{RF}}{eB}\cdot\gamma_n = n;$$

can be satisfied only if $j = \frac{2\pi mc\cdot f_{RF}}{eB}$ is an integer. Let's assume that *j=1*, then *n* takes numbers 2,3,4.. for first second and third orbits. For *j=1, n* starts at 2 and gain 1 at each turn…. It allows us to define required magnetic field

$$B = \frac{2\pi mc\cdot f_{RF}}{j\cdot e} = \frac{1}{j}\cdot\frac{mc^2}{e}\cdot\frac{2\pi f_{RF}}{c}$$

with mc^2=0.511 MeV, $\frac{mc^2}{e}$ gives us rigidity of 1.702 kGs cm. $\lambda\!\!\!^{-}_{RF} = \frac{c}{2\pi f_{RF}}$ = 1.59 cm is the RF wavelength divided by. 2π results in

$$B[kGs] = \frac{1}{j}\cdot\frac{1.703\left[kGs\cdot cm\right]}{\lambda_{RF}[cm]} = 1.07\ kGs$$

At first orbit $E=1.022\ MeV$ electrons are still not moving at the speed of light and pc is slightly different from E:

$$pc=\sqrt{E^2-(mc^2)^2}=0.885\ MeV$$

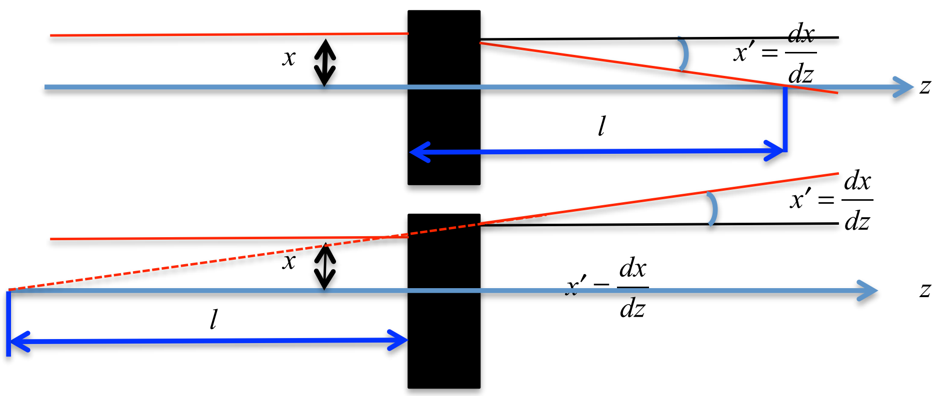
which correspond to radius of the trajectory of

$$\rho[cm]=\frac{pc[MeV]}{0.299792458\cdot B[kGs]}=2.76cm$$

HW 1.4 (5 point): Let's first determine an effective focal length, F, of a paraxial (e.g. small angles!) focusing object (a black-box) as ratio between a parallel displacement of trajectory at its entrance to corresponding change of the angle at its exit (see figure below):

$$F=-\frac{x}{x'};x'\equiv\frac{dx}{dz}$$

see figure below for



Let consider a doublet of two thin lenses: a focusing (F) and defocusing (D) lenses center separated by distance L as in Fig. 1. The lenses have opposite in sign but not equal focal lengths: f_1 for F and f_2 for D lenses.

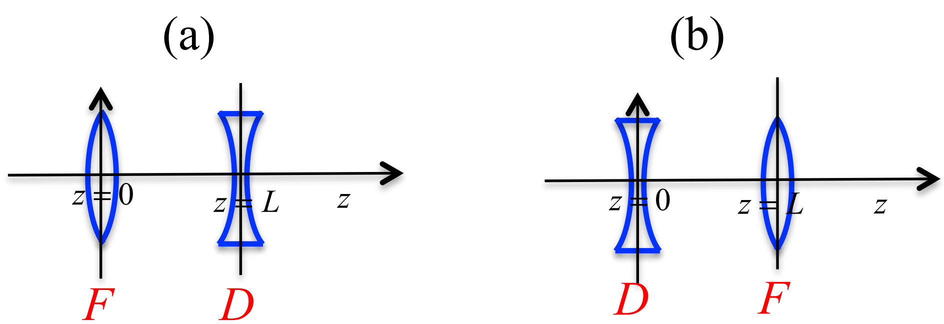


Fig.1. Two combinations of a doublet: *FD* and *DF*.

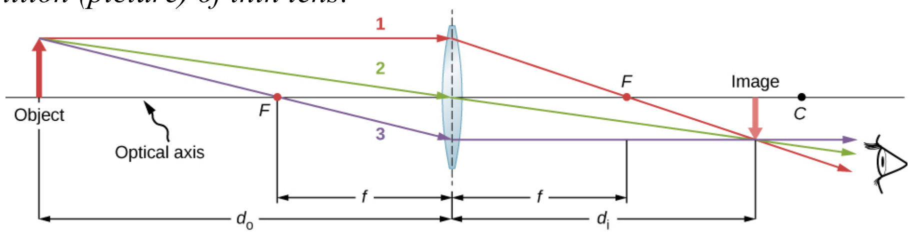
1. (3 points) Find focal lengths of *FD* and *DF* doublets. For the case of $f_1= f_2=f$, show that they are equal and given by following expression:

$$F_{doublet}=\frac{f^2}{L}$$

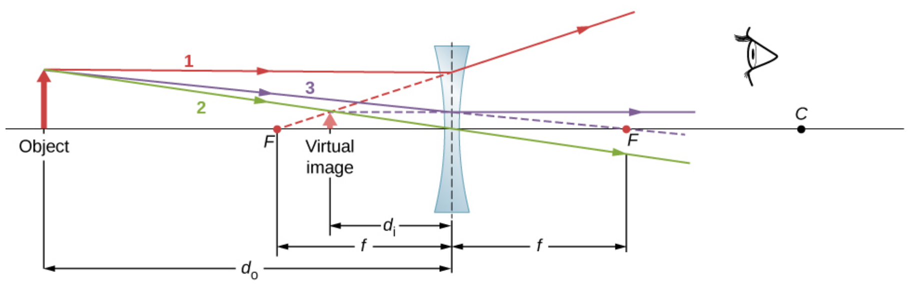
2. (2 points) The ray (trajectory) parallel to the axis is entering the FD or DF system of lenses. Using you calculation of the trajectories in *FD* and *DF* doublets for $f_1= f_2=f$, determine location of to the ray crossing the axis and find their difference between *FD* and

DF doublets. Since a quadrupole focusing in horizontal plane is defocusing in vertical plane - and visa versa –by solving this your find astigmatism of a doublet built from two quadrupoles, i.e. difference between locations of the focal planes for horizontal and vertical direction of motion.

P.S. Definition (picture) of thin lens:



(a)



(b)

Solution: In both cases we start from initial conditions

$$x = x_o; x' = 0;$$

and apply following transformations:

$$F\ lens:\ x_{out} = x_{in};\ x'_{out} = x'_{in} - \frac{x_{in}}{f_F};$$

$$D\ lens:\ x_{out} = x_{in};\ x'_{out} = x'_{in} + \frac{x_{in}}{f_D};$$

$$Drift:\ x_{out} = x_{in} + Lx'_{in};\ x'_{out} = x'_{in};$$

For FD case is gives us

$$x_1 = x_0; x'_1 = -\frac{x_0}{f_F} \to x_2 = x_0 - L\frac{x_0}{f_F}; x'_2 = -\frac{x_0}{f_F} \to$$

$$x_3 = x_0 - L\frac{x_0}{f_F}; x'_3 = -\frac{x_0}{f_F} + \frac{1}{f_D}\left(x_0 - L\frac{x_0}{f_F}\right) = -L\frac{x_0}{f_F \cdot f_D} + \left(\frac{x_0}{f_D} - \frac{x_0}{f_F}\right); \qquad (1)$$

and for DF case

$$x_1 = x_0;\ x'_1 = \frac{x_0}{f_D} \to x_2 = x_0 + L\frac{x_0}{f_D};\ x'_2 = +\frac{x_0}{f_D} \to$$

$$x_3 = x_0 + L\frac{x_0}{f_D}; x'_3 = \frac{x_0}{f_D} - \frac{1}{f_F}\left(x_0 + L\frac{x_0}{f_D}\right) = -L\frac{x_0}{f_F \cdot f_D} + \left(\frac{x_0}{f_D} - \frac{x_0}{f_F}\right); \qquad (2)$$

with x'_3 being the angle at the exit of the "black box" and x_o being the postion at its entrance. For f_1= f_2=f, the answer for the first question is coming from $x'_3 = -L\frac{x_0}{f^2}$ for both FD and DF cases.

The location of the ray crossing the z-axis coming from dividing the position at the exit of the second lens by the angle and adding L (distance from the starting point):

$$\begin{aligned} F:&\quad Z = L - \frac{x_3}{x'_3} = L + \frac{f^2}{L}\left(1 - \frac{L}{f}\right) = L - f + \frac{f^2}{L} \\ D:&\quad Z = L - \frac{x_3}{x'_3} = L + \frac{f^2}{L}\left(1 + \frac{L}{f}\right) = L + f + \frac{f^2}{L} \end{aligned} \tag{3}$$

Hence, the astigmatism of FD set is equal to 2f.