USPAS'23: Hadron Beam Cooling in Particle Accelerators
Solutions for Homework#3: Stochastic Cooling

Problem #1: Numerical model of SC (Total: 8 points)

In this problem we will use a simple model to simulate the effects of Stochastic Cooling. You can use your favourite tool (Python, Matlab, Mathematica, MathCad, etc.) to follow along this exercise.

1. Develop a simple SC model (2 points):
   - Generate an array of N random numbers $[x_1, x_2, .., x_{\mathrm{N}}]$ - these are the initial positions of your particles. Generate at least several thousands of particles.
   - Calculate and record the variance of the array.
   - Slice your bunch into $N_{\mathrm{s}} = 20$ equal samples and calculate the average of each sample (i.e. errors to be corrected).
   - Subtract sample average from the position of its respective sample (i.e. apply correction).
   - Randomize the order of the elements in the corrected array to get a new series.
   - Repeat at least 2000 times.
2. Characterize your system and discuss:
   - (2 points) Plot the particle distribution before and after cooling. Check how the beam variance changes with time.
   - (2 points) Repeat the procedure for $N_{\mathrm{s}} = 10, 300, 600$. How does it affect the cooling process and why?
   - (2 points) How can you change this model to take the gain into account? Implement that and study how gain affects the cooling time.

Solution:

1. Example of a simple script in Matlab:

```
close all;
clear all;

Np=6000;                    % number of particles in the beam
Ns=10;                      % number of particles in each slice
slice=floor(Np/Ns);         % number of slices
Nturn=2000;                 % number of turns

stddata=zeros(Nturn,1);     % container for the variation
xpart=(2*rand(Np,1)-1);     % random particle positions
data=zeros(Np,1);           % container for data storage

for i=1:Np
    data(i,1)=xpart(i);     % assign positions to the particles
end;

data0=data;                 % data0 is our initial configuration

 for j=1:Nturn
    dataslice=reshape(data(:,1),[Ns, slice]);                                %
 reshape the data into slices
    for islice=1:slice
        avgslice=mean(dataslice(:,islice));                                  %
 calculate mean for every slice
            for i=1:Ns
                data((islice-1)*Ns+i,1)=data((islice-1)*Ns+i,1)-avgslice;    %
 apply the kick to every position
            end;
    end;
    stddata(j,1)=std(data(:,1));                                             %
 calculate new variance
    data2=data;                                                              %
 new data
    data=data(randperm(Np),:);                                               %
 mix up the particles!
 end;

 fig = figure()
 subplot(2,1,1)
 plot([1:1:Np],data0(:,1),'.','linew',1.5,'DisplayName','initial'); hold on;
 plot([1:1:Np],data2(:,1),'.','linew',1.5,'DisplayName','final');
 xlabel('Particle number')
 ylabel('Particle position')
 legend('Location','Best')
 title('Number of particles in a slice = 10 particles, Number of slices =
 600')

 subplot(2,1,2)
 plot([1:1:Nturn],stddata(:,1),'linew',1.5); hold on;
 xlabel('Turn')
 ylabel('Variance')
```

Particle distribution before and after cooling and the beam variance vs. time are shown in Fig. 1(a). The similar plots are obtained for $N_s = 10,\ 300,\ 600$ (Fig. 1(b), (c) and (d) respectively). The results indicate that as the number of slices increase, the sample size decreases which results in a faster cooling process.

One can introduce the gain as a multiplication factor at the stage of the correction. In the code shown above we will replace the line
$\mathrm{data}((\mathrm{islice}-1)*N_s+i,1) = \mathrm{data}((\mathrm{islice}-1)*N_s+i,1) - \mathrm{avgslice}$

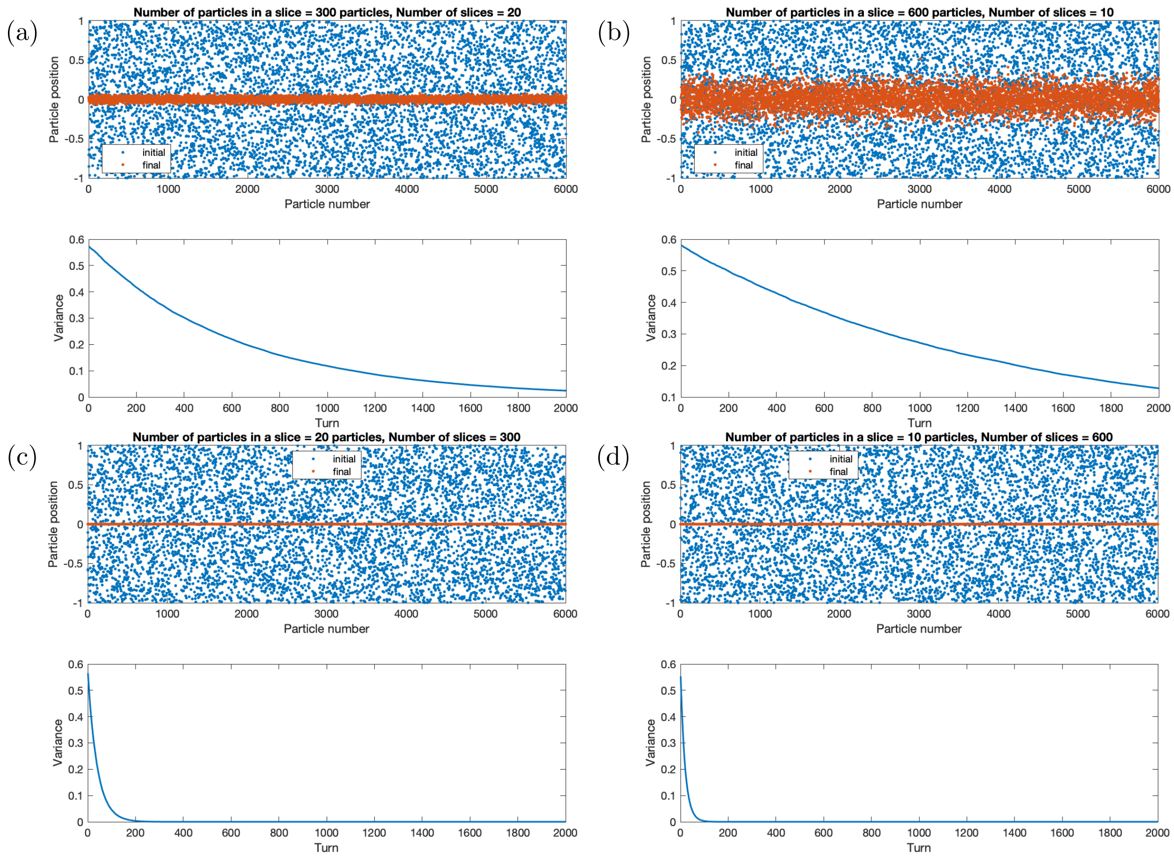

Figure 1: Particle positions before (blue) and after (orange) cooling and beam variance as a function of time for various number of samples N_s: (a) $N_s = 20$; (b) $N_s = 10$; (c) $N_s = 300$; (d) $N_s = 600$.

with
$\mathrm{data}((\mathrm{islice}-1)*N_s+i,1) = \mathrm{data}((\mathrm{islice}-1)*N_s+i,1) - \textbf{gain} * \mathrm{avgslice}.$

The results of a simple simulation that includes the gain is shown in Fig. 2. As expected, for the partial correction ($g < 1$), the cooling will slow down, and for the gain>1 cooling stops and the process is eventually turned into heating.

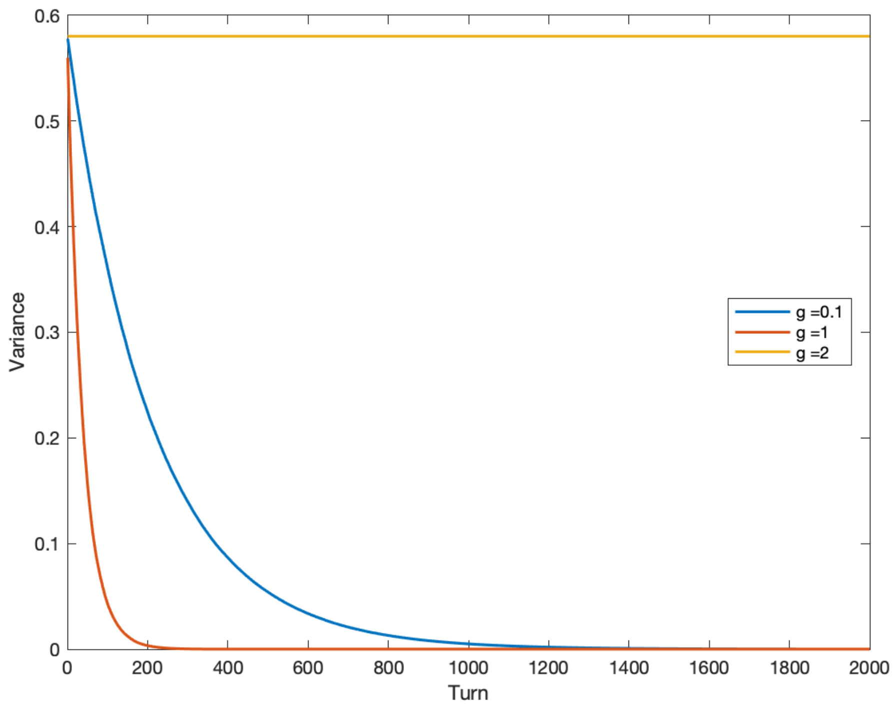

Figure 2: Beam variance as a function of time for the gain = [0.1, 1, 2]

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Problem #2: Mixing, cooling rate, and bandwidth
(Total: 8 points)

In the lectures, we have introduced an expression for the r.m.s. cooling rate that includes Kicker-to-Pick-Up mixing, M, and Pick-Up-to-Kicker mixing, $\tilde{M}$:

$$\frac{1}{\tau} = \frac{W}{N}\left[2g\left(1 - \tilde{M}^{-2}\right) - g^2\left(M + U\right)\right], \tag{1}$$

1. (3 points) Using Eq. 1, find the expressions for the maximum achievable cooling rate and the respective optimum gain. If you have an ideal system, how long would it take to cool $N = 10^9$ particles if the system bandwidth is $W = 1$ GHz? How about $N = 4 \times 10^{13}$?

2. (2 points) Assuming that the time-of-flight dispersion between Pick-Up and Kicker and between Kicker and Pick-Up are such that the unwanted mixing is 1/2 of the wanted mixing, plot cooling time as a function of number of particles ($N \in [10^5, 10^{13}]$) for a system with $W = 1$ GHz. Explore all combinations of the following parameters: $M = 1,\ 10,\ 50$; $U = 0,\ 10$. Discuss.

3. (3 points) It appears that the unwanted mixing imposes a limit on the upper frequency of the cooling band. Using the parameters of the "first generation cooling experiment" at the Antiproton Accumulator Ring at CERN (1984), obtain the upper frequency of the cooling band for that machine by following the steps below:

   - Assume a band-pass with flat response from $f_{\min}$ to $f_{\max}$. Find an expression for the useful width of the correction pulse T_c.
   - Express the time of flight error δt_{PK} in terms of the momentum spread $\Delta p/p$ and local Pick-Up-to-Kicker slip factor η_{PK}.
   - What is the upper frequency of the cooling band for given momentum spread $\Delta p/p = 2 \times 10^{-2}$, slip factor $\eta_{PK} \approx \eta = 0.1$, flight time (Pick-Up-to-Kicker/circumference) $\alpha_T = 0.5$, and revolution frequency $f_{rev} = 1.5$ MHz?

Solution:

1. One can simply differentiate Eq. 1 with respect to the gain and find the optimum value:

$$g_0 = \frac{1 - \tilde{M}^{-2}}{M + U}$$

   Then the maximum cooling rate is:

$$\frac{1}{\tau_0} = \frac{W}{N}\frac{\left(1 - \tilde{M}^{-2}\right)^2}{M + U}$$

   In the best case scenario, $M = 1, U = 0, \tilde{M}^{-2} = 0$, which results in $\tau = N/W$. Then for $W = 1$ GHz and $N = 10^9$ we obtain cooling time of 1s, and for $N = 4 \times 10^{13}$ cooling time of 11 hours.

2. By using the result from part 1 with $\tilde{M} = 2M$, one can plot the maximum achievable cooling rate as a function of the number of particles (see Fig. 3).

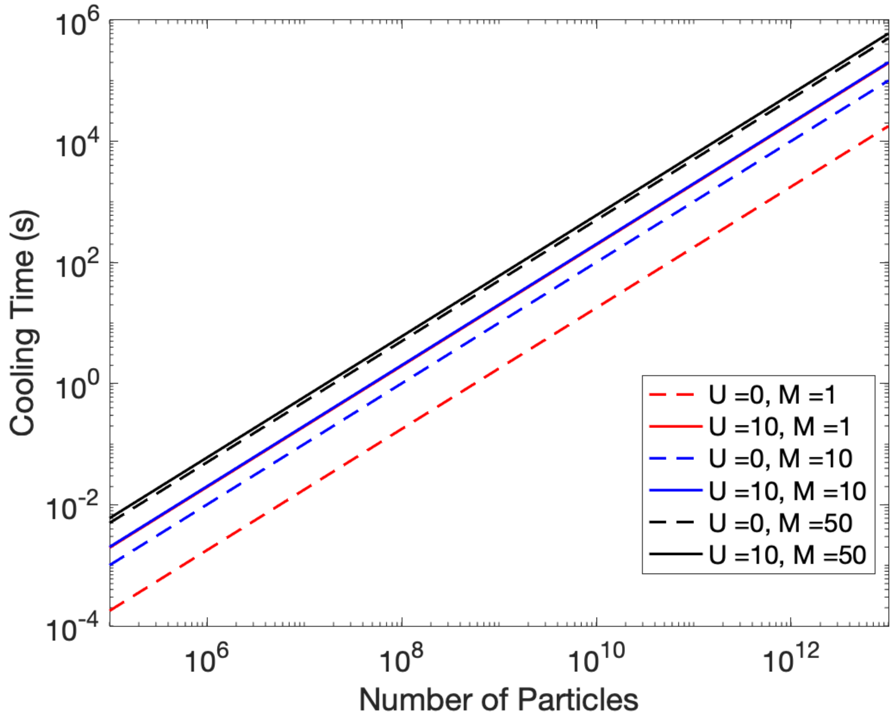

Figure 3: Cooling time as a function of number of particles for various combinations of $M = 1,\ 10,\ 50$; $U = 0,\ 10$.

3. The useful width of the correction pulse:

$$T_c \approx 1/\left[2(f_{\max} - f_{\min})\right]$$

Expressing the time of flight error:

$$\delta t_{\mathrm{PK}} = t_{\mathrm{PK}}\eta_{\mathrm{PK}}\frac{\Delta p}{p} = \alpha_T T_{\mathrm{rev}}\eta_{\mathrm{PK}}\frac{\Delta p}{p}$$

The condition $\delta t_{\mathrm{PK}} < T_c$ yields:

$$1/\left[2(f_{\max} - f_{\min})\right] > \alpha_T T_{\mathrm{rev}}\eta_{\mathrm{PK}}\frac{\Delta p}{p}$$

Then for the given parameters one obtains $f_{\max} - f_{\min} \approx f_{\max} < 0.75$ GHz.

Problem #3: Optical Stochastic Cooling (Total: 9 points)

Let us consider the application of optical stochastic cooling to three types of particles: electrons/positrons, protons/antiprotons, and heavy ions.

1. **Electrons:** (9 points) Since electrons already have a good damping mechanism due to synchrotron radiation, examine what OSC can do in low energy regime.
   Consider a 150 MeV ring of 60 m circumference with 2 cooling insertions: one for longitudinal-horizontal cooling and one for longitudinal-vertical. Assume the following beam parameters: $N = 5 \times 10^9$, normalized transverse emittances $\varepsilon_{x,n} = \varepsilon_{y,n} = 5 \times 10^{-4}$ m, bunch length $l_b = 2.5$ cm, and relative energy spread of 10^{-3}. For the amplifier with central wavelength of 0.8 μm and a bandwidth of 10%, calculate the optimal amplification factor g (3 points), the damping time for betatron oscillations $\tau_{x,y}$ (3 points), and the damping time for energy oscillations τ_δ (2 points). Verify that with one bunch in the ring and with the obtained amplification factor, the average output power of the amplifier is about 5 W in each cooling insertion (1 point).

2. **Protons:** (dropped - 0 points) Assume once again a machine with two cooling insertions, but this time we'll be cooling six bunches with 1×10^{11} protons/bunch with relative momentum spread of 3×10^{-4}, and a revolution frequency of 47.7 kHz. Consider an amplifier with an average output power of 100 W and a central wavelength $\lambda = 0.8\mu$m. The undulator radiation with this wavelength could be obtained in an undulator with a peak magnetic field of 8 T and $\lambda_u = 1.5$m. Estimate the damping times for betatron and synchrotron oscillations.

3. **Heavy Ions:** (dropped - 0 points) Consider damping of lead ions at an energy of 32.8 TeV. Assume 124 bunches of 1×10^8 ions/bunch, a relative momentum spread of 3×10^{-4}, and revolution frequency of 43 kHz. With two cooling insertions, and an undulator with a peak magnetic field of 8 T and $\lambda_u = 0.3$m, and the same optical amplifier as above, calculate the damping times for betatron and synchrotron oscillations.

Solution:

1. **Electrons:**

   To find the optimal amplification factor, g, one can use the following approximation:

$$g \simeq \frac{1}{4}\frac{\varepsilon_{||}}{r_0}\frac{1}{N}\frac{\Delta f}{f}, \text{ with } r_0 = q^2/mc^2, \varepsilon_{||} = \gamma l_b \Delta E/E$$

   Then in our case $g \approx 13$.

   Alternatively, you can use a slightly different expression for the transit-time OSC, and obtain a similar result if we assume F = 1 :

$$g \simeq \frac{1}{\sqrt{e}}\frac{\varepsilon_{||}}{r_0}\frac{\Gamma F}{N}, \text{ with } \Gamma = \Delta f/f$$

   The damping time:

$$\tau \simeq T_{\text{rev}}\frac{eN}{\Gamma}\frac{\lambda}{F l_b}$$

   Then one obtains that for betatron cooling the damping time will be ∼870 ms.

Since there are two cooling insertions, the longitudinal cooling time will be twice shorter than the betatron cooling time, resulting in $\tau_\delta \sim 475$ ms.

Of course, it's a very crude approximation, and one should expect the optimal gain of this system to be ~350, and cooling times for betatron and momentum cooling on the order of 30 ms and 15 ms, respectively.

To estimate the average power, one can assume $K \approx 1$, and find that the it is, indeed, below 5 W.

$$\frac{\tau_x}{T} = \left[\frac{N\lambda}{\overline{\overline{W}}cTK^2}\frac{\sigma_\delta^2(E_b/q)^2}{Z_0}\right]^{1/2}$$

2. **Protons:**

We will use the following equation that connects the optimal damping time and the average output power of the amplifier.

$$\frac{\tau_x}{T} = \left[\frac{N\lambda}{\overline{\overline{W}}cTK^2}\frac{\sigma_\delta^2(E_b/q)^2}{Z_0}\right]^{1/2}$$

One can first find the undulator factor K that can be expressed in terms of the magnetic field and the period of the undulator λ_u:

$$K = \frac{qB\lambda_u}{2\pi mc^2}$$

The wavelength of the EM radiation from the pick-up undulator depends on the undulator factor, undulator period and relativistic factor of the beam:

$$\lambda = \left[\lambda_u(1 + K^2/2)\right]/2\gamma^2,$$

After we determine the energy of the beam, all other parameters are known in order to find the damping time. The resulting damping time for betatron oscillations is ~ 5 mins, and due to the presence of two cooling section, damping time for momentum cooling is ~ 2.5 mins.

3. **Heavy Ions:**

Following the same procedure as in part 2, and taking into account the fact that we are dealing with lead ions (Z = 82), one finds the damping time for betatron oscillations is ~ 2 mins, and the damping time for momentum cooling is ~ 1 min.