Homework 7.

Problem 1 – 10 points.

Consider a flat (no torsion, $\kappa = 0$) trajectory in a bending magnet, with constant radius. Assume that there is no electric field ($\vec{E} = 0$, energy is constant) and there in no x-y coupling (skew quadrupole components and longitudinal magnetic field are zero: $\frac{\partial B_x}{\partial x} = \frac{\partial B_y}{\partial y} = 0$; $B_s = 0$ - this also make mechanical and Canonical momenta equal). Assume that there is a quadrupole component allowed in the magnet

$$\frac{\partial B_y}{\partial x} = \frac{\partial B_x}{\partial y} \neq 0$$

The particle's Hamiltonian is then split in two uncoupled parts: (y, p_y) and (x, p_x, τ, δ)

$$\tilde{h} = h_H(x,p_x,\tau,\delta) + h_V(y,p_y)\ h_H = \frac{p_x^2}{2\cdot p_o} + F\cdot\frac{x^2}{2} + \frac{\delta^2}{2\cdot p_o}\cdot\left(\frac{mc}{p_o}\right)^2 + g_x\cdot x\cdot\delta;\ h_V = \frac{p_y^2}{2\cdot p_o} + G\cdot\frac{y^2}{2};$$

with (see Lecture 4 formulae (140 and 141)

$$K_1 = -\frac{G}{p_o} = -\frac{e}{p_o c}\frac{\partial B_x}{\partial y};\ \frac{F}{p_o} = K^2 + K_1;\ K \equiv K_o = -\frac{eB_y}{p_o c};\ \beta_o = \frac{p_o c}{E_o};\ g_x = -\frac{K}{\beta_o};\ \frac{mc}{p_o} = \frac{1}{\gamma_o\beta_o};$$

where I introduced K_o and K_1 notions frequently used in accelerator literature. Because p_o is constant, I would recommend you switch to switch to dimensionless momenta

$$p_{x,y} \to \frac{p_{x,y}}{p_o};\ \delta \to \frac{\delta}{p_o};$$

$$h_H = \frac{p_x^2}{2} + \left(K_o^{\ 2} + K_1\right)\cdot\frac{x^2}{2} + \frac{\delta^2}{2\cdot(\gamma_o\beta_o)^2} - \frac{K_o}{\beta_o}\cdot x\cdot\delta;\ h_V = \frac{p_y^2}{2} - K_1\cdot\frac{y^2}{2};$$

Since vertical motion is completely separated from that horizontal plane, there is a little incentive to use 6x6 matrices.

3 point. Unless you want to use 6x6 matrices, derive explicit 2x2 matrices for vertical motion without any assumptions about the sign and value of K_1 (i.e. all 3 cases).

7 point. Similarly, derive 4x4 matrices to calculate motion in horizontal plane without any assumptions of the sign and value of K_1 and value of $\beta_o \neq 0$. Compare it with expression we derived in Lecture 8.

Hint: Motion in horizontal plane has zero *det***H**.

Solution: Since *(y, p_y)* part of Hamiltonian is completely separated

$$h_V = \frac{p_y^2}{2\cdot p_o} + G\cdot\frac{y^2}{2};\ \mathbf{H} = \begin{bmatrix} G & 0 \\ 0 & \frac{1}{p_o} \end{bmatrix};\ \mathbf{D} = \begin{bmatrix} 0 & \frac{1}{p_o} \\ -G & 0 \end{bmatrix};$$

we have 3 cases: G<0; G=0 and G>0 with already many times derived solutions with only difference that we did not normalized momentum:

$$det(\mathbf{D} - \lambda\cdot\mathbf{I}) = det\begin{bmatrix} -\lambda & \frac{1}{p_o} \\ -G & -\lambda \end{bmatrix} = \lambda^2 + \frac{G}{p_o} = 0;$$

$$G > 0;\ \lambda_{1,2} = \pm i\omega;\ \omega = \sqrt{\frac{G}{p_o}};$$

$$\mathbf{M} = e^{Ds} = e^{i\omega s}\frac{\mathbf{D} + i\omega\cdot\mathbf{I}}{2i\omega} + e^{i\omega s}\frac{\mathbf{D} - i\omega\cdot\mathbf{I}}{-2i\omega} = \begin{bmatrix} \cos(\omega s) & \frac{\sin(\omega s)}{\sqrt{Gp_o}} \\ -\sqrt{Gp_o}\cdot\sin(\omega s) & \cos(\omega s) \end{bmatrix}$$

$$G = 0;\ \lambda_{1,2} = 0;\ \mathbf{M} = e^{Ds} = \mathbf{I} + \mathbf{D}\cdot s = \begin{bmatrix} 1 & \frac{s}{p_o} \\ 0 & 1 \end{bmatrix};$$

$$G < 0;\ \lambda_{1,2} = \pm\omega;\ \omega = \sqrt{-\frac{G}{p_o}};$$

$$\mathbf{M} = e^{Ds} = e^{\omega s}\frac{D + \omega\cdot I}{2\omega} + e^{\omega s}\frac{D - \omega\cdot I}{-2\omega} = \begin{bmatrix} cosh(\omega s) & \frac{sinh(\omega s)}{\sqrt{-Gp_o}} \\ \sqrt{-Gp_o}\cdot sinh(\omega s) & cosh(\omega s) \end{bmatrix}$$

Using normalized momentum simplifies this into

$$\lambda^2 = K_1$$

$$K_1 < 0;\ \lambda_{1,2} = \pm i\omega;\ \omega = \sqrt{-K_1};\ \ \mathbf{M} = \begin{bmatrix} \cos(\omega s) & \frac{\sin(\omega s)}{\omega} \\ -\omega\cdot\sin(\omega s) & \cos(\omega s) \end{bmatrix}$$

$$K_1 = 0;\ \lambda_{1,2} = 0;\ \mathbf{M} = \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix};$$

$$K_1 > 0;\ \lambda_{1,2} = \pm\omega;\ \omega = \sqrt{K_1};\ \ \mathbf{M} = \begin{bmatrix} cosh(\omega s) & \frac{sinh(\omega s)}{\omega} \\ \omega\cdot sinh(\omega s) & cosh(\omega s) \end{bmatrix}$$

Finishing with trivial case, let's focus on always coupled and most of the time degenerated motion in the orbit plane:

$$h_H\left(x,p_x,\tau,\delta\right)=\frac{p_x^2}{2}+f\cdot\frac{x^2}{2}+\frac{\delta^2}{2\cdot\left(\gamma_o\beta_o\right)^2}-K_o\beta_o^{-1}\cdot x\cdot\delta;\ f=K_o^2+K_1;$$

$$\mathbf{H}=\begin{bmatrix} f & 0 & 0 & -K_o\beta_o^{-1} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -K_o\beta_o^{-1} & 0 & 0 & (\gamma\beta)^{-2}\end{bmatrix};\mathbf{D}=\begin{bmatrix} 0 & 1 & 0 & 0 \\ -f & 0 & 0 & K_o\beta_o^{-1} \\ -K_o\beta_o^{-1} & 0 & 0 & (\gamma\beta)^{-2} \\ 0 & 0 & 0 & 0\end{bmatrix};$$

with obvious (line of zeros) that *det*$\mathbf{D}$=0. Since eigne values are coming in opposite sign pairs we naturally expect λ^2 in the characteristic equation and indeed

$$det(\mathbf{D}-\lambda\cdot\mathbf{I})=\lambda^2\cdot\left(\lambda^2+f\right)=0\rightarrow\lambda_{1,2}=\pm\sqrt{-f};\ \lambda_3=\lambda_4=0$$

we have at least two zero eigen values and two other depend of the sign of F. Hence there are also 3 cases: f<0; f=0 and f>0. We have to use the most general Sylvester formula:

$$f(\mathbf{D})=\sum_{i=1}^{m}\left[\left(\sum_{j=0}^{l_i-1}\frac{\phi_i^{(j)}\left(\lambda_i\right)}{j!}\left(\mathbf{D}-\lambda_i I\right)^j\right)\prod_{k\neq i}\left(\mathbf{D}-\lambda_k\mathbf{I}\right)^{l_k}\right];$$

$$\phi_i(\lambda)=f(\lambda)/\prod_{j\neq i}\left(\lambda-\lambda_j\right)^{l_j};\ \phi_i^{(j)}\left(\lambda_i\right)=\left.\frac{d^j\phi_i(\lambda)}{d\lambda^j}\right|_{\lambda=\lambda_i}.$$

F>0, two distinct and one degenerated eigen vector with height ≤ 2:

$$\lambda_{1,2}=\pm i\omega;\ \omega=\sqrt{f};\ \lambda_3=\lambda_4=0,\ h\leq 2;\ \varphi=\omega\cdot s;$$

$$\phi_3(\lambda)=\frac{e^{\lambda s}}{\left(\lambda-\lambda_1\right)\left(\lambda-\lambda_2\right)}=\frac{e^{\lambda s}}{\left(\lambda^2+\omega^2\right)};\phi'_3(\lambda)=s\cdot\phi_3(\lambda)-2\lambda\cdot\frac{e^{\lambda s}}{\left(\lambda^2+\omega^2\right)^2};$$

$$\phi_3\left(\lambda_3\right)=\frac{1}{\omega^2};\ \phi'_3\left(\lambda_3\right)=\frac{s}{\omega^2};$$

$$\left(\phi_3\left(\lambda_3\right)\cdot\mathbf{I}+\phi'_3\left(\lambda_3\right)\cdot\left(\mathbf{D}-\lambda_3\mathbf{I}\right)\right)\cdot\left(\mathbf{D}-\boldsymbol{\lambda}_1\mathbf{I}\right)\left(\mathbf{D}-\boldsymbol{\lambda}_2\mathbf{I}\right)=\frac{\mathbf{I}+\mathbf{D}\cdot s}{\omega^2}\left(\mathbf{D}^2+\omega^2\cdot\mathbf{I}\right)$$

$$\mathbf{M}=e^{\mathbf{D}s}=e^{i\varphi}\cdot\frac{\mathbf{D}+i\omega\mathbf{I}}{2i\omega}\cdot\left(\frac{\mathbf{D}}{i\omega}\right)^2+e^{-i\varphi}\cdot\frac{\mathbf{D}-i\omega\mathbf{I}}{-2i\omega}\cdot\left(\frac{\mathbf{D}}{-i\omega}\right)^2+\frac{\mathbf{I}+\mathbf{D}\cdot s}{\omega^2}\left(\mathbf{D}+\omega^2\cdot\mathbf{I}\right)$$

and further expending it we have

$$\mathbf{M} = -\left(\mathbf{I}\cdot\cos\varphi + \mathbf{D}\cdot\frac{\sin\varphi}{\omega}\right)\cdot\left(\frac{\mathbf{D}}{\omega}\right)^2 + \frac{\mathbf{I}+\mathbf{D}\cdot s}{\omega^2}(\mathbf{D}^2 + \omega^2\cdot\mathbf{I})$$

It takes a bit of calculations to evaluate the overall expression, but there is a lot of ones and zeros in $(\mathbf{D}/\omega)^2$ and the final 4x4 matrix is

$$\mathbf{M} = \begin{bmatrix} \cos\varphi & \frac{\sin\varphi}{\omega} & 0 & \frac{K_o}{\beta_o}\cdot\frac{1-\cos\varphi}{\omega^2} \\ -\omega\cdot\sin\varphi & \cos\varphi & 0 & \frac{K_o}{\beta_o}\cdot\frac{\sin\varphi}{\omega} \\ -\frac{K_o}{\beta_o}\cdot\frac{\sin\varphi}{\omega} & -\frac{K_o}{\beta_o}\cdot\frac{1-\cos\varphi}{\omega^2} & 1 & \frac{s}{(\gamma\beta)^2} + \left(\frac{K_o}{\omega\beta_o}\right)^2\cdot\left(\frac{\sin\varphi}{\omega} - s\right) \\ 0 & 0 & 0 & 1 \end{bmatrix};$$

Similarly for ***F<0***, we have two distinct and one degenerated eigen vector with height ≤ 2:

$$\lambda_{1,2} = \pm\omega;\ \omega = -f;\ \lambda_3 = \lambda_4 = 0,\ h \le 2;\ \varphi = \omega\cdot s;$$

$$\phi_3(\lambda) = \frac{e^{\lambda s}}{(\lambda-\lambda_1)(\lambda-\lambda_2)} = \frac{e^{\lambda s}}{(\lambda^2 - \omega^2)};\phi'_3(\lambda) = s\cdot\phi_3(\lambda) - 2\lambda\cdot\frac{e^{\lambda s}}{(\lambda^2-\omega^2)^2};$$

$$\phi_3(\lambda_3) = -\frac{1}{\omega^2};\ \phi'_3(\lambda_3) = -\frac{s}{\omega^2};$$

$$\left(\phi_3(\lambda_3)\cdot\mathbf{I} + \phi'_3(\lambda_3)\cdot(\mathbf{D}-\lambda_3\mathbf{I})\right)\cdot(\mathbf{D}-\lambda_1\mathbf{I})(\mathbf{D}-\lambda_2\mathbf{I}) = -\frac{\mathbf{I}+\mathbf{D}\cdot s}{\omega^2}(\mathbf{D}^2 - \omega^2\cdot\mathbf{I})$$

$$\mathbf{M} = e^{\mathbf{D}s} = e^{\varphi}\cdot\frac{\mathbf{D}+\omega\mathbf{I}}{2\omega}\cdot\left(\frac{\mathbf{D}}{\omega}\right)^2 + e^{-\varphi}\cdot\frac{\mathbf{D}-\omega\mathbf{I}}{-2\omega}\cdot\left(\frac{\mathbf{D}}{-\omega}\right)^2 - \frac{\mathbf{I}+\mathbf{D}\cdot\mathbf{s}}{\boldsymbol{\omega}^2}(\mathbf{D}^2 - \boldsymbol{\omega}^2\cdot\mathbf{I})$$

getting

$$\mathbf{M} = \begin{bmatrix} \cosh\varphi & \frac{\sinh\varphi}{\omega} & 0 & \frac{K_o}{\beta_o}\cdot\frac{\cosh\varphi - 1}{\omega^2} \\ \omega\cdot\sinh\varphi & \cosh\varphi & 0 & \frac{K_o}{\beta_o}\cdot\frac{\sinh\varphi}{\omega} \\ -\frac{K_o}{\beta_o}\cdot\frac{\sinh\varphi}{\omega} & -\frac{K_o}{\beta_o}\cdot\frac{\cosh\varphi - 1}{\omega^2} & 1 & \frac{s}{(\gamma\beta)^2} - \left(\frac{K_o}{\omega\beta_o}\right)^2\cdot\left(\frac{\sinh\varphi}{\omega} - s\right) \\ 0 & 0 & 0 & 1 \end{bmatrix};$$

With Selvester taking all necessary integrals for you. You can easily check that all equations of motions are solved correctly.

Now, let's do the "nastiest" but actually easiest case when $Ko \neq 0$ and all eigen values are zeros with degeneration level up to four:

$$f = 0;\ \lambda_{1,2,3,4} = 0\ ;$$

$$\phi(\lambda) = e^{\lambda s};\ \phi'(\lambda) = s \cdot \phi(\lambda);\ \phi''(\lambda) = s^2 \cdot \phi(\lambda); \phi'''(\lambda) = s^3 \cdot \phi(\lambda);$$

$$\phi(0) = 1;\ \phi'(0) = s;\ \phi''(0) = s^2; \phi'''(0) = s^3;\ or\ u\text{sing}\ \mathbf{D}^4 = \mathbf{0};$$

$$\mathbf{M} = e^{\mathbf{D}s} = \mathbf{I} + \mathbf{D} \cdot s + \frac{\mathbf{D}^2 \cdot s^2}{2!} + \frac{\mathbf{D}^3 \cdot s^3}{3!}$$

with

$$\mathbf{M} = \begin{bmatrix} 1 & s & 0 & \frac{K_o}{\beta_o} \cdot \frac{s^2}{2} \\ 0 & 1 & 0 & \frac{K_o}{\beta_o} \cdot s \\ -\frac{K_o}{\beta_o} \cdot s & -\frac{K_o}{\beta_o} \cdot \frac{s^2}{2} & 1 & \frac{s}{(\gamma\beta)^2} - \left(\frac{K_o}{\beta_o}\right)^2 \cdot \frac{s^3}{6} \\ 0 & 0 & 0 & 1 \end{bmatrix};$$

where Sylvester formula and just a simple Taylor series provides with all necessary expansions up to s^3.