About these notes

These notes are primarily taken from Introduction to plasma physics and controlled fusion by Francis F. Chen and from my my own plasma physics class, taught by Warren B. Mori in the year 2010.

Definition of plasma

There is no perfect definition for plasma, but there are a few that are generally correct:

1. Loosely speaking, plasma is the fourth state of matter:

$$\text{Solid} \xrightarrow{\text{heat}} \text{Liquid} \xrightarrow{\text{heat}} \text{Gas} \xrightarrow{\text{heat}} \text{plasma}$$

(hot gas/ionized gas)

phase transitions

<u>not</u> a phase transition (continuous transition)

2. A system containing mobile charges— positive, negative, or both — in which electromagnetic interactions between the particles play the dominant role in the dynamics of the system. The caveats to this definition is that the plasma need not be fully ionized (could contain neutral particles), and need not be at equilibrium.

However, as a starting point, consider the equilibrium situation. In equilibrium, the fraction of ionized particles at a particular temperature is given by the <u>Saha</u> equation:

$$\frac{n_i}{n_n} \cong 2.4 \times 10^{21} \frac{T^{3/2}}{n_i} e^{-U_i/kT}$$

n_i : number density of ions in m^{-3} (is equal to number of e^- in singly ionized case)

n_n : number density of remaining neutral atoms in m^{-3}

$n_n + n_i = n_0$ initial number density of neutral atoms in m^{-3}

$K = 1.38 \times 10^{-23}$ J/Kelvin Boltzman Constant

T: temperature in Kelvin

U_i: ionization energy of gas, usually expressed in units of electron volt (eV)

$1\,eV = 1.6 \times 10^{-19}$ J

e.g. Hydrogen, $U_i = 13.6$ eV

Nitrogen, $U_i = 14.5$ eV

$$\frac{n_i}{n_n} = \begin{cases} \text{increases with increasing temperature} \\ \text{decreases with increasing } n_i \end{cases}$$

Examples of Plasma

1. Air at room temperature,

$n_0 = 2.66 \times 10^{25}\ m^{-3}$ (from ideal gas Law)

$U_i = 14.5$ eV (for nitrogen)

$T = 300°$ K ($KT = 0.025$ eV)

What about n_n? We already know from daily life experience that most of the atoms in the atmosphere are neutral atoms. So we are going to make an approximation: the number density of the remaining neutrals is basically the same as the number density of initially unionized atoms. In math, this means

$$n_n = n_o = 2.66 \times 10^{25}\ m^{-3}$$

Is this approximation justified? In plasma physics, we make a LOT of approximations to make the calculations simpler or sometimes even possible to perform. What we need to do is to keep these approximations in mind, and once we get the final answer, check that our answer is consistent with our assumptions. So using this approximation,

$$n_i^2 = 2.4 \times 10^{21} \times (300)^{3/2}\, e^{-14.5/0.025} \times 2.66 \times 10^{25}$$

$$= \underbrace{3.75 \times 10^{-203}}_{\text{very small!}}$$

very small ! plasma is weakly ionized, meaning that the assumption $n_n \sim n_o$ is valid.

2. Interplanetary space

$n_i \sim 10^6\ m^{-3}$

$U_i = 13.6\ eV$ (mostly hydrogen)

$T = 110^\circ\ K \Rightarrow KT = 0.01\ eV$

$\frac{n_i}{n_n} \approx 0$ from Saha equation, but interplanetary space is actually mostly ionized. The reason for this contradiction is that interplanetary space is not in equilibrium, so the application of Saha equation is not valid. Interplanetary plasma is created by solar wind, which is radiation generated by stars.

3. Stars

$$\frac{n_i}{n_n} \gg 1 \quad \text{(fully ionized)}$$

$$U_i = 13.6 \text{ eV}$$

$$T = 2\text{KeV at core}, \; 200\text{ eV on surface.}$$

$$n_i = 10^{16} - 10^{23} \text{ at surface}$$

$$10^{26} - 10^{32} \text{ at core}$$

This number density for ions corresponds to

$$\rho = \text{mass density} \sim 100 \text{ g/cm}^3 \text{ or } 10^5 \text{ kg/m}^3$$

In comparison, fusion targets such as solid Deuterium are much denser and have a mass density of 0.3 g/cm^3 or 300 kg/m^3

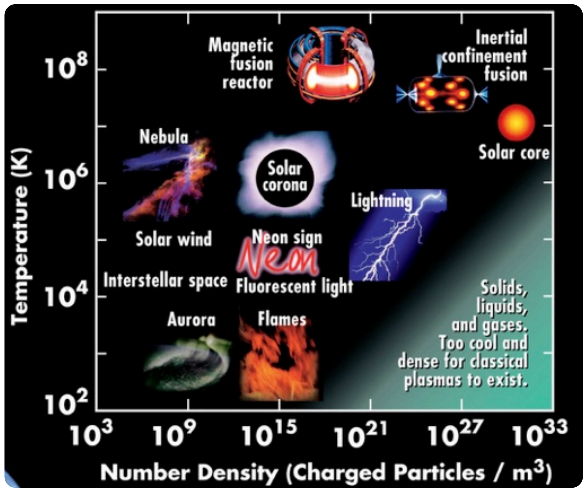
Because of the exponential dependence of Saha equation on the temperature, the high ionization state in a star implies

$$KT \gtrsim U_i$$

i.e. $$KT \gtrsim 10\text{eV} \approxeq \underline{\underline{10^5 \,°K}}$$

HOT Gas!

The spectrum of plasma in the universe can be represented on a continuum of number density and temperature. The figure below is created by the Contemporary Physics Education Project



Type	Source	Applications
1. Hot gas	$U_i/kT \sim 1$	stars, fusion (Lasers/ magnetic)
2. Discharge	Applied Electric field	Flourescent Light bulbs, Lightening, plasma processing
3. Intersteller gas interplanetary space	Radiation from stars, solar wind	
4. Ionosphere, magnetosphere	solar radiation & solar wind	Radio communication
5. Tunnel ionized gas	Electric field, e.g. from a laser	Laser matter interaction, plasma-based acceleration
6. Semiconductors (electron-hole pairs), metals	$U_i \sim 0.1 eV$	

7. Electrolytes (ions in liquid structure) — Different equilibrium equations

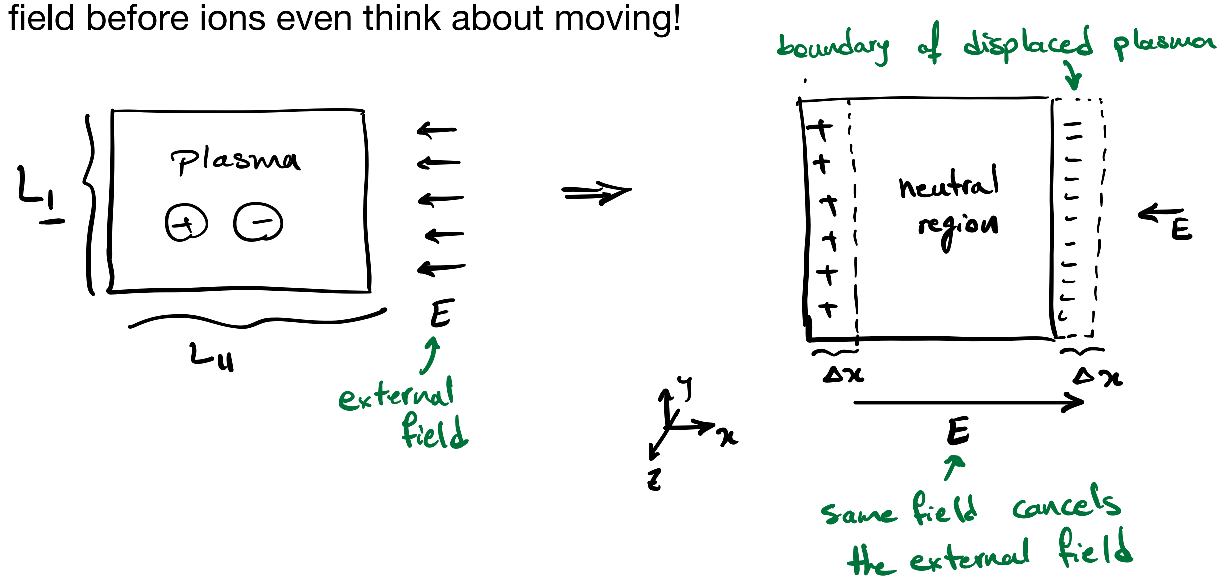
8. Early Universe — hot gas (Recombination phase) — Understanding the primordial background radiation

Next, we will need to review the concept of temperature and discuss important time and spatial scales in a plasma

Plasma Oscillation Frequency

Plasma frequency is a natural frequency of oscillation for electrons. It is a fundamental timescale in plasma physics and can be derived by considering how quickly the plasma electrons in a neutral plasma move to shield out an external electric field (here, plasma behaves like a perfect metal).

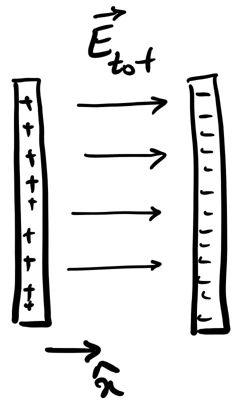
We start by imposing an electric field on the plasma, resulting in a displacement of plasma electrons. We attribute the entire motion to electrons because ions are thousands of time more massive, so electrons can shield the field before ions even think about moving!



Assume $\Delta x \ll L_\parallel \ll L_\perp$ so that each region of uncovered charge can be modelled as an infinite plate with charge density σ. Using Gauss's Law (see Griffiths, example 2.5), The field for each uncovered region is

$$\vec{E} = \frac{\sigma}{2\epsilon_0}\hat{x}$$

surface charge density



Total field is given by superposition of the two fields,

$$\vec{E}_{tot} = \frac{2\sigma}{2\epsilon_0}\hat{x} = \frac{\sigma}{\epsilon_0}\hat{x}$$

note: the two regions have the same amount of charge density as the original plasma was neutral & the same amount of +ve charge exists as -ve.

σ = charge per unit area = charge per unit volume $\times \Delta x$

$= e n_0 \Delta x$

charge of each e^- — number density

$$\vec{F} = -eE = -\frac{e^2 n_0 \Delta x}{\epsilon_0} \quad (\text{force on } e^-)$$

Newton's second law of motion written in terms of Δx:

$$m\frac{d^2\Delta x}{dt^2} = F = -\frac{e^2 n_0 \Delta x}{\epsilon_0}$$

$$\frac{d^2\Delta x}{dt^2} + \frac{e^2 n_0}{m\epsilon_0}\Delta x = 0$$ ← simple harmonic motion equation

Define $\omega_p^2 \equiv \frac{e^2 n_0}{m \epsilon_0}$, plasma frequency

$$\omega_p = 5.6 \times 10^4 \sqrt{n\,(cm^{-3})} = 56 \sqrt{n\,(m^{-3})}$$

As our calculation above shows, plasma electrons can move "as fast as" ω_p. We will show later that whether an electromagnetic wave can propagate in plasma will depend on the relation between its temporal radial frequency and plasma frequency:

$\omega > \omega_p$: wave propagates in plasma (plasma is a good conductor)

$\omega < \omega_p$: plasma will not support wave propagation & the wave will be reflected.

like a metal

Plasma frequency and conductivity

In electrodynamics, the concept of conductivity is usually introduced in relation to Ohm's law:

conductivity. Note: conductivity & surface charge density share the same symbol.

Ohm's Law $\quad \vec{J} = \sigma \vec{E}$

volume current density. $d\vec{I} = \vec{J} \cdot d\vec{a}$ (see Appendix for a refresher on $\vec{J}$)

volume charge density

Continuity equation $\quad \frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J} = 0$

assume σ is uniform

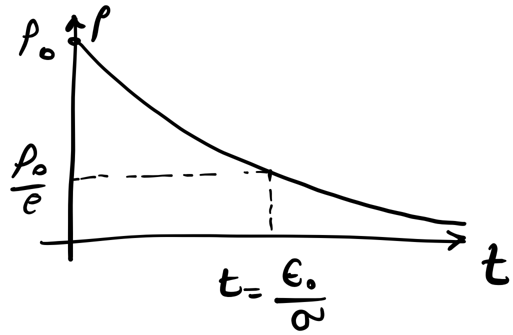
$$\therefore \frac{\partial \rho}{\partial t} + \sigma \underbrace{\vec{\nabla} \cdot \vec{E}}_{} = 0$$

$\vec{\nabla} \cdot \vec{E} = \rho / \epsilon_0$ (Gauss's Law/Maxwell's eqn)

$$\frac{\partial \rho}{\partial t} + \frac{\sigma}{\epsilon_0} \rho = 0$$

$$\Rightarrow \rho = \rho_0 \, e^{-\frac{\sigma}{\epsilon_0} t}$$

initial charge density dissipates in exponential decay:



e.g. copper $\sigma = 5 \times 10^{7} \frac{S}{m} \Rightarrow t = \frac{8.854 \times 10^{-12}}{5.8 \times 10^{7}} = 1.5 \times 10^{-19}\,s$

$= 0.15\, as$

atto seconds.

The description above works well in metals because Ohm's law describes the relation between volume current density and electric field properly. The question of interest here is whether Ohm's law is valid in plasma. Let's examine the underlying assumptions Ohm's law, starting with the definition of volume current density:

$$\vec{J} = \rho \vec{v} = -e n_0 \vec{v}$$

for e^-

(assumes current is carried by e^-)

velocity

volume charge density $dq = \rho \, d\tau$

Volume element

Equation of motion: $\frac{d\vec{v}}{dt} = -\frac{e\vec{E}}{m} - \nu \vec{v}$ ... ①

collisional drag coefficient. represented by Greek letter "\nu"

Ohm's Law implies that e^- velocity is described by a constant, "terminal" velocity. In math, this means $\frac{d\vec{v}}{dt} = 0$ & therefore the field $\vec{E} \propto \vec{v}$ from eqn 1.

$$\frac{e\vec{E}}{m} = -\nu\vec{v}$$

$$\Rightarrow \vec{v} = -\frac{e\vec{E}}{\nu m} \quad \ldots (2)$$

In this case, conductivity would be

$$-n_0 e\vec{v} = \rho\vec{v} \underset{\text{def'n}}{=} \vec{J} \underset{\text{Ohm's}}{=} \sigma\vec{E}$$

$$\xrightarrow{(2)} \frac{n_0 e^2}{m\nu}\vec{E} = \sigma\vec{E}$$

$$\therefore \sigma = \frac{e^2 n_0}{m\epsilon_0}\frac{\epsilon_0}{\nu} = \frac{\epsilon_0}{\nu}\omega_p^2 \quad \ldots (3)$$

The question now is whether the assumption of constant drift velocity, which allows us to relate the electric field to velocity is valid. In math, this translates to whether the factor of $\frac{d\vec{v}}{dt}$ can be ignored in Eqn 1.

$$(1): \frac{d\vec{v}}{dt} = -\frac{e\vec{E}}{m} - \nu\vec{v}$$

There are 3 terms in this equation. Ignoring $\frac{d\vec{v}}{dt}$ regardless of the value of $\vec{v}$ means that we are assuming that it is much smaller than the other velocity term:

$$\frac{d\vec{v}}{dt} \overset{?}{\ll} |\nu\vec{v}| \quad \ldots (4)$$

$\frac{d\vec{v}}{dt}$: under Ohm's law assumption $v \propto E$

$$\rho = \rho_0 e^{-\sigma/\epsilon_0 t} \Rightarrow E \propto e^{-\sigma/\epsilon_0 t}$$

($\vec{\nabla}\cdot\vec{E} = \rho/\epsilon_0$, they should have the same temporal dependence)

$$\therefore v \propto e^{-\sigma/\epsilon_0 t} \Rightarrow \left|\frac{dv}{dt}\right| \propto \frac{\sigma}{\epsilon_0} v$$

$$(4) \Rightarrow \sigma/\epsilon_0 \, v \overset{?}{\ll} \nu v$$

Calculate this in a bit

$$\sigma/\epsilon_0 \overset{?}{\ll} \nu \xrightarrow{(3)} \frac{\omega_p^2}{\nu} \ll \nu \Rightarrow \left(\frac{\nu}{\omega_p}\right)^2 \gg 1 \quad \cdots (5)$$

will discuss this in a bit

Equivalently, $\frac{\sigma}{\epsilon_0} \overset{?}{\ll} \frac{\epsilon_0 \omega_p^2}{\sigma} \Rightarrow \sigma^2 \overset{?}{\ll} \epsilon_0^2 \omega_p^2$

For copper at solid density

$$n_0 \sim 10^{29} \, m^{-3}, \quad \sigma = 5\times10^7 \, \frac{S}{m} \Rightarrow 3\times10^{15} \not\ll 2\times10^{10}$$

Therefore this condition does not generally hold for a plasma, and so conductivity in plasma is not simply described by equation 3 and is in general frequency dependent, rather than just being a constant of proportionality. We will show later how one can define conductivity for plasma modeled as a fluid.

Length scales

One way to get the length scales is to decide a velocity by a time. We already found an important time scale ω_p^{-1}

What are the important velocities?

1. The speed of light, c
2. The thermal velocity of particles, I.e. average velocity for a distribution of particles, $V_{th} \Rightarrow \frac{1}{2} m V_{th}^2 = \langle KE \rangle$

So the two length scales are

$$\frac{v_{th}}{\omega_p} \quad \& \quad \frac{c}{\omega_p}$$

We will discuss the significance of the first one here and the second one later. Let's review the concept of distribution function and the temperature: a gas in thermal equilibrium has particles of all energies or velocities. The velocity distribution is given by a Gaussian function:

1D: $f(v) = A e^{-\frac{1}{2}mv^2/kT}$ ≡ Maxwellian distribution function of velocities

$K = 1.38 \times 10^{-23}\ J/°K$ Boltzmann constant. May also be expressed in $eV = 1.6 \times 10^{-19}\ J$

$f(v)dv$: # density of particles with velocity between v & $v+dv$

A: normalization constant:

$$n = \int_{-\infty}^{\infty} dv\, f(v) = A\int_{-\infty}^{\infty} dv\, e^{-\frac{1}{2}mv^2/kT} = A\sqrt{\pi \cdot \frac{2kT}{m}}$$

$$\Rightarrow \boxed{A = n\sqrt{\frac{m}{2\pi kT}}}$$ ← see Appendix / Warren's notes

So, what is the average kinetic energy in this distribution?
In a general case, suppose you have a quantity g(v). The average of this quantity is calculated by

$$\bar{g}(v) = \frac{\int dv\, g(v)\, f(v)}{\int f(v)\, dv} = \underbrace{\frac{\sum g(v_i) f(v_i)\, \Delta v_i}{\sum f(v_i)\, \Delta v_i} = \frac{\sum g(v_i) \cdot \#\text{ at } v_i}{\sum \#\text{ at } v_i}}_{\text{discrete averages}}$$

We are interested in kinetic energy, so

set $g(v) = \frac{1}{2} m v^2$,

$$\bar{g}(v) = \frac{A \int_{-\infty}^{\infty} dv \, \frac{1}{2} m v^2 \, e^{-\frac{1}{2} m v^2 / KT}}{A \int_{-\infty}^{\infty} dv \, e^{-\frac{1}{2} m v^2 / KT}} = E_{av}$$

Let $\alpha = \frac{1}{2} \frac{m}{KT}$

$$-\frac{d}{d\alpha} \int_{-\infty}^{\infty} dv \, e^{-\alpha v^2} = -\frac{d}{d\alpha} \sqrt{\frac{\pi}{\alpha}}$$

$$E_{av} = \frac{KT\alpha \overbrace{\int_{-\infty}^{\infty} dv \, v^2 e^{-\alpha v^2}}}{\underbrace{\int_{-\infty}^{\infty} dv \, e^{-\alpha v^2}}_{\sqrt{\pi/2}}}$$

$$\therefore E_{av} = \frac{KT\alpha \left[-\frac{d}{d\alpha} \sqrt{\frac{\pi}{\alpha}} \right]}{\sqrt{\frac{\pi}{2}}} = \frac{KT\alpha}{2} \cdot \frac{1}{\alpha} = \frac{KT}{2}$$

$$\therefore E_{av} \equiv \frac{KT}{2} = \frac{1}{2} m v_{th}^2$$ ← Define a v for average energy.

So now, $f(v) = e^{-v^2/2v_{th}^2}$, $v_{th} = \sqrt{\frac{KT}{m}}$

Note: F. Chen book uses $v_{th} = \sqrt{\frac{2KT}{m}}$, so that $f(v) \propto e^{-v^2/v_{th}^2}$. In many books & papers, symbol a_e is used for this quantity

$$\langle KE \rangle = \frac{1}{4} m a_e^2 \text{ or } \frac{1}{4} m V_{th}^2 \text{ (in Chen)}$$

$$= \frac{KT}{2}$$

What is unambiguous is that

$$\langle KE \rangle = \frac{1}{2} KT \text{ per degree of freedom}$$

in 3D,

$$V^2 \longrightarrow V_x^2 + V_y^2 + V_z^2$$

$$\langle KE \rangle = \frac{3}{2} KT \quad \text{(see Chen's book)}$$

In plasma physics, the temperature of the plasma is very often stated in terms of average energy in plasma since K, the Boltzman constant, is just a number. e.g. plasma with 1eV temperature:

$$\left.\begin{array}{l} 1eV = 1.6 \times 10^{-19}\ J = \underbrace{KT} \\ K = 1.38 \times 10^{-23}\ J/{}^\circ K \end{array}\right\} \Rightarrow T_{eV} = \frac{1.6 \times 10^{-19}\ J}{1.38 \times 10^{-23}\ J}\ {}^\circ K = 11{,}600\,{}^\circ K$$

linear relation: one to one mapping between average energy & plasma temperature.

This derivation assumes an <u>isotopic</u> plasma (I.e., Tx=Ty=Tz). The case of <u>anisotropic</u> plasma, where this is not the case, is an important topic in plasma physics, particularly in laser driven plasma, where the direction of polarization breaks the symmetry of plasma heating. For the time being, let's keep our focus on an isotopic plasma.

<u>Debye Shielding</u>

The length scale derived from the thermal velocity of plasma is called the Debye length and is a very important length scale in plasma physics as we will see. This length scale appears when we study the distance over which the plasma can shield out a DC (or low frequency) electric field.

First, recall that in presence of a potential, the distribution function changes as follows:

$$f(v) = e^{-(KE + PE)/kT}$$

in equilibrium

potential energy due to electric field:

$$W = q\phi \quad , \quad \text{where} \quad E = -\nabla\phi \quad \text{in electrostatics}$$

scalar potential

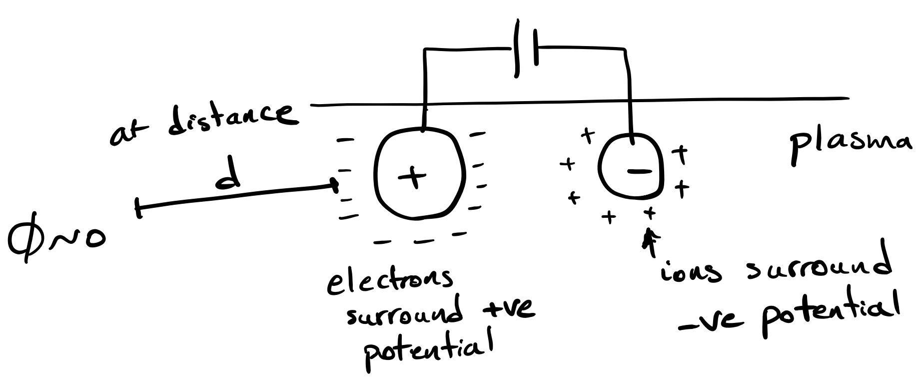
Note: in electrodynamics, we often use the symbol "V" to represent the scalar potential. In plasma physics, the scalar potential is most commonly represented by ϕ.

$$\text{so,} \quad n = \int dv\, f(v) = n_0 e^{e\phi/kT} \quad \text{electrons}$$

$$n_0 e^{-e\phi/kT} \quad \text{ions}$$

n_0 = density where $\phi = 0$ at ∞

Consider what happens physically if we put a source of potential, i.e. a charge in plasma. Since the ions and electrons are both mobile in plasma, they flow to and surround the source of the potential, such that at some distance "d" away, the potential of the source is no longer observed. This phenomenon is referred to as Debye shielding in plasma and the distance is called Debye length. Let's work out what this length is.



<u>Note</u>: we are interested in steady state solution (i.e. after the equilibrium was reached

Poisson's equation:

$$-\epsilon_0 \nabla^2 \phi = \rho = \rho_i - \rho_e$$ ← Boundary condion: $\phi = \phi_0$ at point $\vec{r} = \vec{r_0}$

$$= e(n_i - n_e)$$ ← assume singly ionized plasma

$$= e\left(n_0 e^{-e\phi/kT_i} - n_0 e^{+e\phi/kT_e}\right)$$

$$\Rightarrow \quad \nabla^2 \phi = \frac{e n_0}{\epsilon_0}\left(e^{e\phi/kT_e} - e^{-e\phi/kT_i}\right)$$

This is a nonlinear differential equations. The exact solution can be found in 1D if Te=Ti. To simplify, we are going to look for solutions where the potential is small, i.e. we are sufficiently far away from the electrodes. This will allow us to Taylor expand and only keep a small number of terms:

$$\frac{e\phi}{kT_e}, \ \frac{e\phi}{kT_i} \ll 1$$

$$\nabla^2 \phi = \frac{e n_0}{\epsilon_0}\left(\left[1 + \frac{e\phi}{kT_e} + \cdots\right] - \left[1 - \frac{e\phi}{kT_i} + \cdots\right]\right)$$

$$\approx \frac{e n_0}{\epsilon_0}\left(\frac{e\phi}{kT_e} + \frac{e\phi}{kT_i}\right)$$

$$\cong \frac{e^2 n_0}{\epsilon_0 k T_e}\left(1 + \frac{T_e}{T_i}\right)\phi$$

$$\frac{e^2 n_0}{\epsilon_0 k T_e} \equiv \frac{1}{\lambda_D^2} = \frac{e^2 n_0}{\epsilon_0 k T_e}\frac{m}{m} = \frac{\omega_p^2}{v_{th}^2}$$

$$\lambda_D \equiv \text{Debye Length} = \frac{v_{th}}{\omega_p} = \frac{1}{k_D}$$

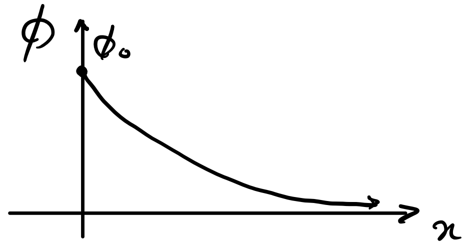
first studdied this
for electrolytes

Poisson equation becomes

$$\nabla^2\phi - \frac{1}{\lambda_D^2}\left(1 + \frac{T_e}{T_i}\right)\phi = 0 \quad \cdots \quad ⑥$$

for $T_e = T_i = T$ in 1-D $\left(\nabla^2\phi = \frac{d^2}{dx^2}\phi\right)$

$$\phi = \phi_0 \, e^{-|x|\sqrt{2}/\lambda_D}$$



Engineering formula:

$$\lambda_D = 69 \left(\frac{T\,[°K]}{n\,[m^{-3}]}\right)^{1/2} m$$

$$= 7430 \left(\frac{kT\,[eV]}{n\,[m^{-3}]}\right) m$$

$\lambda_D \uparrow$ as $T \uparrow$: higher T means particles can move away from potential

$\downarrow$ as $n \uparrow$: more e^- to shield out potential

Quasi-neutrality

If the size of the plasma (L) is large compared to the Debye length, the plasma can be considered quasi-neutral, i.e.

$$n_i \sim n_e \sim n$$

Where n is the common density called the plasma density. This is because any potentials that arise due to a charge imbalance (e.g. fluctuations in density due to tempreture) are shielded out over a distance that is short compared to plasma, leaving the bulk of the plasma free of large electric potential and fields.

Note: it takes only a small charge imbalance to result in potentials on the order of KT/e. The plasma therefore is quasi-neutral, meaning that

Another perspective on the Debye length is that it describes the distance over which the local variations in potential (e.g. due to charge density fluctuations) are shielded out. Therefore, the density on ions and electrons are equal in bulk of the plasma, but there are small regions of electromagnetic activity, or as Frank Chen puts it “not so neutral that all the interesting electromagnetic forces vanish!”

Note: quasi-neutrality is often considered a basic requirement for plasma, i.e.

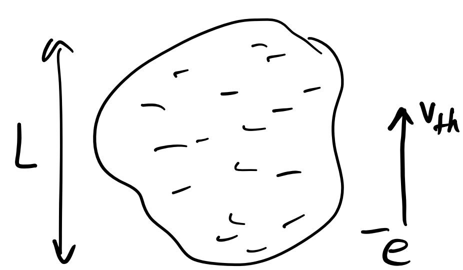
$$\boxed{\begin{gathered} n_e \sim n_i \sim n \\ L \gg \lambda_D \end{gathered}}$$

The physical picture of the relation between λ_D & ω_p

Suppose electrons are bounded on a size “L” and consider a typical electron which moves past the plasma at thermal velocity v_{th}.

Time it takes the electron to move past the plasma

$$\Delta t \sim \frac{L}{v_{th}}$$



But, the plasma will shield out or smear out the fluctuation on a time scale of ω_p^{-1}

$\therefore$ if $\frac{L}{V_{th}} = \Delta t > \omega_p^{-1}$ the plasma shields out the e^-

if $\frac{L}{V_{th}} = \Delta t < \omega_p^{-1}$ the plasma can't shield it out

critical distance is $L_c = \frac{V_{th}}{\omega_p} = \lambda_D$

same value as the rigorous analytical solution.

Note: this treatment assumes that there are "enough" electrons in the Debye length so that they actually can shield out a potential. In other words, a built in assumption in this treatment is that the number of particles in Debye length are:

$$N_D = \frac{4}{3}\pi n \lambda_D^3 \gg 1$$

$$= 1.38 \times 10^6 \frac{(T\,[°K])^{3/2}}{(n\,[m^{-3}])^{1/2}}$$

$$4\pi n \lambda_D^3 \equiv \Lambda : \text{plasma parameter}$$

In a more advanced treatment of Debye shielding, one can consider how the fields from one extra charge are screened out by the many charges. To do so, one can use the charge density for a single charge as a source term in Equation 6:

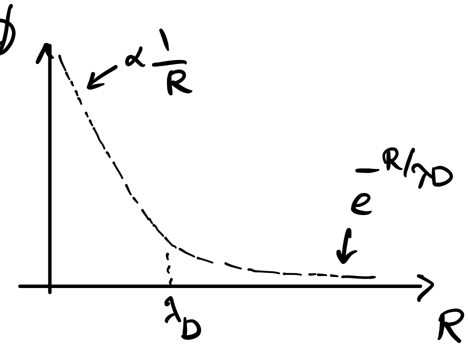
$$\nabla^2 \phi - \frac{1}{\lambda_D^2}\phi = \frac{1}{\epsilon_0} q\,\delta(\vec{x} - \vec{x}')$$

This equation assumes cold ions, $T_i \approx 0$ $\quad e^{-e\phi/kT} \to e^{-\infty} \to 0$

Solution:

$$\phi = \frac{q}{4\pi\epsilon_0 R} \underbrace{e^{-R/\lambda_D}}_{\text{Debye screening}}$$

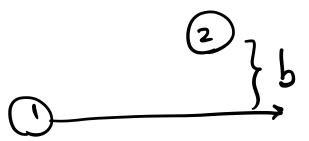
$$|\vec{x} - \vec{x}'| = R$$



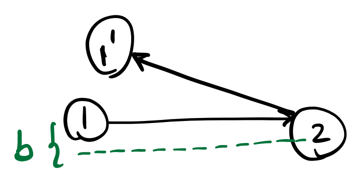
Collisions

We close this introduction by a discussion of collisions. This course is primarily concerned with collisionless plasma, but to understand what collisionless means, we first need to define what we mean by collisions.

Neutral atoms: two neutral particles collide is one passes within the radius of the other:



don't collide



collide

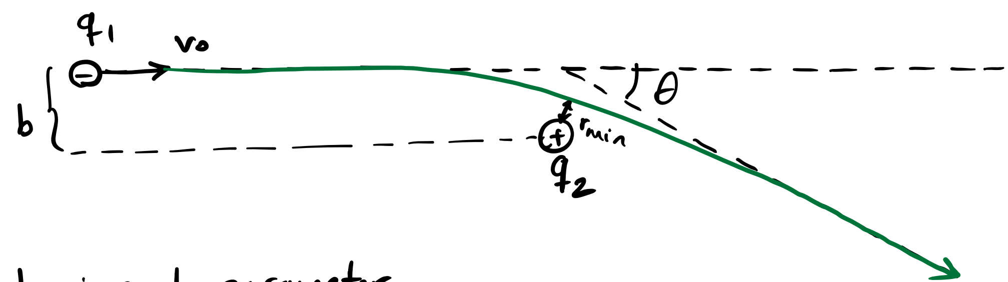
If $b < r_0$, where r_0 is the radius of particle ②, then there is a collision, like billiard balls

Each particle assumed as a sphere has a cross section $\sigma = \pi r_0^2$

same symbol, but not conductivity or charge density

Charged particles: in contrast to neutral particles, charge particles exert the Lorentz force on each other even when they are far from each other. So the cross section of collision is no longer a simple function of the size of the particle.

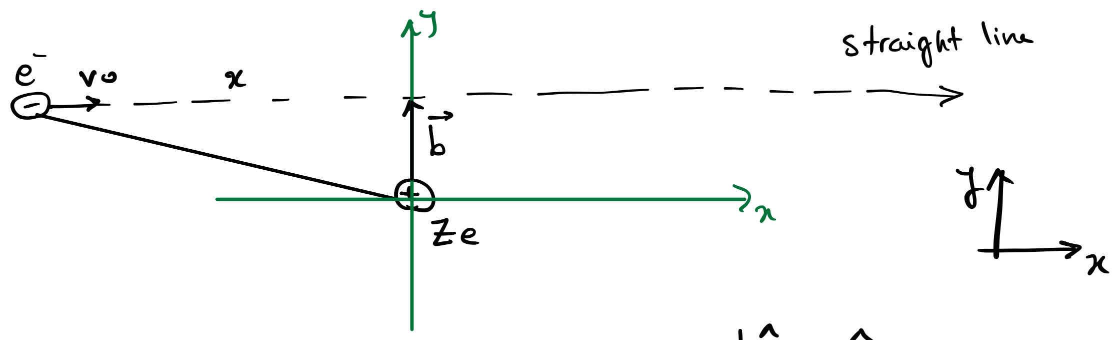
e.g. collision between charges with opposite sign in the center of mass frame of the positive charge)



$b \equiv$ impact parameter

It is possible to derive an exact formula for θ in terms of b & v_0

However, it is possible to get an approximate answer with much less algebra by simplifying the problem. We are going to look for solutions where θ is small (small angle scattering). If the angle is small, we can make the assumption that to the zero order, the trajectory is a straight line



$$\text{Coulomb force} = \frac{1}{4\pi\epsilon_0} \frac{-e \cdot Ze}{b^2 + x^2} \cdot \frac{b\hat{y} + x\hat{x}}{\sqrt{x^2 + b^2}}$$

$b \simeq$ Constant, $\quad x$ goes from $-\infty$ to ∞

$$\frac{d}{dt} V_x = \frac{F_x}{m} = -\frac{1}{m} \frac{Ze^2}{4\pi\epsilon_0} \cdot \frac{x}{(x^2+b^2)^{3/2}}$$

$$\Rightarrow \Delta V_x = \frac{-Ze^2}{4\pi\epsilon_0 m} \int_{-\infty}^{\infty} \frac{x}{(x^2+b^2)^{3/2}} dt$$

$V_x \approx V_0 \Rightarrow dx = V_0 dt$ (The two charges weakly interact)

$$\therefore \Delta V_x = -\frac{Ze^2}{4\pi\epsilon_0 m V_0} \int_{-\infty}^{\infty} \frac{x\,dx}{(x^2+b^2)^{3/2}} = 0 \quad \text{(odd function)}$$

as expected based on the assumptions

$$\frac{dv_y}{dt} = \frac{F_y}{m} = \frac{1}{m} \frac{1}{4\pi\epsilon_0} \frac{-e \cdot Zeb}{(b^2+x^2)^{3/2}}$$

$$\Rightarrow \Delta v_y = \frac{-Ze^2}{4\pi\epsilon_0 m V_0} \int_{-\infty}^{\infty} \frac{b}{(b^2+x^2)^{3/2}} dx$$

integral $= \dfrac{x}{b\sqrt{b^2+x^2}}$

$$= \frac{-Ze^2}{4\pi\epsilon_0 m V_0 b} \left[\frac{x}{\sqrt{b^2+x^2}} \right]_{-\infty}^{\infty}$$

$x \to \pm\infty \Rightarrow x^2+b^2 \approx x^2$

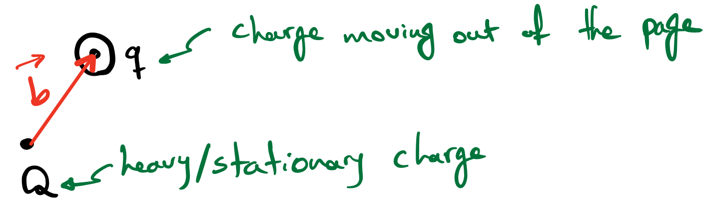
$$= \frac{-Ze^2}{4\pi\epsilon_0 m V_0 b} \left(1 - (-1)\right)$$

$$= \frac{2q_1 q_2}{4\pi\epsilon_0 m V_0 b}$$

$$\therefore \tan\theta \approx \theta = \frac{\Delta v_y}{v_o} = \frac{2 q_1 q_2}{4\pi\epsilon_o m V_o^2 b}$$

Exact answer: $\tan\frac{\theta}{2} = \frac{q_1 q_2}{4\pi\epsilon_o m V_o^2 b}$ ... same answer for small angle collisions!

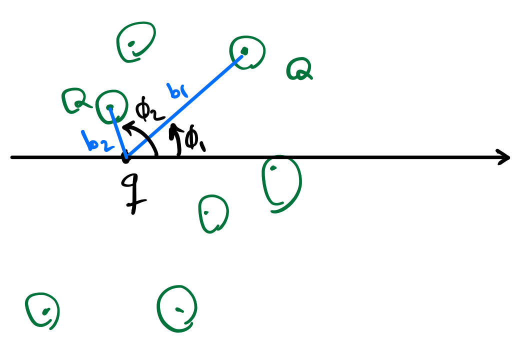
Since only one component of velocity changes, we can write the change in velocity in vector form. To do so, we define vector $\vec{b}$:



$$\Delta\vec{V} = V_o \tan\theta\, \hat{b} \sim V_o\theta\, \hat{b} = \frac{2Qq}{4\pi\epsilon_o m V_o^2} \frac{\vec{b}}{b^2} V_o$$

note: vector $\vec{b}$ point away from Q. If q & Q have opposite charges, $\Delta\vec{V}$ is towards Q & if the charges have the same sign, it would be away from Q.

This is the result for a single collision. We are interested in a collision frequency, so we need to consider the impact of multiple independent collisions. Consider collisions that occur during a time Δt for a single electron:

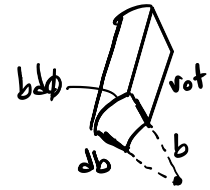


Use cylindrical coordinate system

charge q is moving into the paper with velocity V_o

all other charges are stationary

After a time Δt, the volume element at the impact parameter "b" is



Volume: $b\, d\phi\, db\, v_0 \Delta t$

number of ions: $n_i b\, db\, d\phi\, v_0 \Delta t$

deflection at (b, ϕ):

$$d\left[\Delta \vec{V}(b,\phi)\right] = \underbrace{\frac{2Qe}{4\pi\epsilon_0 m v_0^2}\,\frac{\hat{b}}{b}\,\hat{v_0}}_{\Delta v \text{ per scatterer}}\;\underbrace{n_i b\, db\, d\phi\, v_0 \Delta t}_{\text{number of scatterers}}$$

$$= \frac{2Qe\, n_i\, \hat{b}}{4\pi\epsilon_0 m}\Delta t\, db\, d\phi$$

If ion density is uniform, i.e. n_i does not depend on b & ϕ,

$$\therefore \Delta \vec{V}(b,\phi) = \frac{2Qe n_i}{4\pi\epsilon_0 m}\Delta t \int_{b_{min}}^{b_{max}} \int_0^{2\pi} \hat{b}\, db\, d\phi$$

$\cos\phi\, \hat{x} + \sin\phi\, \hat{y}$

both integrate to zero over 2π

$$\boxed{\Delta \vec{V}(b,\phi) = 0}$$

On average, there are the same number of ions on one side as the other.

In plasma, we have a population of electrons with a velocity distribution. What we are interested in is the impact of collisions on this distribution. So, consider "N" electrons with initial velocity $\vec{V_0}$. Because the collision of each electron is independent from the other, the average change in transverse momentum is expressed as

$$\overset{\text{average}}{\langle \Delta \vec{V} \rangle} = \frac{1}{N} \sum_{i=1}^{N} \Delta \vec{V_i} \quad = 0$$

One might suppose then that the collisions have no effect on the transverse momentum of electron population, but this is not the case. Consider the average change on momentum squared (the temperature)

$$\langle \Delta V^2 \rangle = \frac{1}{N} \sum_{i=1}^{N} \Delta V_i^2$$

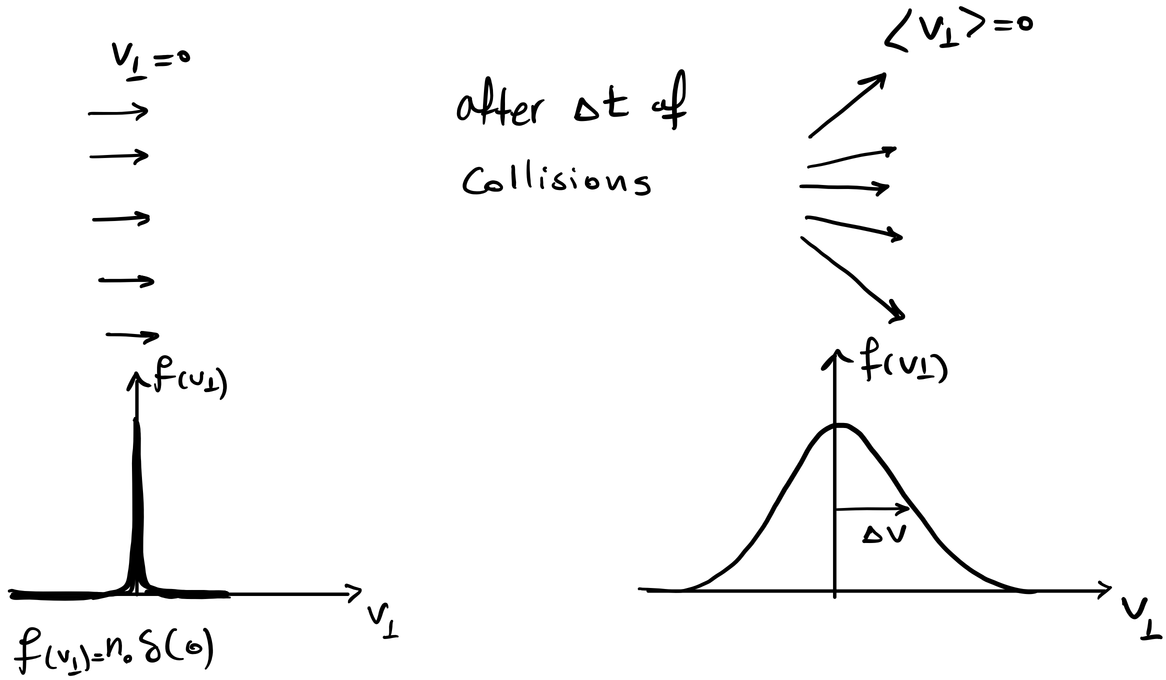
$$d\left[\Delta V^2(b,\phi)\right] = \underbrace{\left[\frac{2Qe}{4\pi \epsilon_0 m V_0^2} \frac{\hat{b}}{b} V_0\right]}_{\Delta v \text{ per scatterer}}^2 \underbrace{n_i \cdot b \, db \, d\phi \, V_0 \, \Delta t}_{\text{number of scatterers}}$$

$$= \frac{4}{(4\pi)^2} \left(\frac{Qe}{m\epsilon_0}\right)^2 \frac{n_i}{V_0} \cdot \frac{\overset{\hat{b}\cdot\hat{b}=1}{1}}{b} \, db \, d\phi \, \Delta t$$

$$\therefore \frac{\Delta V^2}{\Delta t} = \int_{b_{min}}^{b_{max}} \int_0^{2\pi} \frac{4}{(4\pi)^2} \left(\frac{Qe}{m\epsilon_0}\right)^2 \frac{n_i}{V_0} \cdot \frac{1}{b} \, db \, d\phi$$

$$\boxed{\frac{\Delta V^2}{\Delta t} = \frac{8\pi}{(4\pi\epsilon_0)^2} \frac{(Qe)^2}{m^2} \frac{n_i}{V_0} \ln \frac{b_{max}}{b_{min}}}$$

What is the physical interpretation of this conclusion? A stream of particles (e.g. electrons) moving through the plasma with the same initial velocity will have the same average velocity at the end, but with a larger momentum spread:



This is the hand-wavy description of this process. The proper way of doing this, which involves stochastic mathematics and the study of random walk process (gets to the same result!) is a topic for graduate school!

We can define a collision frequency based on this description by choosing the time interval when $\Delta V = V_0$ (i.e. 90 degree scattering due to many small collisions) as the collision time.

$$\frac{1}{\Delta t} = \nu_{ei} = \frac{8\pi}{(4\pi\epsilon_0)^2} \frac{Q^2 e^2 n_i}{m^2 V_0^3} \ln \frac{b_{max}}{b_{min}}$$

where $Q^2 = e^2$ is assumed (singly ionized ion)

$$= \frac{1}{2\pi} \frac{\omega_p^4}{n_i V_0^3} \ln\left(\frac{b_{max}}{b_{min}}\right)$$

$$\therefore \frac{\nu_{ei}}{\omega_p} = \frac{1}{2\pi} \frac{1}{n_i \lambda_D^3} \ln\left(\frac{b_{max}}{b_{min}}\right) \qquad \text{for } V_0 = V_{th}$$

$b_{max} = \lambda_D$: since Coulomb force is shielded out in this distance

b_{min} : One can define as the distance where particle is scattered at 90° in a single collision (Also, can be defined using uncertainty principle)

$$\tan\left[\frac{(\pi/2)}{2}\right] = \frac{Qe}{4\pi\epsilon_0 m V_{th}^2 b_{min}} \qquad \text{for } v_0 = v_{th}$$

$$\Rightarrow b_{min} = \underbrace{\frac{e^2 n}{4\pi\epsilon_0 m\, n}}_{1/\omega_p^2} \cdot \frac{1}{V_{th}^2} = \frac{1}{4\pi n \lambda_D^2}$$

$$\therefore \frac{b_{max}}{b_{min}} = 4\pi n \lambda_D^3 = \Lambda \quad (= 3N_D \text{ number of particles in a Debye sphere})$$

$$\therefore \frac{\nu_{ei}}{\omega_p} = \frac{1}{2\pi} \frac{1}{n_i \lambda_D^3} \ln\Lambda = \boxed{\frac{2}{\Lambda} \ln\Lambda}$$

The requirement for quasi-neutrality was

$$N_D \gg 1$$

$$\Rightarrow \Lambda \gg 1$$

$$\Rightarrow \frac{\nu_{ei}}{\omega_p} \ll 1$$

$\frac{1}{\Lambda}$ goes to zero faster than $\ln\Lambda \rightarrow \infty$ as $\Lambda \rightarrow \infty$

This last condition implies that a quasi-neutral plasma is also a "collisionless" one. This is the type of plasma that we will discuss in the rest of the class. To summarize, these conditions are

1. $\lambda_D \ll L_s$

2. $N_D \gg 1$

3. $\Lambda \gg 1$ & $\nu_{ei} \ll \omega_p$

Incidentally, when $\Lambda \ll 1 \Rightarrow \frac{\nu_{ei}}{\omega_p} \gg 1$, the plasma is called a strongly coupled plasma. In that case, the conductivity of plasma is

$$\sigma = \frac{\omega_p^2 \epsilon_0}{\nu_{drag}}$$

It turns out that a more detailed calculation gives

$$\nu_{drag} = \frac{1}{2} \nu_{ei} \quad (v_0 \cong 2 v_{th})$$

$$\therefore \sigma_{plasma} = \omega_p^2 \epsilon_0 \frac{32\pi \, n \lambda_D^3}{(\ln \Lambda) \cdot \omega_p}$$

$$\boxed{\sigma_{plasma} = \omega_p \epsilon_0 \, 32 \frac{n_e \lambda_D^3}{\ln \Lambda}}$$

$$= \frac{2 \times 10^4 \, (T[eV])^{3/2}}{\underbrace{\ln \Lambda}_{\text{weak function of } n}}$$

≈ independent of density!

$\ln \Lambda \approx 10\text{-}20$ for a wide range of parameters

Appendix 1: current densities

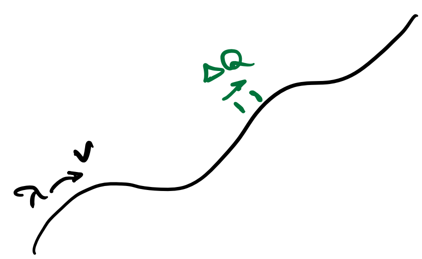
Current = charge per unit time passing a point

$$I = \frac{\partial Q(\vec{r},t)}{\partial t}$$

units $\quad \overset{\text{Ampere}}{1A} = 1\,C/s \leftarrow$ Coulomb per second

If you know what charge you have & how fast it is moving, then charge passing a point can be related to charge & its velocity:

for a line charge (e.g. a wire)



$$\Delta Q = \lambda\,\Delta L$$

$$\Delta Q = \lambda\, v\, \Delta t$$

$$\Rightarrow \quad I = \frac{\Delta Q}{\Delta t} = \lambda v$$

Note: current is actually a vector pointing in the direction of charge flow:

$$\vec{I} = \lambda \vec{v}$$

$\therefore$ Force on a wire carrying charge is

$$\vec{F}_{mag} = \int dq\,(\vec{v}\times\vec{B})$$

$$= \int \lambda\, dL\,(\vec{v}\times\vec{B})$$

$$= \int dl \, (\vec{I} \times \vec{B})$$

Since current points along dL

$$\boxed{F_{mag} = \int I \, (d\vec{L} \times \vec{B})} \quad (5.16)$$

in Griffiths

In many problems including circuitry, we are concerned about a current-carrying wire, but what about moving surface charge or volume charge?

<u>Surface current density:</u>
For moving charge over a surface, and in analogy with line current, we define the surface current density as

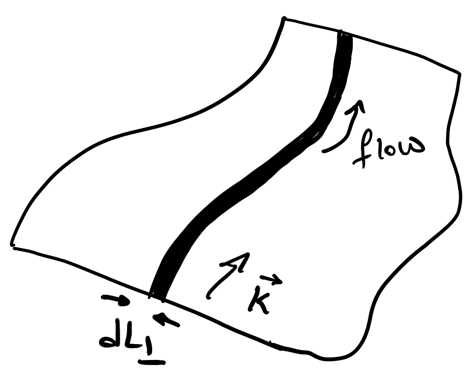
$$\vec{K} = \sigma \vec{v}$$

Look at units of $\vec{K} = \sigma \vec{v}$

$$[C/m^2][m/s] = [C/s]\left[\frac{1}{m}\right]$$

σ is charge divided 2 perpedicular length segments, one of which is compensated by $\vec{v}$. In other words, $\vec{K}$ is current per unit length in the direction perpendicular to $\vec{v}$:

$$\vec{K} = \frac{d\vec{I}}{dL_{\perp}}$$



In other words, **K** is current per unit width.

In general, **K** varies from point to point over the surface, reflecting variations in σ and/or $\vec{v}$.

The magnetic force on the surface current is

$$\vec{F}_{mag} = \int dq\,(\vec{v} \times \vec{B})$$

$$= \int da\ \sigma\,(\vec{v} \times \vec{B})$$

$$\vec{F}_{mag} = \int (\vec{K} \times \vec{B})\ da \qquad (5.24)$$

Volume current density:

This is the general formulation for current in three dimensional space and is frequently used where charge can freely move in space, such as modeling intergalactic plasma. Formulation is the same as the other two types:

$$\vec{J} = \rho\,\vec{v}$$

Again, looking at units of $\vec{J}$ we see

$$[J] = [C/m^3][m/s] = [C/s]\left[\frac{1}{m^2}\right]$$

$\vec{J}$ is current per area, in particular, it is current per unit area perpendicular to direction of flow (the length in direction of flow is accounted for by $\vec{v}$)

$$\vec{J} = \frac{d\vec{I}}{da_\perp}$$

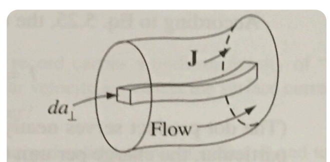


FIGURE 5.14

Similarly, $\vec{F}_{mag} = \int dq\,(\vec{v} \times \vec{B})$

$$= \int d\tau\,(\rho\,\vec{v} \times \vec{B})$$

$$\vec{F}_{mag} = \int d\tau\,(\vec{J} \times \vec{B})$$

One can see that if there is motion along the direction of $\vec{J} \times \vec{B}$, work will be done on the charge particle. Indeed, this term is an important source of heating in plasma physics.

Appendix 2: solving for the Gaussian normalization factor

$$f(v) = A e^{-\frac{1}{2}mv^2/kT}$$

$$n = \int_{-\infty}^{\infty} dv\, f(v) = A\int_{-\infty}^{\infty} dv\, e^{-\frac{1}{2}mv^2/kT}$$

v is a "dummy" variable of a definite integral & can be changed with any symbol, say x or y. Specifically,

$$n^2 = A^2 \int_{-\infty}^{\infty} \underbrace{dv}_{\text{call this } x}\, e^{-\frac{1}{2}mv^2/kT} \int_{-\infty}^{\infty} \underbrace{dv_2}_{\text{call this } y}\, e^{-\frac{1}{2}mv_2^2/kT}$$

$$= A^2 \int_{-\infty}^{\infty} dx\, e^{-\frac{1}{2}mx^2/kT} \int_{-\infty}^{\infty} dy\, e^{-\frac{1}{2}my^2/kT}$$

$$n^2 = A^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\frac{m}{2kT}(x^2+y^2)}\, dx\, dy$$

equivalent polar representation

$x^2+y^2 = r^2$

$dx\,dy = r\,d\phi\,dr$

$r: 0 \to \infty$, $\phi: 0 \to 2\pi$

$$= A^2 \int_0^{2\pi}\int_0^{\infty} e^{-mr^2/2kT}\, r\,dr\, d\phi$$

$$= 2\pi A^2 \left[-\frac{kT}{m}\right] e^{-mr^2/2kT}\Big|_0^{\infty}$$

$\infty \to e^{-\infty} \to 0$

$0 \to e^0 \to 1$

$$n^2 = \frac{2\pi kT}{m} A^2 \Rightarrow \boxed{A = n\sqrt{\frac{m}{2\pi kT}}}$$