PHY 564
Advanced Accelerator Physics
Lectures 25

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Introduction to Free Electron Lasers

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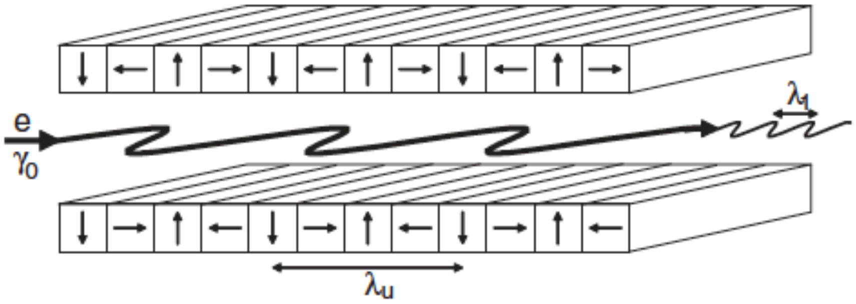
Outline

- Introduction
- Electrons’ trajectory and resonant condition
- Analysis of FEL process at small gain regime (Oscillator)
- Analysis of FEL process at high gain regime (Amplifier)

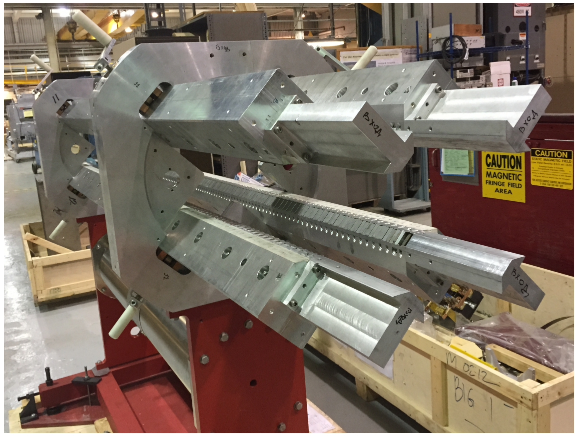
Introduction I: Basic Setup

Planar undulator

$$B_y(x,y,z) = B_0 \sin(k_u z)$$
$$\text{for } x,y << \text{gap size}$$

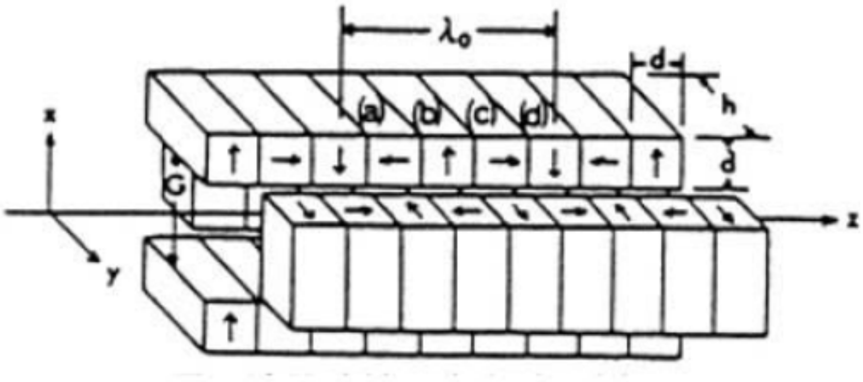


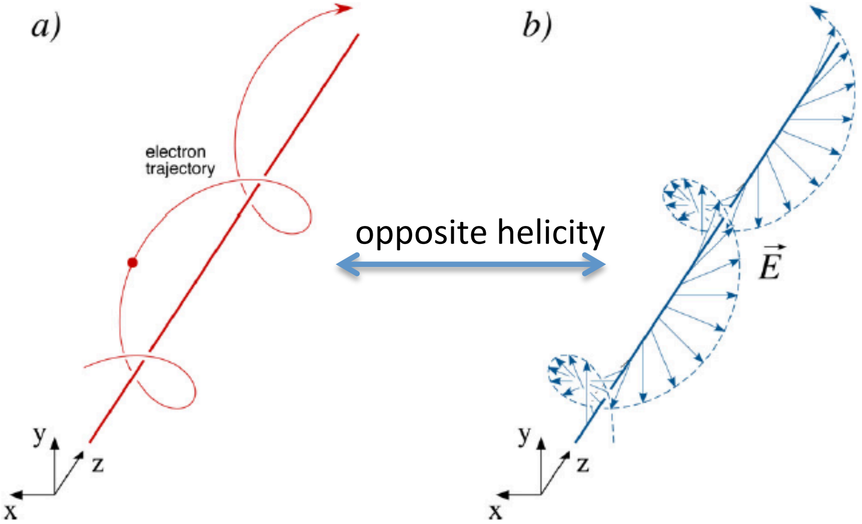
Helical wiggler for CeC PoP



Helical undulator

$$B_x(x,y,z) = B_0 \cos(k_u z)$$
$$B_y(x,y,z) = B_0 \sin(k_u z)$$
$$\text{for } x,y << \text{gap size}$$

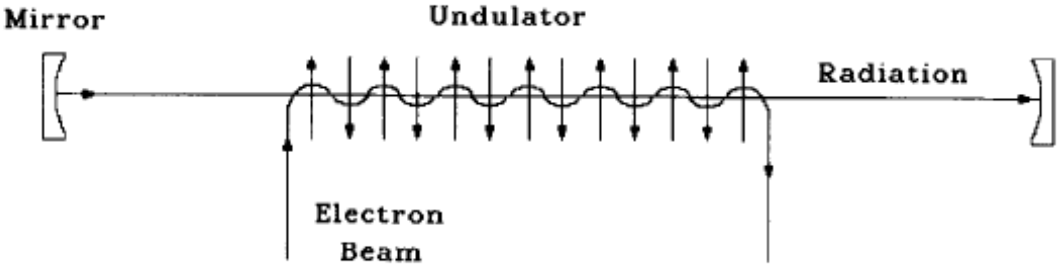




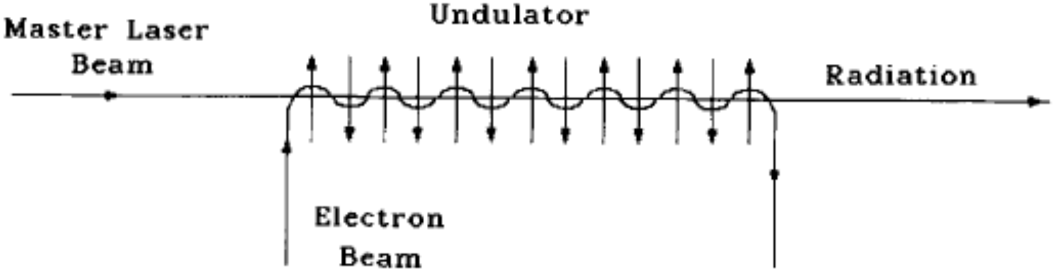
opposite helicity

Introduction II: different types of FEL

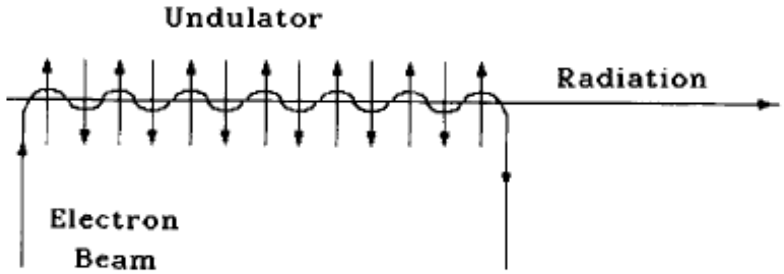
FEL Oscillator
(Low gain regime)



FEL Amplifier
(High gain regime)



SASE FEL
(High gain regime)



Self-Amplified Spontaneous Emittion (SASE)

Unperturbed Electron motion in helical wiggler (in the absence of radiation field)

$$\vec{B}_w(x,y,z)=B_w\left[\cos(k_u z)\hat{x}-\sin(k_u z)\hat{y}\right]$$

$$\vec{F}(x,y,z)=-e\vec{v}\times\vec{B}=-ev_z\hat{z}\times\vec{B}=-ev_z B_w\left[\cos(k_u z)\hat{y}+\sin(k_u z)\hat{x}\right]$$

$$\frac{d(m\gamma v_x)}{dt}=m\gamma\frac{dv_x}{dt}=-ev_z B_w\sin(k_u z) \qquad \frac{d(m\gamma v_y)}{dt}=m\gamma\frac{dv_y}{dt}=-ev_z B_w\cos(k_u z)$$

$$\gamma=\frac{1}{\sqrt{1-v^2/c^2}} \qquad v=\sqrt{v_x^2+v_y^2+v_z^2} \qquad \tilde{v}\equiv v_x+iv_y$$

$$m\gamma\frac{d\tilde{v}}{dt}=-iev_z B_w\left(\cos(k_u z)-i\sin(k_u z)\right)=-iev_z B_w e^{-ik_u z}$$

$$m\gamma\frac{d\tilde{v}}{dz}=m\gamma\frac{dz}{dt}\frac{d\tilde{v}}{dt}=-iev_z B_w e^{-ik_u z}\Rightarrow m\gamma\frac{d\tilde{v}}{dz}=-ieB_w e^{-ik_u z}$$

Undulator parameter, also called a_w

$$K\equiv\frac{eB_w\lambda_w}{2\pi mc}$$

Electron rotation angle in undulator:

$$\theta_s=K/\gamma$$

$$\frac{\tilde{v}(z)}{c}=\frac{-ieB_w}{mc\gamma}\int e^{-ik_u z_1}dz_1=\frac{eB_w}{mc\gamma k_u}e^{-ik_u z_1}=\frac{K}{\gamma}e^{-ik_u z_1}$$

Assume the initial velocity of the electron make the integral constant vanishing.

$$\vec{v}_\perp(z)=\frac{cK}{\gamma}\left[\cos(k_u z)\hat{x}-\sin(k_u z)\hat{y}\right] \qquad v_z=const. \qquad \vec{x}(z)=\int_0^z\vec{v}(t_1)dt_1+\vec{x}(z=0)$$

Energy change of electrons due to radiation field

$$\vec{v}_\perp(z) = \frac{cK}{\gamma}\left[\cos(k_u z)\hat{x} - \sin(k_u z)\hat{y}\right]$$

Consider a circularly polarized electromagnetic wave (plane wave is an assumption for 1D analysis, which is usually valid for near axis analysis) propogating along z direction

$$\vec{E}_\perp(z,t) = E\left[\cos(kz-\omega t)\hat{x} + \sin(kz-\omega t)\hat{y}\right] \qquad E_z = 0$$

$$= E\left[\cos(k(z-ct))\hat{x} + \sin(k(z-ct))\hat{y}\right] \qquad \omega = kc$$

Energy change of an electron is given by

$$\frac{d\mathcal{E}}{dt} = \vec{F}\cdot\vec{v} = -e\vec{v}_\perp \cdot \vec{E}_\perp$$

$$\frac{d\mathcal{E}}{dz} = -eE\theta_s \frac{c}{v_z}\cos(\psi) \approx -eE\theta_s \cos(\psi)$$

Pondermotive phase:

$$\psi = k_u z + k(z-ct)$$

To the leading order, electrons move with constant velocity and hence $z = v_z(t-t_0)$

Resonant Radiation Wavelength

$$\frac{d\mathcal{E}}{dz} = -eE\theta_s \cos\left[\left(k_w + k - k\frac{c}{v_z}\right)z + \psi_0\right]$$

We define the resonant radiation wavelength such that

$$k_w + k_0 - k_0\frac{c}{v_z} = 0 \Rightarrow \lambda_0 = \lambda_w\left(\frac{c}{v_z} - 1\right) \approx \frac{\lambda_w}{2\gamma_z^2}$$

$$\gamma_z^{-2} \equiv 1 - v_z^2/c^2 = 1 - \left(v_z^2 + v_\perp^2\right)/c^2 + v_\perp^2/c^2 = \gamma^{-2} + \theta_s^2 = \gamma^{-2}\left(1 + K^2\right)$$

FEL resonant frequency:

$$\boxed{\lambda_0 \approx \frac{\lambda_w\left(1 + K^2\right)}{2\gamma^2}}$$

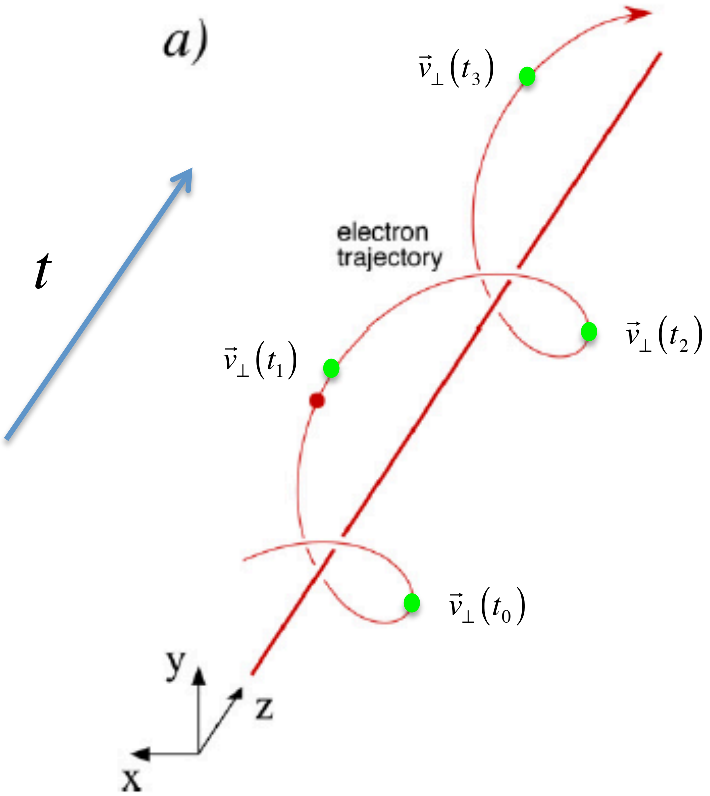
$$K \equiv \frac{eB_w\lambda_w}{2\pi mc}$$

At resonant frequency, the rotation of the electron and the radiation field is synchronized in the x-y plane and hence the energy exchange between them is most efficient.

Helicity of radiation at synchronization

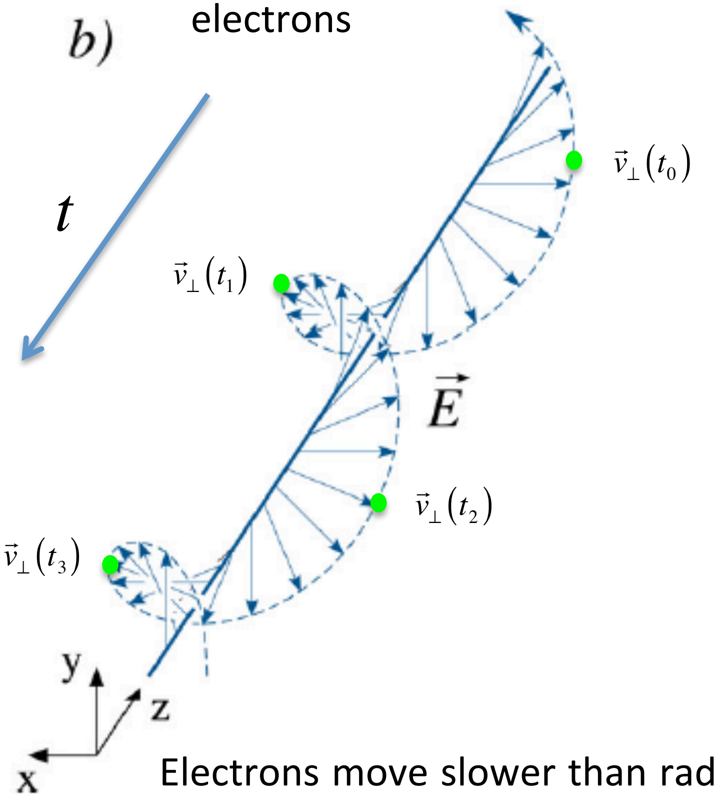
The synchronization requires opposite helicity of radiation with respect to the electrons' trajectories.

Electrons' trajectories



$$t_0 < t_1 < t_2 < t_3$$

Radiation field observed by electrons



Electrons move slower than radiation and hence see the radiation wave slipping ahead. As a result, the rotation direction of the radiation field seen by an electron is the same as its own rotation direction.

Longitudinal equation of motion

In the presence of the radiation field, the longitudinal equation of motion of an electron read

$$\frac{d\mathcal{E}}{dz} = -eE\theta_s \cos(\psi) \qquad \psi = k_w z + k(z - ct)$$

$\mathcal{E}_0$ is the average energy of the beam.

$$\frac{d}{dz}\psi = k_w + k - \frac{\omega}{v_z(\mathcal{E})}$$

$$\approx k_w + k - \omega\left[\frac{1}{v_z(\mathcal{E}_0)} + (\mathcal{E} - \mathcal{E}_0)\frac{d}{d\mathcal{E}}\frac{1}{v_z}\right] \Leftarrow$$

$$\approx k_w + k - \frac{\omega}{v_z(\mathcal{E}_0)} + \frac{\omega}{\gamma_z^2 c}\frac{(\mathcal{E} - \mathcal{E}_0)}{\mathcal{E}_0}$$

$$\frac{d}{d\mathcal{E}}\frac{1}{v_z} = \frac{1}{mc^3}\frac{d}{d\gamma}\frac{1}{\beta_z} = \frac{1}{mc^3}\frac{d\gamma_z}{d\gamma}\frac{d}{d\gamma_z}\frac{1}{\beta_z}$$

$$\gamma_z^2 = \frac{\gamma^2}{(1 + K^2)} \qquad \frac{d\gamma_z}{d\gamma} = \frac{\gamma}{\gamma_z(1 + K^2)}$$

$$\frac{d}{d\gamma_z}\frac{1}{\beta_z} = -\frac{1}{2\beta_z^3}\frac{d}{d\gamma_z}\left(1 - \frac{1}{\gamma_z^2}\right) = -\frac{1}{\beta_z^3\gamma_z^3}$$

$$\Rightarrow \begin{cases} \dfrac{dP}{dz} = -eE\theta_s \cos(\psi) \\ \dfrac{d}{dz}\psi \approx C + \dfrac{\omega}{\gamma_z^2 c\mathcal{E}_0} P \end{cases}$$

Energy deviation: $P \equiv \mathcal{E} - \mathcal{E}_0$

Detuning parameter: $C \equiv k_w + k - \dfrac{\omega}{v_z(\mathcal{E}_0)}$

Low Gain Regime: Pendulum Equation

$$\left.\begin{aligned}\frac{dP}{dz}&=-eE\theta_s\cos(\psi)\\ \frac{d}{dz}\psi&=C+\frac{\omega}{\gamma_z^2 c\mathcal{E}_0}P\end{aligned}\right\}\Rightarrow \qquad \frac{d^2}{dz^2}\psi+\frac{eE\theta_s\omega}{\gamma_z^2 c\mathcal{E}_0}\cos(\psi)=0$$

We assume that the change of the amplitude of the radiation field, E, is negligible and treat it as a constant over the whole interaction.

$$\frac{d^2}{d\hat{z}^2}\psi+\hat{u}\cos(\psi)=0 \qquad \hat{u}=\frac{l_w^2 eE\theta_s\omega}{\gamma_z^2 c\mathcal{E}_0} \qquad \hat{z}=\frac{z}{l_w}$$

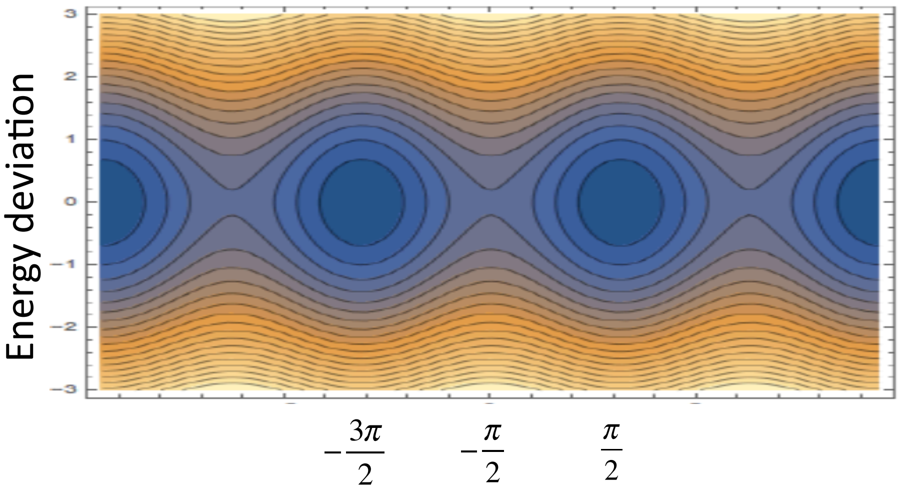
Pendulum equation:

$$\boxed{\frac{d^2}{d\hat{z}^2}\left(\psi+\frac{\pi}{2}\right)+\hat{u}\sin\left(\psi+\frac{\pi}{2}\right)=0}$$

Low Gain Regime: Similarity to Synchrotron Oscillation

FEL

ψ is the angle between the transverse velocity vector and the radiation field vector and hence there is no energy kick for $\psi = \pi / 2$



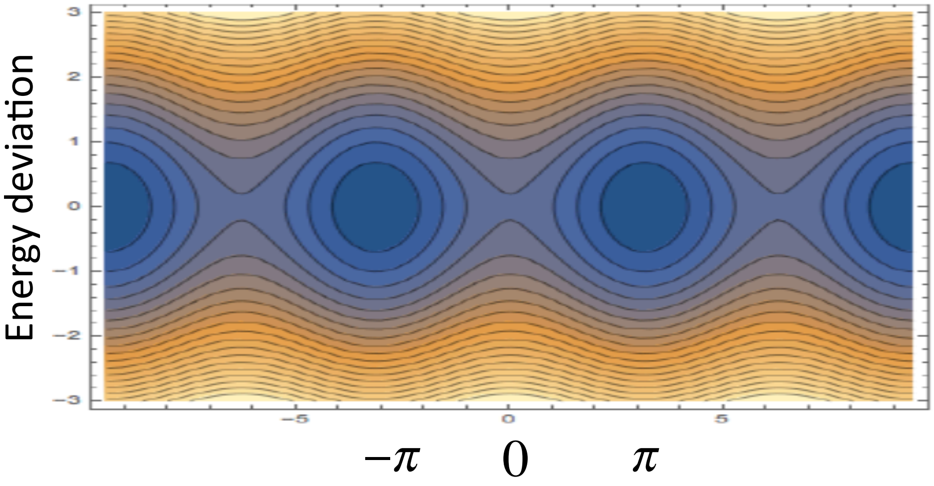
Pondermotive phase, ψ

$$\frac{d^2}{d\hat{z}^2}\left(\psi + \frac{\pi}{2}\right) + \hat{u}\sin\left(\psi + \frac{\pi}{2}\right) = 0$$

$$\hat{u} = \frac{l_w^2 eE\theta_s\omega}{\gamma_z^2 c\mathcal{E}_0} \qquad \psi = k_u z + k(z - ct)$$

Synchrotron Oscillation

$$\frac{d\tau}{ds} = \eta_\tau \pi_\tau; \quad \frac{d\pi_\tau}{ds} = \frac{1}{C}\frac{eV_{RF}}{p_o c}\sin\left(k_o h_{rf}\tau\right);$$

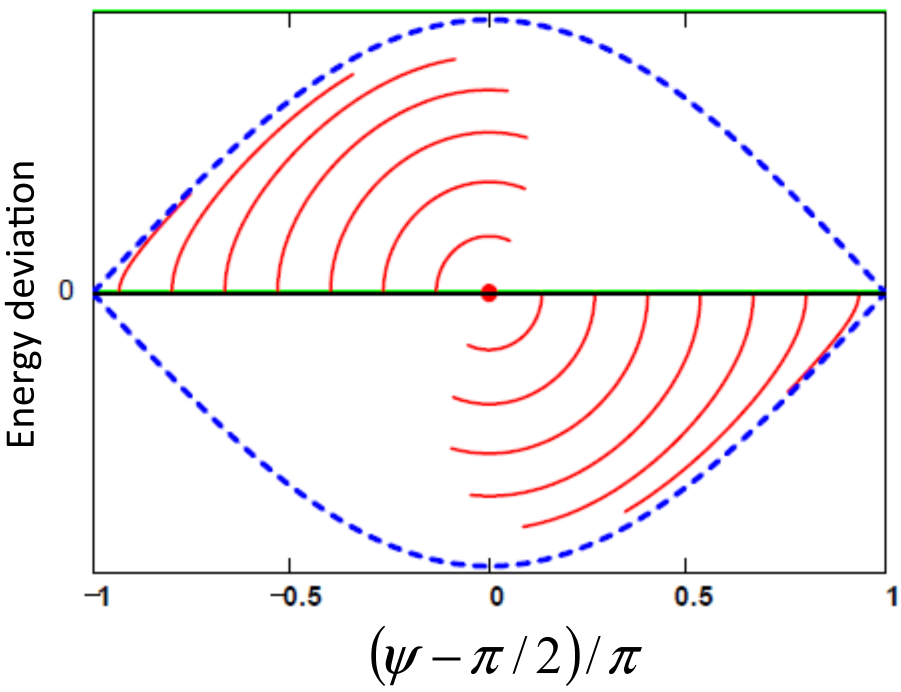


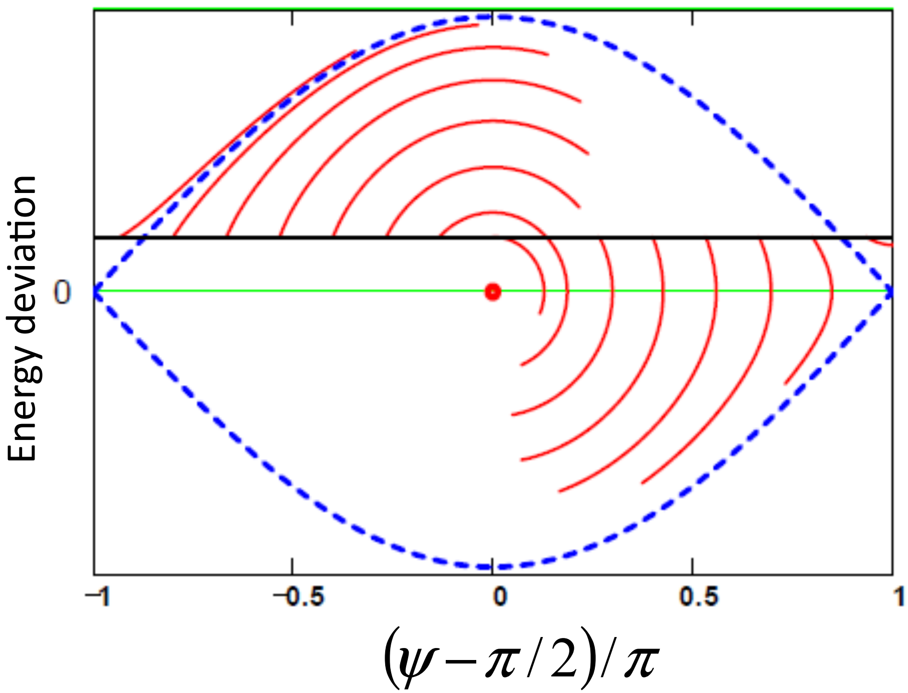
RF phase, ϕ_{rf}

$$\frac{d^2\phi_{rf}}{ds^2} = u_{rf}\sin\phi_{rf}$$

$$u_{rf} = \eta\frac{1}{C}\frac{eV_{RF}k_0 h_{rf}}{p_0 c} \qquad \phi_{rf} = k_0 h_{rf}\tau$$

Low Gain Regime: Qualitative Observation





The average energy of the electrons is right at resonant energy:

$$\lambda_0 \approx \frac{\lambda_w\left(1+K^2\right)}{2\gamma^2} \quad \Rightarrow \gamma = \gamma_0 = \sqrt{\frac{\lambda_w\left(1+K^2\right)}{2\lambda_0}}$$

*Plots are taken from talk slides by Peter Schmuser.

The average energy of the electrons is slightly above the resonant energy:

$$\gamma = \gamma_0 + \Delta\gamma$$

With positive detuning, there is net energy loss by electrons.

Low Gain Regime: Derivation of FEL Gain

Change in radiation power density (energy gain per seconds per unit area):

$$\Delta\Pi_r = c\varepsilon_0\left(E_{ext}+\Delta E\right)^2 - c\varepsilon_0 E_{ext}^2 \approx 2c\varepsilon_0 E_{ext}\Delta E$$

Average change rate in electrons' energy per unit beam area:

$$\Delta\Pi_e = \frac{j_0\langle P\rangle}{e}$$

*The average, <…>, is over all electrons in the beam.

Energy deviation at entrance

Pondermotive phase at entrance

$$\langle P(z)\rangle = \int_{-\infty}^{\infty} dP_0 \int_0^{2\pi} d\psi_0 f\left(P_0,\psi_0\right) P\left(P_0,\psi_0,z\right)$$

Assuming radiation has the same cross section area as the electron beam, we obtain the change in electric field amplitude:

$$\Delta\Pi_r + \Delta\Pi_e = 0 \Rightarrow \boxed{\Delta E = -\frac{j_0\langle P\rangle}{2c\varepsilon_0 E_{ext} e}}$$

$$\left.\begin{aligned} \frac{dP}{dz} &= -eE\theta_s\cos(\psi) \\ \frac{d}{dz}\psi &= C + \frac{\omega}{\gamma_z^2 c\mathcal{E}_0}P \end{aligned}\right\} \Rightarrow \langle P\rangle = -eE\theta_s\left\langle \int_0^1 \cos\left[\psi(\hat{z})\right] d\hat{z} \right\rangle$$

Low Gain Regime: Derivation of FEL Gain

$$\frac{d^2}{d\hat{z}^2}\psi + \hat{u}\cos\psi = 0$$

$$\psi(\hat{z}) = \psi(0) + \psi'(0)\hat{z} - \hat{u}\int_0^{\hat{z}} d\hat{z}_1 \int_0^{\hat{z}_1} \cos\psi(\hat{z}_2) d\hat{z}_2 \qquad (1)$$

Assuming that all electrons have the same energy and uniformly distributed in the Pondermotive phase at the entrance of FEL: $P_0 = 0$ and $f(\psi_0) = \frac{1}{2\pi}$.

The zeroth order solution for phase evolution is given by ignoring the effects from FEL interaction:

$$\left.\begin{aligned} \frac{dP}{dz} &= -eE\theta_s \cos(\psi) \\ \frac{d}{dz}\psi &= C + \frac{\omega}{\gamma_z^2 c\mathcal{E}_0} P \end{aligned}\right\} \Rightarrow \frac{d}{d\hat{z}}\psi = \hat{C} \Rightarrow \begin{cases} \psi(\hat{z}) = \psi_0 + \hat{C}\hat{z} \\ \psi'(0) = \hat{C} \end{cases} \qquad \hat{C} \equiv Cl_w$$

Inserting the zeroth order solution back into eq. (1) yields the 1st order solution:

$$\psi(\hat{z}) = \psi_0 + \hat{C}\hat{z} + \Delta\psi(\psi_0, \hat{z}) \qquad \Delta\psi(\psi_0, \hat{z}) \equiv -\hat{u}\int_0^{\hat{z}} d\hat{z}_1 \int_0^{\hat{z}_1} \cos\left[\psi_0 + \hat{C}\hat{z}_2\right] d\hat{z}_2$$

Low Energy Regime: Derivation of FEL Gain

$$\Delta\psi(\psi_0,\hat{z}) \equiv -\hat{u}\int_0^{\hat{z}} d\hat{z}_1 \int_0^{\hat{z}_1} \cos\left[\psi_0 + \hat{C}\hat{z}_2\right] d\hat{z}_2$$

$$= -\frac{\hat{u}}{\hat{C}^2}\left\{\int_0^{\hat{C}\hat{z}} \sin(\psi_0 + x_1)\,dx_1 - \hat{C}\hat{z}\sin\psi_0\right\} = \frac{\hat{u}}{\hat{C}^2}\left[\cos\left(\psi_0 + \hat{C}\hat{z}\right) - \cos\psi_0 + \hat{C}\hat{z}\sin\psi_0\right]$$

$$\langle P\rangle = -eEl_w\theta_s\left\langle\int_0^1 \cos\left[\psi_0 + \hat{C}\hat{z} + \Delta\psi(\psi_0,\hat{z})\right] d\hat{z}\right\rangle$$

← Average energy loss of electrons

$$= eE\theta_s l_w\left\langle\int_0^1 \sin\left[\psi_0 + \hat{C}\hat{z}\right]\sin(\Delta\psi(\psi_0,\hat{z}))\,d\hat{z}\right\rangle - eE\theta_s l_w\left\langle\int_0^1 \cos\left[\psi_0 + \hat{C}\hat{z}\right]\cos(\Delta\psi(\psi_0,\hat{z}))\,d\hat{z}\right\rangle$$

$$\approx eE\theta_s l_w\left\langle\int_0^1 \Delta\psi(\psi_0,\hat{z})\sin\left[\psi_0 + \hat{C}\hat{z}\right] d\hat{z}\right\rangle - \cancel{\frac{eE\theta_s l_w}{2\pi}\int_0^1 d\hat{z}\int_0^{2\pi} \cos\left[\psi_0 + \hat{C}\hat{z}\right] d\psi_0}$$

$$= \frac{eE\theta_s l_w}{2\pi}\int_0^1 d\hat{z}\left\{\cos\left(\hat{C}\hat{z}\right)\int_0^{2\pi} \Delta\psi(\psi_0,\hat{z})\sin\psi_0\,d\psi_0 + \sin\left(\hat{C}\hat{z}\right)\int_0^{2\pi} \Delta\psi(\psi_0,\hat{z})\cos\psi_0\,d\psi_0\right\}$$

$$= \frac{eE\theta_s l_w}{2\pi}\frac{\hat{u}}{\hat{C}^2}\int_0^1 d\hat{z}\left\{\hat{C}\hat{z}\cos\left(\hat{C}\hat{z}\right)\int_0^{2\pi} \sin^2\psi_0\,d\psi_0 - \sin\left(\hat{C}\hat{z}\right)\int_0^{2\pi} \cos^2\psi_0\,d\psi_0\right\}$$

$$= -eE\theta_s l_w\frac{\hat{u}}{\hat{C}^3}\left(1 - \frac{\hat{C}}{2}\sin\hat{C} - \cos\hat{C}\right)$$

Low Energy Regime: Derivation of FEL Gain

Growth in the amplitude of radiation field:

$$\Delta E = -\frac{j_0\langle P\rangle}{2c\varepsilon_0 E_{ext} e} = \frac{\pi j_0 \theta_s^2 \omega}{c\gamma_z^2\gamma}\frac{l_w^3 E_{ext}}{I_A}\frac{2}{\hat{C}^3}\left(1-\frac{\hat{C}}{2}\sin\hat{C}-\cos\hat{C}\right)$$

$$\hat{u} = \frac{l_w^2 e E_{ext}\theta_s \omega}{\gamma_z^2 c \gamma m c^2}$$

$$I_A = \frac{4\pi\varepsilon_0 m c^3}{e}$$

The gain is defined as the relative growth in radiation power:

$$\boxed{g_s = \frac{\left(E_{ext}+\Delta E\right)^2 - E_{ext}^2}{E_{ext}^2} \approx \frac{2\Delta E}{E_{ext}} = \tau\cdot f\left(\hat{C}\right)}$$

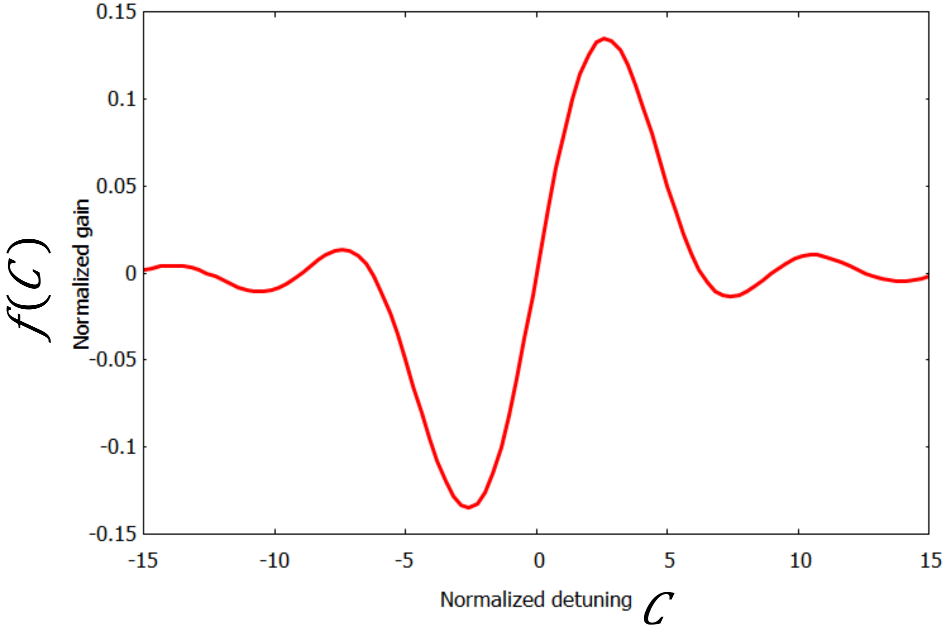
As observed earlier, there is no gain if the electrons has resonant energy.

$$\tau \equiv \frac{2\pi j_0 \theta_s^2 \omega}{c\gamma_z^2\gamma}\frac{l_w^3}{I_A}$$

↗ Cubic in FEL length

$$f\left(\hat{C}\right) = \frac{2}{\hat{C}^3}\left(1-\cos\hat{C}-\frac{\hat{C}}{2}\sin\hat{C}\right) \longrightarrow$$

$$= -2\frac{d}{d\hat{C}}\frac{\sin^2\left(\hat{C}/2\right)}{\hat{C}^2}$$



High Gain Regime: 1-D FEL Theory

- Ignoring the space charge effects, the Hamiltonian for electrons in a FEL can be written as (see additional material):

$$\boxed{H(\psi,P,z)=CP+\frac{\omega}{2c\gamma_z^2E_0}P^2-\left(U(z)e^{i\psi}+U^*(z)e^{-i\psi}\right)}$$

$$U=-\frac{e\theta_s\widetilde{E}(z)}{2i} \qquad\qquad E_x+iE_y=\widetilde{E}(z)\exp[i\omega(z/c-t)]$$

$$\Rightarrow\begin{cases}\dfrac{dP}{dz}=-\dfrac{\partial H}{\partial\psi}=2\dfrac{\partial}{\partial\psi}\mathrm{Re}\left[Ue^{i\psi}\right]=-\mathrm{Re}\left[e\theta_s\widetilde{E}(z)e^{i\psi}\right]=-e\theta_s\left|\widetilde{E}(z)\right|\cos(\psi+\overset{\text{Slow varying phase}}{\overset{\searrow}{\varphi(z)}})\\[2ex]\dfrac{d\psi}{dz}=\dfrac{\partial H}{\partial P}=C+\dfrac{\omega}{c\gamma_z^2E_0}P\end{cases}$$

Linearization of Vlasov Equation

Vlasov equation: $$\frac{\partial f}{\partial z}+\frac{\partial H}{\partial P}\frac{\partial f}{\partial \psi}-\frac{\partial H}{\partial \psi}\frac{\partial f}{\partial P}=0$$

$$f(\psi,P,z)=f_0(P)+\widetilde{f}_1(P,z)e^{i\psi}+\widetilde{f}_1^{*}(P,z)e^{-i\psi} \qquad \psi=k_u z+k(z-ct)$$

Linearized Vlasov equation: $$\frac{\partial \widetilde{f}_1}{\partial z}+i\left[C+\frac{\omega}{c\gamma_z^2E_0}P\right]\widetilde{f}_1+iU\frac{\partial f_0}{\partial P}=0$$

$$\frac{\partial}{\partial z}\left\{\widetilde{f}_1\exp\left[i\left(C+\frac{\omega}{c\gamma_z^2E_0}P\right)z\right]\right\}+iU\exp\left[i\left(C+\frac{\omega}{c\gamma_z^2E_0}P\right)z\right]\frac{\partial f_0}{\partial P}=0$$

Assuming that there is no initial modulation in the electrons, i.e. $\widetilde{f}_1(0)=0$

$$\widetilde{f}_1(z)=-in_0\frac{\partial F_0(P)}{\partial P}\int_0^z dz_1U\exp\left[i\left(C+\frac{\omega}{c\gamma_z^2E_0}P\right)(z_1-z)\right]dz_1 \qquad f_0(P)=n_0F(P)$$

Integrate over energy deviation: $-ec\int_{-\infty}^{\infty}\widetilde{f}_1(P,z)dP=\widetilde{j}_1(z) \qquad j_z=-j_0+\widetilde{j}_1e^{i\psi}++\widetilde{j}_1^{*}e^{-i\psi} \qquad j_0=en_0c$

$$\widetilde{j}_1(z)=ij_0\int_0^z dz_1U(z_1)\int_{-\infty}^{\infty}\frac{\partial F_0(P)}{\partial P}\exp\left[i\left(C+\frac{\omega}{c\gamma_z^2E_0}P\right)(z_1-z)\right]dP$$

Wave Equation

$$\psi = k_u z + k\left(z - ct\right)$$

1-D theory and hence $\partial / \partial x = 0$ and $\partial / \partial y = 0$

Wave equation for transverse vector potential: $$\frac{\partial^2 \vec{A}_\perp}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 \vec{A}_\perp}{\partial t^2} = -\mu_0 \vec{j}_\perp \qquad (1)$$

Transverse current perturbation: $$j_x + ij_y = \frac{1}{v_z}\left(v_x + iv_y\right)j_{z,1} = \theta_s e^{-ik_w z}\left(\widetilde{j}_1 e^{i\psi} + + \widetilde{j}_1^{\,*} e^{-i\psi}\right) \qquad (2)$$

We seek the solution for vector potential of the form:

$$A_{x,y}(z,t) = \widetilde{A}_{x,y}(z)e^{i\omega(z/c - t)} + \widetilde{A}_{x,y}{}^{*}(z)e^{-i\omega(z/c-t)} \qquad (3)$$

Inserting eq. (2) and (3) into eq. (1) yields

$$e^{i\omega(z/c-t)}\left\{\frac{2i\omega}{c}\frac{\partial}{\partial z}\begin{pmatrix}\widetilde{A}_x\\ \widetilde{A}_y\end{pmatrix} + \frac{\partial^2}{\partial z^2}\begin{pmatrix}\widetilde{A}_x\\ \widetilde{A}_y\end{pmatrix}\right\} + C.C. = -\mu_0\theta_s\begin{pmatrix}\cos(k_w z)\\ -\sin(k_w z)\end{pmatrix}\left(\widetilde{j}_1 e^{i\psi} + C.C.\right)$$

$$\left\{\frac{2i\omega}{c}\frac{\partial}{\partial z}\begin{pmatrix}\widetilde{A}_x\\ \widetilde{A}_y\end{pmatrix} + \frac{\partial^2}{\partial z^2}\begin{pmatrix}\widetilde{A}_x\\ \widetilde{A}_y\end{pmatrix}\right\} = -\frac{\mu_0\theta_s}{2}\begin{pmatrix}e^{ik_w z} + e^{-ik_w z}\\ ie^{ik_w z} - ie^{-ik_w z}\end{pmatrix}\widetilde{j}_1 e^{ik_w z}$$

1. Ignoring fast oscillating term $\sim e^{2ik_w z}$

2. Ignoring second derivative by assuming that the variation of $\widetilde{A}_x'$ is negligible over the optical wave length.

Wave Equation

After neglecting the fast oscillation terms, we get the following relation between the current perturbation and the vector potential of the radiation field:

$$\frac{\partial}{\partial z}\tilde{A}_x = -\frac{c\mu_0\theta_s}{4i\omega}\tilde{j}_1 \qquad \frac{\partial}{\partial z}\tilde{A}_y = \frac{\mu_0 c\theta_s}{4\omega}\tilde{j}_1$$

In order to relate the vector potential to the electric field, we use the Maxwell equation:

$$\nabla\times\vec{E}+\frac{\partial\vec{B}}{\partial t}=0 \Rightarrow \nabla\times\left(\vec{E}+\frac{\partial\vec{A}}{\partial t}\right)=0 \Rightarrow \left(\vec{E}+\frac{\partial\vec{A}}{\partial t}\right)=\vec{\nabla}\varphi \Rightarrow E_{x,y}=-\frac{\partial A_{x,y}}{\partial t}$$

$$\Rightarrow \tilde{E}e^{i\omega(z/c-t)} = E_x + iE_y = -\frac{\partial}{\partial t}\left[\left(\tilde{A}_x + i\tilde{A}_y\right)e^{i\omega(z/c-t)}\right]$$

$$\Rightarrow \tilde{E} = i\omega\left(\tilde{A}_x + i\tilde{A}_y\right)$$

Finally, the relation between the radiatio field and the current modulation is obtained:

$$\boxed{\frac{d}{dz}\tilde{E} = i\omega\left(\frac{\partial}{\partial z}\tilde{A}_x + i\frac{\partial}{\partial z}\tilde{A}_y\right) = -\frac{c\mu_0\theta_s}{2}\tilde{j}_1}$$

Integra-differential Equation

Let's put together what we achieved so far…

$$\widetilde{j}_1(z) = i j_0 \int_0^z dz_1 U(z_1) \int_{-\infty}^{\infty} \frac{\partial F_0(P)}{\partial P} \exp\left[i\left(C + \frac{\omega}{c\gamma_z^2 E_0} P \right)(z_1 - z)\right] dP$$

$$\frac{d}{dz}\widetilde{E}(z) = -\frac{c\mu_0\theta_s}{2}\widetilde{j}_1(z) \qquad U \equiv -\frac{e\theta_s \widetilde{E}(z)}{2i}$$

After inserting the latter two equations back into the first equation, we arrive at

$$\frac{d}{d\hat{z}}\widetilde{E}(\hat{z}) = \int_0^{\hat{z}} d\hat{z}_1 \widetilde{E}(\hat{z}_1) \int_{-\infty}^{\infty} \frac{dF_0(\hat{P})}{d\hat{P}} \exp\left[i\left(\hat{C} + \hat{P}\right)(\hat{z}_1 - \hat{z})\right] d\hat{P}$$

where the following normalized variables are used to make the equation more compact:

Gain parameter: $\Gamma = \left[\frac{\pi j_0 \theta_s^2 \omega}{c\gamma_z^2 \gamma I_A}\right]^{1/3}$ Pierce Parameter: $\rho = \gamma_z^2 \Gamma c / \omega$

$$\hat{C} = C/\Gamma \qquad \hat{z} = z\Gamma \qquad \hat{P} = \frac{E - E_0}{E_0 \rho}$$

Solution for Cold Beam

After integration by parts: $$\frac{d}{d\hat{z}}\widetilde{E}(\hat{z}) = -i\int_0^{\hat{z}} d\hat{z}_1 \widetilde{E}(\hat{z}_1)(\hat{z}_1 - \hat{z})\int_{-\infty}^{\infty} F_0(\hat{P})\exp\left[i(\hat{C}+\hat{P})(\hat{z}_1 - \hat{z})\right]d\hat{P}$$

For cold beam: $$F_0(\hat{P}) = \delta(\hat{P})$$

$$e^{i\hat{C}\hat{z}}\frac{d}{d\hat{z}}\widetilde{E}(\hat{z}) = -i\int_0^{\hat{z}} \widetilde{E}(\hat{z}_1)(\hat{z}_1 - \hat{z})e^{i\hat{C}\hat{z}_1}d\hat{z}_1$$

Taking derivative: $$\frac{d}{d\hat{z}}\left[e^{i\hat{C}\hat{z}}\frac{d}{d\hat{z}}\widetilde{E}(\hat{z})\right] = i\int_0^{\hat{z}} \widetilde{E}(\hat{z}_1)e^{i\hat{C}\hat{z}_1}d\hat{z}_1$$

Taking another derivative: $$\frac{d^2}{d\hat{z}^2}\left[e^{i\hat{C}\hat{z}}\frac{d}{d\hat{z}}\widetilde{E}(\hat{z})\right] = i\widetilde{E}(\hat{z})e^{i\hat{C}\hat{z}}$$

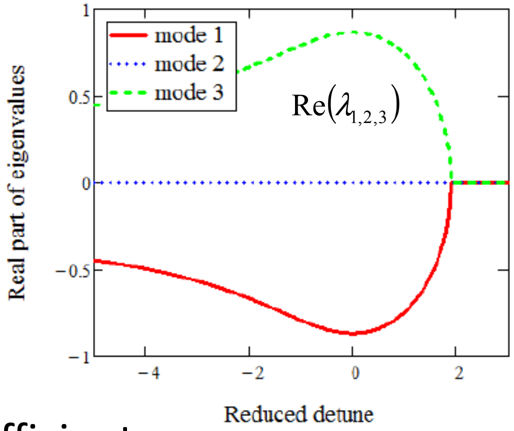
We obtain a third order homogenous ODE: $$\frac{d^3}{d\hat{z}^3}\widetilde{E}(\hat{z}) + 2i\hat{C}\frac{d^2}{d\hat{z}^2}\widetilde{E}(\hat{z}) - \hat{C}^2\frac{d}{d\hat{z}}\widetilde{E}(\hat{z}) = i\widetilde{E}(\hat{z})$$

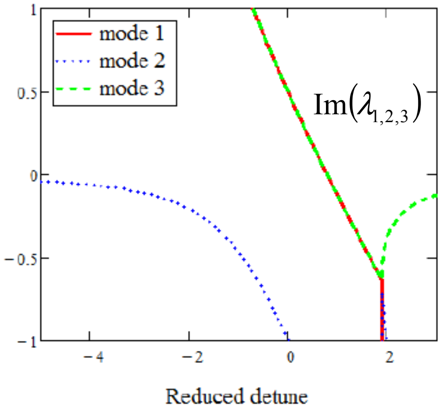
Solution for Cold Beam

The general solution of the ODE reads:

$$\tilde{E}(\hat{z}) = \sum_{k=1}^{3} B_k e^{i\lambda_k \hat{z}}$$

$$\lambda^3 + 2i\hat{C}\lambda^2 - \hat{C}^2\lambda = i$$





Applying initial condition to get the coefficients

$$\begin{pmatrix} \tilde{E}(0) \\ \tilde{E}'(0) \\ \tilde{E}''(0) \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ i\lambda_1 & i\lambda_2 & i\lambda_3 \\ -\lambda_1^2 & -\lambda_2^2 & -\lambda_3^2 \end{pmatrix} \begin{pmatrix} B_1 \\ B_2 \\ B_3 \end{pmatrix} \Rightarrow \begin{pmatrix} B_1 \\ B_2 \\ B_3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ i\lambda_1 & i\lambda_2 & i\lambda_3 \\ -\lambda_1^2 & -\lambda_2^2 & -\lambda_3^2 \end{pmatrix}^{-1} \begin{pmatrix} \tilde{E}(0) \\ \tilde{E}'(0) \\ \tilde{E}''(0) \end{pmatrix}$$

For $\tilde{E}(0) = E_{ext}$ and $\tilde{E}'(0) = \tilde{E}''(0) = 0$, the solution can be explicitly written as

$$\tilde{E}(\hat{z}) = E_{ext}\left[\frac{\lambda_2\lambda_3 e^{\lambda_1\hat{z}}}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)} + \frac{\lambda_1\lambda_3 e^{\lambda_2\hat{z}}}{(\lambda_2-\lambda_3)(\lambda_2-\lambda_1)} + \frac{\lambda_1\lambda_2 e^{\lambda_3\hat{z}}}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)}\right]$$