PHY 564
Advanced Accelerator Physics
Lecture 5
Hamiltonian Method for Accelerators

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Power of Hamiltonian method: Phase space and invariants

While in accelerators we are usually dealing with three degrees of freedom - frequently reduced to one or two – it is useful to consider a general case on ***n*** degrees of freedom.

As we discussed before, there can be invariants of motion. But in Hamiltonian system there are always invariants of motion– the Poincaré invariants. Let' us consider a Hamiltonian system with a Hamiltonian $H(q,P,s)$ for a system described by the set of coordinates $q=\{q^1,...,q^n\}$, Canonical momenta, $P=\{P_1,...,P_n\}$ and the independent monotonic variable *s (whatever it is!).*

There are two mathematically equivalent but different in appearance definitions of phase space vector X: one (we will call is "a traditional" convention) is a listing sequence of canonical pairs $\{q^i,P_i\}$ while the other (we will call it either "a mathematical" or Dragt' convention):

$$\underline{X} \equiv \begin{bmatrix} x_1 \\ x_2 \\ ... \\ x_{2n-1} \\ x_{2n} \end{bmatrix} = \begin{bmatrix} q^1 \\ P_1 \\ ... \\ q^n \\ P_n \end{bmatrix} \quad or \ \overline{X} \equiv \begin{bmatrix} x_1 \\ x_2 \\ ... \\ x_{2n-1} \\ x_{2n} \end{bmatrix} = \begin{bmatrix} q \\ P \end{bmatrix} \equiv \begin{bmatrix} q^1 \\ ... \\ q^n \\ P_1 \\ ... \\ P_n \end{bmatrix} \tag{5.1}$$

It is obvious that mathematically these two conventions are identical and connected by simple relations with unit-determinant matrices*:*

$$\underline{X} = \mathbf{C}_{mt} \cdot \overline{X}; \ \overline{X} = \mathbf{C}_{tm} \cdot \underline{X}; \ \left[\mathbf{C}_{mt}\right]_{ik} = \begin{Bmatrix} 1, \ i=2k-1, \ k=1,...,n \\ 1, \ i=2k, \ k=1,...,n \\ 0, \ otherwise \end{Bmatrix}; \left[\mathbf{C}_{tm}\right]_{ki} = \begin{Bmatrix} 1, \ i=2k-1, \ k=1,...,n \\ 1, \ i=2k, \ k=1,...,n \\ 0, \ otherwise \end{Bmatrix}; \tag{5.2}$$

[1] After Prof. Alex J. Dragt from university of Maryland, College Park, the author of famous manuscript "Lie Methods for Nonlinear Dynamics with Applications to Accelerator Physics", who shoes that this convention has multiple advantages in beatifying equations.

We know very well that Hamiltonian equations of motion is *2n*-set of ordinary differential equations:

$$\frac{dq^i}{ds} = \frac{\partial H}{\partial P^i}; \frac{dP^i}{ds} = -\frac{\partial H}{\partial q^i} \Leftrightarrow \frac{dq}{ds} = \frac{\partial H}{\partial P}; \frac{dP}{ds} = -\frac{\partial H}{\partial q}, \tag{5.3}$$

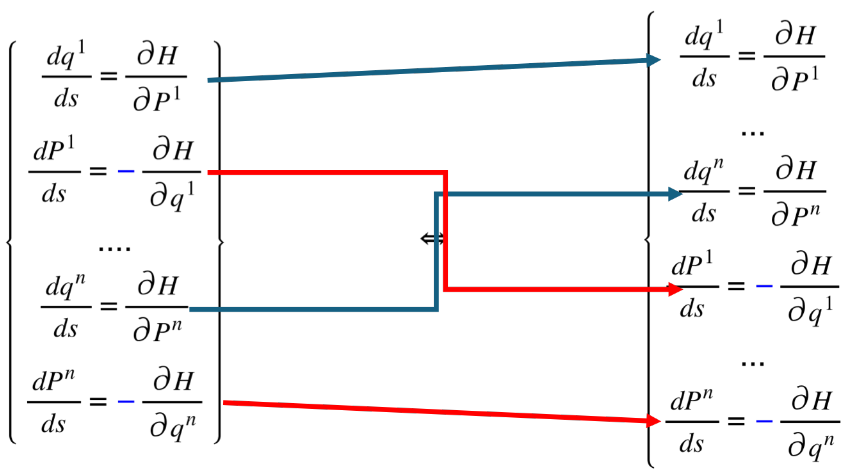
Which can be written in matrix form using conventions (5.1) and (5.2) as

$$\frac{d\underline{X}}{ds} = \underline{\mathbf{S}}\frac{\partial H}{\partial \underline{X}}; \underline{\mathbf{S}} = \begin{bmatrix} \boldsymbol{\sigma} & \ldots & \mathbf{0}_2 \\ \ldots & \ldots & \ldots \\ \mathbf{0}_2 & \ldots & \boldsymbol{\sigma} \end{bmatrix}; \boldsymbol{\sigma} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}; \mathbf{0}_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{5.4}$$

for traditional convention and

$$\frac{d\bar{X}}{ds} = \bar{\mathbf{S}}\cdot\frac{\partial H}{\partial \bar{X}}; \underline{\mathbf{S}} = \begin{bmatrix} \mathbf{0}_n & \mathbf{I}_n \\ -\mathbf{I}_n & \mathbf{0}_n \end{bmatrix}; [\,\mathbf{I_n}]_{ik} = \begin{Bmatrix} 1, i=k \\ 0, \textit{otherwise} \end{Bmatrix}; [\,\mathbf{0_n}]_{\mathbf{ik}} = 0 \tag{5.5}$$

with *2nx2n* asymmetric matrices $\underline{\mathbf{S}}$, $\bar{\mathbf{S}}$ applying Hamiltonian rules to the partial derivatives of the Hamiltonian mechanics. Difference between (5.4) and (5.5) is just reordering the same set of *2n* equations (5.3): just change in the sequence not in the substance



It is easy to show that both **S** -matrices are antisymmetric and

$$\bar{\mathbf{S}}^T = -\bar{\mathbf{S}};\ \underline{\mathbf{S}}^T = -\underline{\mathbf{S}};$$
$$\bar{\mathbf{S}}^2 \equiv \bar{\mathbf{S}}\cdot\bar{\mathbf{S}} = -\mathbf{I};\ \underline{\mathbf{S}}^2 \equiv \underline{\mathbf{S}}\cdot\underline{\mathbf{S}} = -\mathbf{I}, \tag{5.6}$$

we will use further down symbol **I** for unit diagonal and **O** for zero matrices of necessary orders.

$$\underline{\mathbf{S}}^2 = \begin{bmatrix} \sigma & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \sigma & \dots & \mathbf{0} & \mathbf{0} \\ \dots & \dots & \dots & \dots & \dots \\ \mathbf{0} & \mathbf{0} & \dots & \sigma & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \sigma \end{bmatrix}^2 = \begin{bmatrix} \sigma^2 & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \sigma^2 & \dots & \mathbf{0} & \mathbf{0} \\ \dots & \dots & \dots & \dots & \dots \\ \mathbf{0} & \mathbf{0} & \dots & \sigma^2 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \sigma^2 \end{bmatrix}; \sigma^2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \Rightarrow \underline{\mathbf{S}}^2 = -\mathbf{I};\ \boldsymbol{\sigma}^{\mathbf{T}} = -\boldsymbol{\sigma};$$

$$\underline{\mathbf{S}}^{\mathbf{T}} = \begin{bmatrix} \boldsymbol{\sigma}^{\mathbf{T}} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\sigma}^{\mathbf{T}} & \dots & \mathbf{0} & \mathbf{0} \\ \dots & \dots & \dots & \dots & \dots \\ \mathbf{0} & \mathbf{0} & \dots & \boldsymbol{\sigma}^{\mathbf{T}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \boldsymbol{\sigma}^{\mathbf{T}} \end{bmatrix} = -\underline{\mathbf{S}}; \quad \bar{\mathbf{S}}^2 = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{I} & \mathbf{0} \end{bmatrix}^2 = \begin{bmatrix} -\mathbf{I} & \mathbf{0} \\ \mathbf{0} & -\mathbf{I} \end{bmatrix} \equiv -\mathbf{I};\ \overline{\mathbf{S}^{\mathbf{T}}} = \begin{bmatrix} \mathbf{0} & -\mathbf{I} \\ \mathbf{I} & \mathbf{0} \end{bmatrix} = -\bar{\mathbf{S}}^{\mathbf{T}}\ \#$$

Mathematically **S** is a generator (norm) of a symplectic group of matrixes (two different but closely related types of mathematical groups). The space of coordinates and momenta is called phase space of the system with dimension ***2n***.

What is very good that the appearance of many matrix equations and maps are identical for both conventions and unless it is specifically stated we will use X and **S** without under- and over- score for both conventions

$$\frac{dX}{ds} = \mathbf{S}\cdot\frac{\partial H}{\partial X}; \tag{5.7}$$

Let's consider an ensemble of particles in this phase space whose motions satisfy the Hamiltonian equations; then, their motion is completely determined by their initial position in the phase space. This means that in the Hamiltonian system the phase-space trajectories of particles, which initially were separated, will never cross! Consider one trajectory in the phase space $X_o(s)$, which satisfies the Hamiltonian equation (5.7) and another trajectory with an infinitesimally small deviation from X_o

$$X_1 = X_o(s) + \delta X;$$

$$\frac{dX_o}{ds} = S\left.\frac{\partial H(X)}{\partial X}\right|_{X=X_o};H(X+\Delta X) = H(X) + \frac{\partial H(X)}{\partial X}\delta X + \frac{1}{2}\frac{\partial^2 H(X)}{\partial X^2}\delta X^2 + O(\delta X^3);$$

$$\frac{d\left(X_o(s)+\delta X\right)}{ds} = \mathbf{S}\cdot\left.\frac{\partial H(X)}{\partial X}\right|_{X=X_o(s)+\delta X} \cong \mathbf{S}\cdot\left.\frac{\partial H(X)}{\partial X}\right|_{X=X_o} + \mathbf{S}\cdot\mathbf{H}(\mathrm{s})\cdot\delta X + O(\delta X^2); \tag{5.8}$$

$$\mathbf{H}(\mathrm{s}) = \frac{1}{2}\left.\frac{\partial^2 H(X)}{\partial X^2}\right|_{X=X_o(s)};$$

$$\left[\mathbf{H}(\mathrm{s})\right]_{ij} = \left[\frac{\partial^2 H}{\partial x_i \partial x_j}\right]\Bigg|_{X=X_o(s)} \Rightarrow \left[\mathbf{H}(\mathrm{s})\right]_{ij} \equiv \left[\mathbf{H}(\mathrm{s})\right]_{ji} \Rightarrow \mathbf{H}^{\mathbf{T}}(\mathrm{s}) = \mathbf{H}(\mathbf{s})$$

with a symmetric *2n* x *2n* matrix **H(s)**.

Thus, the equation of motion for a small deviation about the known trajectory are linear but s-dependent, and can be expressed via the linear transform matrix *M(s)*:

$$\frac{d\delta X}{ds} = \mathbf{S}\cdot\mathbf{H}(s)\cdot\delta X; \quad \delta X(s) = \mathbf{M}(s)\cdot\delta X(0); \quad \frac{d\mathbf{M}}{ds} = \mathbf{S}\cdot\mathbf{H}(s)\cdot\mathbf{M}$$
$$\Rightarrow \quad \frac{d\delta X}{ds} = \frac{d\mathbf{M}(s)}{ds}\cdot\delta X(0) = \mathbf{S}\cdot\mathbf{H}(s))\cdot\mathbf{M}\cdot\delta X(0) = \mathbf{S}\cdot\mathbf{H}(s)\cdot\delta X\# \qquad (5.9)$$

The matrix **M**(s) is symplectic, which is proven as follows:

$$\mathbf{M}^T(s)\cdot\mathbf{S}\cdot\mathbf{M}(s) = \mathbf{S} \; -\textit{symplectic condition}; \; \mathbf{M}(0) = \mathbf{I} \; \Rightarrow \mathbf{M}^T(0)\cdot\mathbf{S}\cdot\mathbf{M}(0) = \mathbf{S};$$

$$\frac{d\left(\mathbf{M}^T\cdot\mathbf{S}\cdot\mathbf{M}\right)}{ds} = \frac{d\mathbf{M}^T}{ds}\mathbf{S}\cdot\mathbf{M} + \mathbf{M}^T\cdot\mathbf{S}\frac{d\mathbf{M}}{ds};$$

$$\frac{d\mathbf{M}^T}{ds} = (\mathbf{S}\cdot\mathbf{H}\cdot\mathbf{M})^T = \mathbf{M}^T\cdot\mathbf{H}^{\mathbf{T}}\cdot\mathbf{S}^T = -\mathbf{M}^T\cdot\mathbf{H}\cdot\mathbf{S}$$

$$\textit{because} \quad \mathbf{H}^T = \mathbf{H}; \quad \mathbf{S}^T = -\mathbf{S}; \; \mathbf{S}^2 = \mathbf{I};$$

$$\frac{d\left(\mathbf{M}^T\cdot\mathbf{S}\cdot\mathbf{M}\right)}{ds} = -\mathbf{M}^T\cdot\mathbf{H}\cdot\mathbf{S}^2\cdot\mathbf{M} + \mathbf{M}^T\cdot\mathbf{S}^2\cdot\mathbf{H}\cdot\mathbf{M} = \mathbf{M}^T\cdot\mathbf{H}\cdot\mathbf{M} - \mathbf{M}^T\cdot\mathbf{H}\cdot\mathbf{M} \equiv 0;$$

$$\Rightarrow \mathbf{M}^T(s)\cdot\mathbf{S}\cdot\mathbf{M}(s) = \textit{conts} = \mathbf{M}(0)\cdot\mathbf{S}\cdot\mathbf{M}(0) = \mathbf{S}\#$$

The symplectic condition has two asymmetric *2n* x *2n* matrixes on both sides

$$\mathbf{M}^T(s)\cdot\mathbf{S}\cdot\mathbf{M}(s) = \mathbf{S} \qquad (5.10)$$

and imposes *n (2n-1)* conditions on the matrix **M**.

These conditions result in invariants of motion for the ensembles of particles, called Poincaré invariants. Accordingly, for 3-D motion, there are 15 Poincaré invariants! The most well-known one, the conservation of the phase space volume (Liouville's theorem), is a consequence of the unit determinant of the matrix M:

$$\det\left[\mathbf{M}^T(s)\cdot\mathbf{S}\cdot\mathbf{M}(s)\right]=\det\mathbf{S}\rightarrow\ (\det\mathbf{M}(s))^2=1\rightarrow\det M=\pm1;\quad but\quad \det\mathbf{M}(0)=1\rightarrow det\mathbf{M}(s)=1\,\# \tag{5-11}$$

Next, we consider an infinitesimally small phase-space volume ΔV_{2n} around a known trajectory and its transformation:

$$\Delta V_{2n}(s)=det\left[\frac{\partial\delta X(s)}{\partial\delta X(0)}\right]\cdot\Delta V_{2n}(0)=det\mathbf{M}(s)\cdot\Delta V_{2n}(0)=const \tag{5-12}$$

The 6-dimensional volume occupied by the particles often is termed 3-D beam emittance. The rest of the Poincaré invariants represent similar conservation laws for the sum of projections on hyper-surfaces in $2n$-phase space.

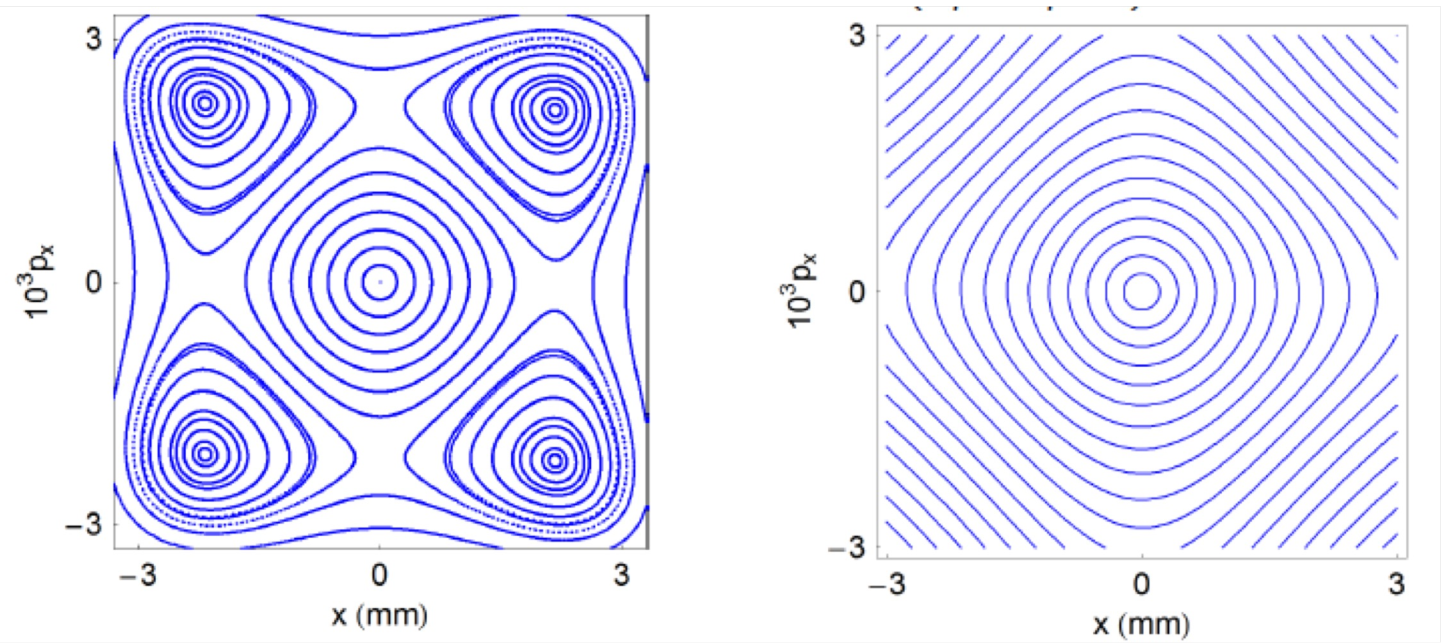
But there is one nice instance wherein the Hamiltonian is decoupled, i.e., it is the direct sum of individual Hamiltonians for each canonical pair.

$$H=H(Q,P,s)=\sum_{k=1}^{3}H_k(Q_k,P_k,s) \tag{5-13}$$

Then, the phase space is two-dimensional, and the area of the space phase occupied by the beam is called beam emittance for a specific dimension – horizontal (x, P_x), vertical (y, P_y) or longitudinal (τ,δ). All three emittances are constants (integrals) of motion.

Phase space. The full set of coordinates and momenta of particle (or a ensemble of particles) *{q,P}* is called phase space. Naturally dimension of the phase space is always even: 2, 4, 6.., 2n. While motion in the coordinate space {q} can be rather arbitrary, the same motion in the phase space satisfies a number of very strong constrains, e.g. there is a number of invariants.

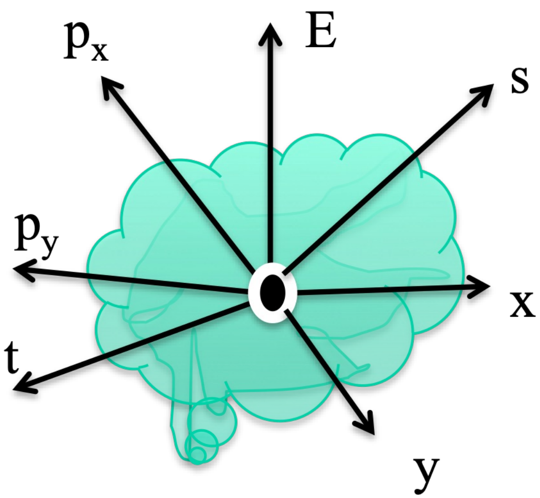
Location or motion of particles in the phase space are called phase-space plots or phase-space diagrams. Naturally we usually can plot on the paper or show on the screen only one coordinate and one momentum – hence, you usually see phase plot for 1D case, or for projections of multi-dimensional phases space plot on one plane.

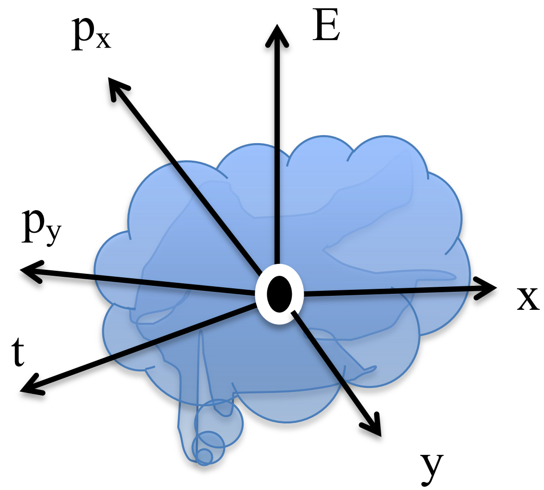


Example of $\{x,P_x\}$ phase-space diagram showing trace of the particles motion in accelerators: a set of particles with initial coordinate were seeded n the plot and then traced for a large number of turns. Stable motion results in periodic and semi-periodic results in "orbits – semi-closed trajectories " in the phase space.

Adding an additional coordinate, *s*, allows one to follow the trajectories of the particles in the phase space. It is important feature if system's Hamiltonian depends on independent variable (*s* or *t*).

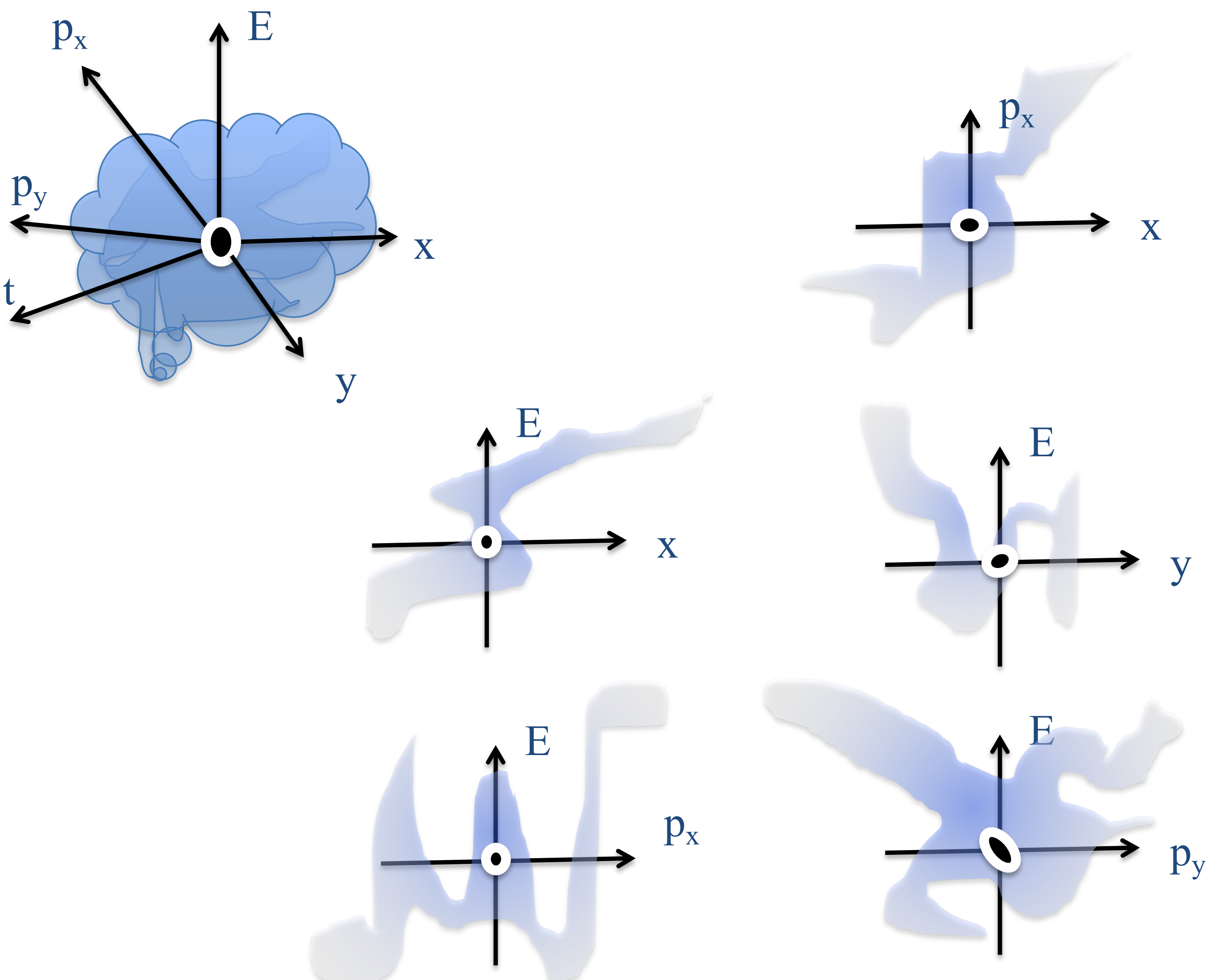
One of very important featured of the particle's trajectories in this version of the phase space that they cannot cross (at the same s!). The later comes from a simple observation that to particles having the same values of coordinates and momenta *{q,P}* at the same moment of time (s), will follow identical trajectories! Note, that this is very general statement – it does not rely on Hamiltonian mechanics, but only on the assumption that full set of coordinates and momenta *{q,P}* fully describes the initial conditions for a particle.





If we take out the independent variable *s* axis from the set of the axes, than trajectories can, in principle, cross if the Hamiltonian is *s* - dependent.

It is hard to draw 6D phase space distribution in PPT.. but it is easy to draw 2D projections



A simple example will be a

$$H = \left\{ \begin{array}{c} \frac{p^2}{2}, s < 0 \\ \frac{p^2}{2} + \frac{x^2}{2}, s < 0 \end{array} \right\} \quad \text{(x)}$$

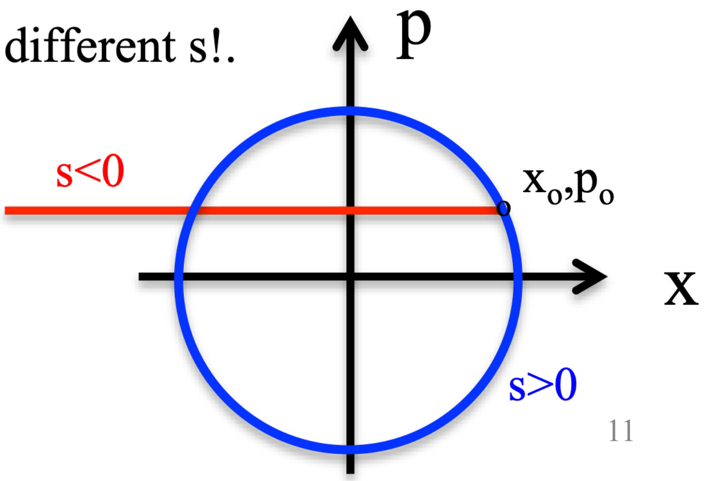
One can easily see that for *s* < *0*, the solution is

$$p = p_o; x = x_o + p_o s \quad \text{(xx)}$$

and for *s* > *0*:

$$p = a \cdot \cos(s + \varphi); x = a \cdot \sin(s + \varphi); \\ a = \sqrt{x_o^2 + p_o^2}; \tan\varphi = \frac{x_o}{p_o} \quad \text{(xxx)}$$

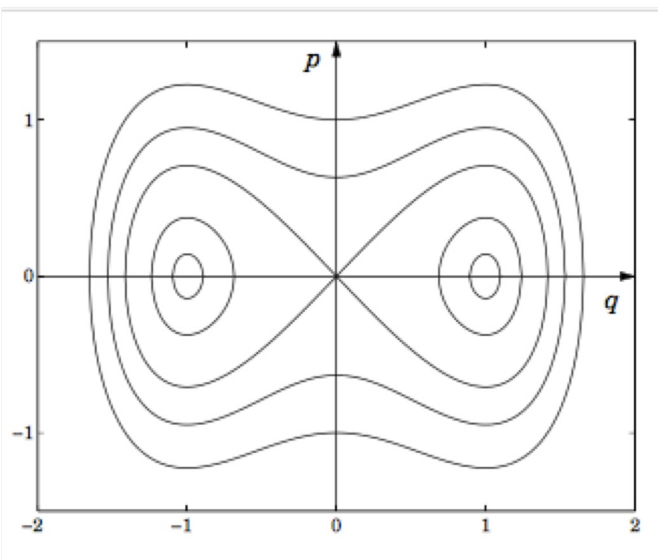
Clearly images of these two trajectories cross – but at different s!.



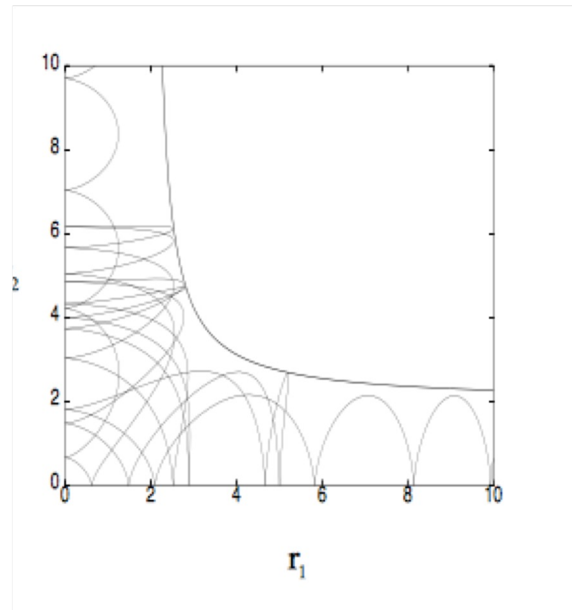
When Hamiltonian does not depend on *s*, situation is simpler and trajectories do not cross in the phase space. The argument is the same – trajectory is determined by the initial conditions and, in this case, simply shifted in *s*, but not in the phase space.

It does not true for motion in coordinate space – particle's trajectory can cross since at the same point they may have different momenta. The same is true for projection of phase diagram for 2D or 3D motion on any subset of coordinate and momenta $\{x, P_x, y, P_y\} \to \{x, y\}$ or $\{x, P_x\}$ or $\{y, P_y\}$…

(a)



(b)

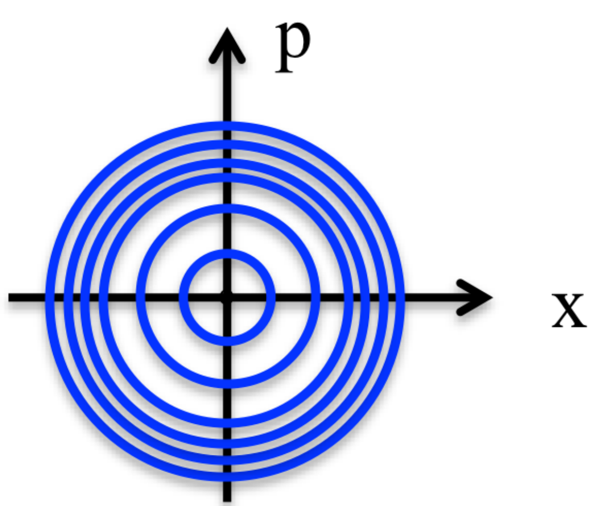


(a) Phase plot of decoupled motion with constant Hamiltonian – no crossing. A special unstable point at zero correspond to a stopping point – e.g. two trajectories approach each other but never cross!; (b) Projection of 4D phase-space trajectories on (q^1, q^2) coordinates – naturally they can cross.

Let's explore this case a notch further. For an oscillator Hamiltonian

$$H = \frac{p^2}{2} + \frac{x^2}{2} \tag{iv}$$

(use $H = \frac{p^2}{2m} + k\frac{x^2}{2}$ if you need more constants) – it just a set of boring concentric circles.



While one can have a lot of fun with the phase space trajectories, invariants (a property of Hamiltonian systems!) are even more important. The most important is conservation of the phase space volume. As we discussed in one of our lectures, any motion of Hamiltonian system is an equivalent to a Canonical transformation.

Maps. Let's follow particle' trajectory originated at an arbitrary point in the phase-space X_1 at s_1 and finishing at X_2 at s_2. Solution X_2 is unique and depends (in general case) on X_1, s_1 and s_2:

$$X_2(s_2) = F(X_1, s_1, s_2)$$

When X_1 runs through the entire phase space $\boldsymbol{R}^{2n}$, than the above equation is nothing that a function defined at the entire phase space. It is frequently called map, e.g. a transformation of the phase in the interval s_1 to s_2:

$$X(s_2) = \mathbb{M}(s_1|s_2)(X(s_1)) \equiv \mathbb{M}:X(s_1) \tag{5-14}$$

which can be locally linearized in proximity of any trajectory $X_o(s)$:

$$\delta X(s_2) = \mathbf{M}_{X_o}(s_1|s_2)\cdot \delta X(s_1) + O(\varepsilon^2)$$

$$\mathbf{M}_{X_o}(s_1|s_2|) = \left.\frac{\partial \mathbb{M}(s_1|s_2):X}{\partial X}\right|_{X=X_o} \tag{5-15}$$

As we discussed above, this matrix is symplectic. We will call map (155) symplectic if it is locally symplectic, e.g.

$$\mathbf{M}^T{}_{X_o}(s_1|s_2)\cdot \mathbf{S}\cdot \mathbf{M}_{X_o}(s_1|s_2) = \mathbf{S};\ \forall\ X_o, s_1, s_2 \tag{5-16}$$

Just to reinforce – any map generated by Hamiltonian motion, is symplectic.

Now, instead of talking about particle motion, we can consider transformation of various volumes in the phase space or transformation of functions, such as particle's density. First, let's consider a space phase volume (dimension 2n) occupied by particles having an arbitrary hyper-surface Ω. Then the hyper-surface can undergo and transformation, but it's the value of the volume inside

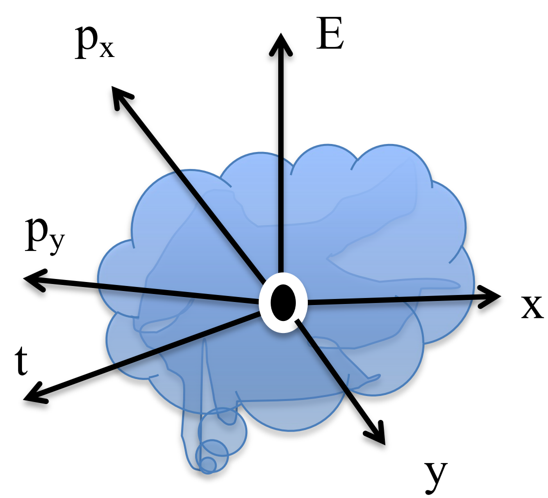
$$\int_{\Omega} \prod_{i=1}^{n} dq_i dP^i = inv \tag{5-17}$$

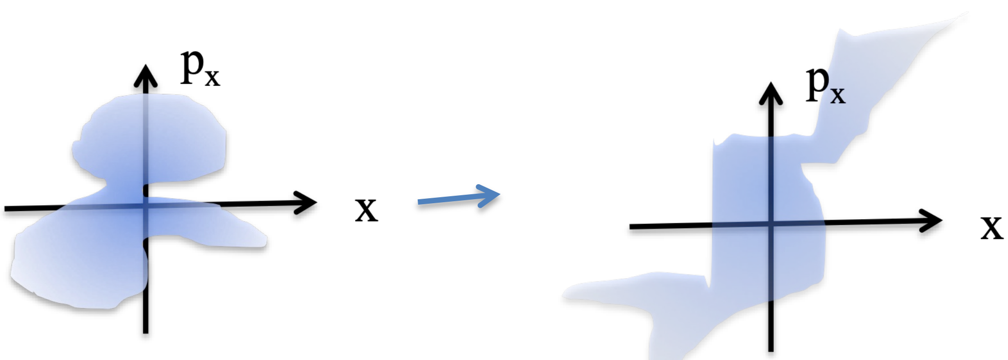
would not change – this is known as Liouville theorem. The prove is easy

$$\begin{gathered} V(s) = \int_{\Omega} \prod_{i=1}^{n} dq_i dP^i \equiv \int_{\Omega} dX(s) \equiv \int_{\Omega} dV(s) \\ V(s_2) = \int_{\Omega} dX(s_2) \equiv \int_{\Omega} \det \mathbf{M}(s_1|s_2) \cdot dX(s_1) = \int_{\Omega} dX(s_1) = V(s_1) \end{gathered} \tag{5-18}$$

where use the fact that transformation is symplectic.

"Poor man" attempt to draw 6D phase-plot of ensemble of particles





Totally non-linear map for 1D case – while boundary can change dramatically, the volume does not change.

Incompressible phase-space liquid

If particles do not decay or disappear in any other way (scatter on residual gas and fly away!), than number of particles inside any hyper-surface transforming according to the map (155) is preserved. Remember, that trajectories can not cross in the phase space – it also means that particle cannot cross a boundary which moving according to the particle's motion. In accelerator physics it is called water-bag. You can deform it, twist and turn, but cannot change its volume. The phase-space liquid is in-compressible.

It means that phase space density of an ensemble of particles is particles is invariant:

$$f(X,s) \underset{def}{=} \frac{dN}{dX^{2n}} \Rightarrow f(\mathbb{M}:X) \equiv f(X) \tag{5-19}$$

$$f\left(\mathbb{M}\left(s_1 \middle| s_2\right):X\left(s_1\right),s_2\right) \equiv f\left(X\left(s_1\right),s_1\right)$$

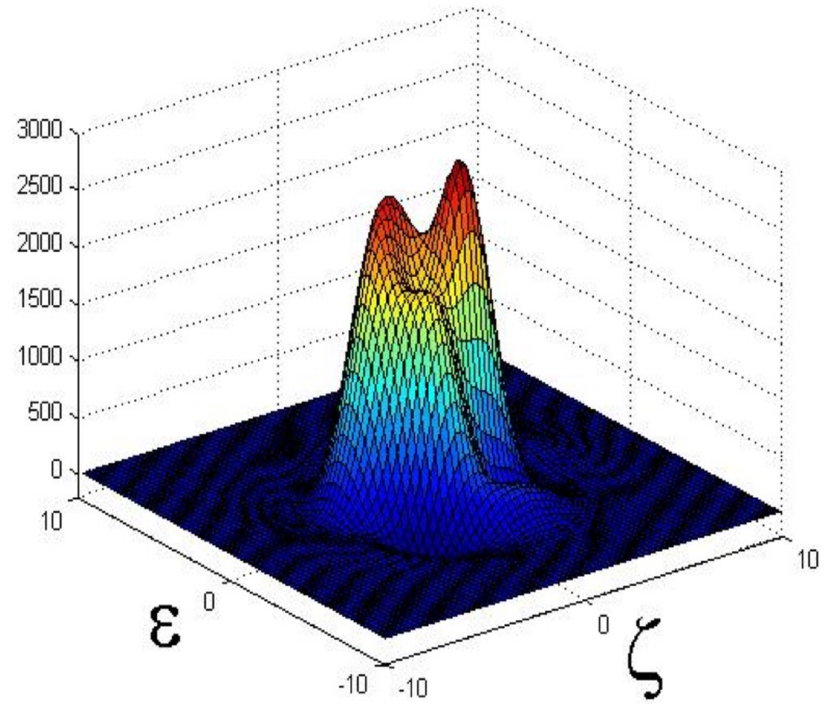
In other words, the phase space density is preserved along the trajectories. This is foundation for one of most used equation in accelerator and plasma physics – Vlasov equation:

$$\frac{df(X(s),s)}{ds} = \frac{\partial f(X,s)}{\partial s} + \frac{\partial f(X,s)}{\partial X}\frac{dX}{ds} = 0$$

$$\frac{dX}{ds} = \mathbf{S}\cdot\frac{\partial H(X,s)}{\partial X} \tag{5-20}$$

$$\frac{\partial f(X,s)}{\partial s} + \frac{\partial f(X,s)}{\partial X}\cdot\mathbf{S}\cdot\frac{\partial H(X,s)}{\partial X} = 0$$

It is also referred to as method of trajectories – now you know what it is about. We will return to this equation when study collective effects.



Distribution function of electron beam (in longitudinal plane) undergoing interaction in a storage ring FEL.

Invariants. *Note: we are using Canonical pairs, i.e. the traditional convention*

Since symplecticity of the map and corresponding matrices, there are $n(2n-1)$ total conditions. One of them is $det\mathbf{M}=1$ we already put in use. The rest of the invariants are called after French mathematician/physicist Poincaré.

The other invariants preserved by symplectic transformations were found by Poincaré and they are the sum of projections onto an appropriate manifold in two, four…. $(2n-2)$ dimensions. In integral form it is

$$\sum_i \iint dq^i dP_i = inv, \sum_{i \neq j} \iint \iint dq^i dP_i dq^j dP_j = inv.... \tag{5-21}$$

If you count the number of Poincaré invariants (including Liouville!) you should not be surprised to find that there is $n*(2n-1)$.

Why these invariants are important? is a very good question. The main reason is that frequently they can be useful to solve problem analytically – the same way as energy conservation completely solves problem in 1D potential. The other important reason is that they actually restrict what one can do with beams of particles, e.g. does not allow us to compress "waterbag".

The look of these invariants is deceivingly simple. Let just discuss one of them – sum of the projections on 2D surfaces for $n=2$ case, e.g. a classical accelerator problem with coupled transverse (x and y) motion:

$$\sum_{i=1}^{2} \iint dq_i dP^i = \iint dx dP_x + \iint dy dP_y = inv \tag{5-22}$$

It states that sum of projections of phase space volume onto two one dimensional "phase-plots" is invariant of motion. But in some cases, one of the projections can have negative value…. We will discuss this in more details later when discussing linear coupling.

To finish our first glance onto the phase space and phase space plots, let's focus on a simple case of time independent Hamiltonian for 1D motion

$$H = H(x,p)$$

It means that since the energy (Hamiltonian) is preserved, all possible trajectories are defined by particle energy level

$$H(x,p) = H_0 \to p = p(x, H_0)$$

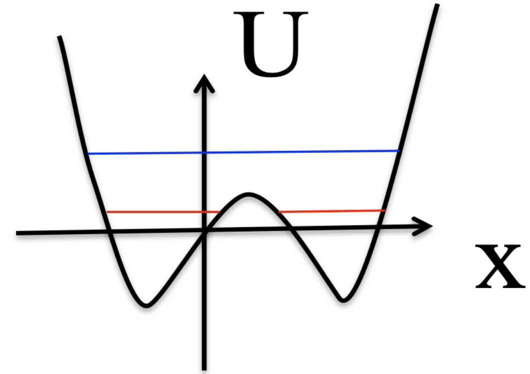
One should note that the above solution may have many branches – e.g. the function $p = p(x, H_0)$ is not unique and number of completely separate (not connected) trajectories can exist. A simplest example is $H = \frac{p^2}{2} + U(x)$ with potential in the figure below. While for high energies (blue) trajectory is unique, for lower energies (red) there are two distantly separated areas of the motion. Stationary points are playing very important role in phase diagram. They are naturally possible solution of Hamiltonian equations

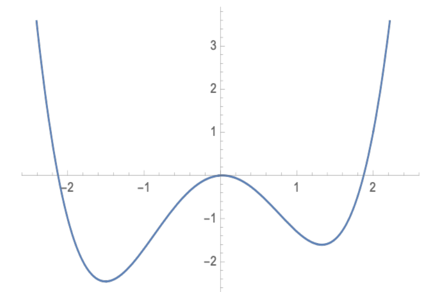
$$\frac{\partial H}{\partial q_i} = 0; \quad \frac{\partial H}{\partial P^i} = 0; \quad i = 1,\ldots,n$$

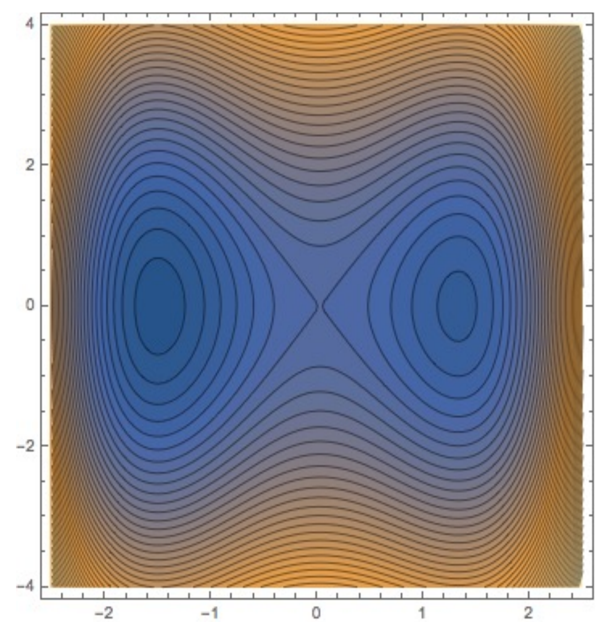
In general, they may exist or not. For 1D case above they are solution of a simple equation

$$p = 0; \quad \frac{dU}{dx} = 0;$$

If stationary point exists, it can be stable or unstable. Expanding Hamiltonian around the stationary point allows to define if solutions are stable (oscillatory) or unstable (exponentially or linearly growing, etc). Homework gives you a chance to explore phase space for 1D case.







Linear equations of motion – the same matrix equations for both conventions

We finished the accelerator Hamiltonian expansion by concluding that the first not-trivial term in the accelerator Hamiltonian expansion is a quadratic term of canonical momenta and coordinates. This Hamiltonian can be written in the matrix form (letting ***n*** be a dimension of the Hamiltonian system with *2n* phase space coordinates)

$$H = \frac{1}{2}\sum_{i=1}^{2n}\sum_{j=1}^{2n} h_{ik}(s)x_i x_k \equiv \frac{1}{2}X^T \cdot \mathbf{H}(s) \cdot X \qquad (5\text{-}23)$$

with the self-evident feature that a symmetric matrix can be chosen

$$\mathbf{H}^T = \mathbf{H} \qquad (5\text{-}24)$$

(to be exact, a quadratic form with any asymmetric matrix has zero value). The equations of motion are just a set of ***2n*** linear ordinary differential equations with ***s***-dependent coefficients:

$$\frac{dX}{ds} = \mathbf{D}(s)\cdot X;\ \mathbf{D}(s) = \mathbf{S}\cdot\mathbf{H}(s) \qquad (5\text{-}25)$$

One important feature of this system is that

$$Trace[\mathbf{D}] = 0 \qquad (5\text{-}26)$$

(the trivial proof is based on $Trace[\mathbf{A}\cdot\mathbf{B}] = Trace[\mathbf{B}\cdot\mathbf{A}]$; $Trace\left[\mathbf{A}^T\right] = Trace[\mathbf{A}]$ and $(\mathbf{S}\cdot\mathbf{H})^T = -\mathbf{H}\cdot\mathbf{S}$)

$$Trace[\mathbf{D}] = Trace\left[\mathbf{D}^{\mathbf{T}}\right] = Trace\left[(\mathbf{S}\cdot\mathbf{H})^{\mathbf{T}}\right] = -Trace\left[\mathbf{H}\cdot\mathbf{S}\right] = -Trace[\mathbf{D}] = 0,$$

i.e. the Wronskian determinant of the system (http://en.wikipedia.org/wiki/Wronskian) is equal to one. The famous Liouville theorem comes from well-known operator formula $\frac{d}{ds}det[\mathbf{W}(s)] = Trace[\mathbf{D}(s)]$; we do not need it here because we will have an easier method of proof. You also have it as a homework problem.

The solution of any system of first-order linear differential equations can be expressed through its ***2n*** initial conditions $\boldsymbol{X_o}$ at azimuth $\boldsymbol{s_o}$

$$X = X_o \tag{5-27}$$

through the transport matrix $\mathbf{M}(s_o|s)$:

$$X(s) = \mathbf{M}\left(s_o\middle|s\right)\cdot X_o \tag{5-28}$$

There are two simple proofs of this theorem. The first is an elegant one: Let us consider the matrix differential equation

$$\mathbf{M}' = \frac{d\mathbf{M}}{ds} = \mathbf{D}(s)\cdot\mathbf{M} \tag{5-29}$$

with a unit matrix as its initial condition at azimuth $\boldsymbol{s_o}$

$$\mathbf{M}\left(s_o\middle|s_o\right) = \mathbf{I} \tag{5-30}$$

Such solution exists* and then we readily see that

$$X(s) = \mathbf{M}\left(s_o\middle|s\right)\cdot X_o \tag{5-31}$$

satisfies equations of motions

$$\frac{dX}{ds} = \frac{d\mathbf{M}(s)}{ds}\cdot X_o = \mathbf{D}(s)\cdot\mathbf{M}(s)\cdot X_o = \mathbf{D}(\mathbf{s})\cdot X_o \quad \# \tag{5-32}$$

* *Mathematically, it is nothing else but*

$$\mathbf{M}(s) = \lim_{N\to\infty}\prod_{k=1}^{N}\left(\mathbf{I}+\mathbf{D}\left(s_k\right)\cdot\Delta s\right);\ \Delta s = \frac{s-s_o}{N};\ s_k = s_o + \Delta s\cdot(k-0.5)$$

A more traditional approach to the same solution is to use the facts that a) there exists a solution for a system of ordinary linear differential equation with arbitrary initial conditions (less-trivial statement); and b) any linear combination of the solutions also is a solution of eq. (165) (very trivial one). Considering a set of solutions of eq. (5-25) $C_k(s)$, k=1,…2n, with initial conditions at azimuth s_o, then

$$C_1(s_o) = \begin{bmatrix} 1 \\ 0 \\ \dots \\ 0 \end{bmatrix}; C_2(s_o) = \begin{bmatrix} 0 \\ 1 \\ \dots \\ 0 \end{bmatrix}; \dots C_{2n}(s_o) = \begin{bmatrix} 0 \\ 0 \\ \dots \\ 1 \end{bmatrix}; \quad \frac{d}{ds} C_k(s) = \mathbf{D}(s) \cdot C_k(s);$$

and their linear combination

$$X(s) = \sum_{k=1}^{2n} x_{k,0} \cdot C_k(s)$$

which satisfies the initial condition

$$X(s_o) = \sum_{k=1}^{2n} x_{k,0} \cdot C_k(s_o) = \begin{bmatrix} x_{1,0} \\ x_{2,0} \\ \dots, \\ x_{2n,0} \end{bmatrix}$$

Now, we recognize that this solution is nothing other than the transport matrix with matrix **M(s)** being a simple combination of $2n$ columns $C_k(s)$:

$$\mathbf{M}(s) = \left[C_k(s)\right] \equiv \left[C_1(s), C_2(s), \dots, C_{2n}(s)\right]$$

with obvious $\mathbf{M}(s_o|s_o) = \mathbf{I}$ at azimuth s_o.

Kinda boring but correct approach…

In differential calculus, the solution is defined as

$$\mathbf{M}(s) = \lim_{N\to\infty} \prod_{k=1}^{N} \left(\mathbf{I} + \mathbf{D}(s_k)\cdot \Delta s\right);\ \Delta s = \frac{s - s_o}{N};\ s_k = s_o + \Delta s \cdot (k - 0.5) \qquad (5\text{-}33)$$

The fact that the transport matrix for a linear Hamiltonian system has unit determinant (i.e., the absence of dissipation!)

$$det\mathbf{M}(s) = exp\left[\int_{so}^{s} Trace(\mathbf{D}(z))\mathrm{d}z\right] = 1 \qquad (5\text{-}34)$$

is the first indicator of the advantages that follow. Let us consider the invariants of motion characteristic of linear Hamiltonian systems, i.e., invariants of the symplectic phase space Starting from the bilinear form of two independent solutions of eq. (165), $X_1(s)$ and $X_2(s)$, *(it is obvious that* $X^T\mathbf{S}X=0$*)* we show that

$$X_2^T(s)\cdot\mathbf{S}\cdot X_1(s) = X_2^T(s_o)\cdot\mathbf{S}\cdot X_1(s_o) = inv \qquad (5\text{-}35)$$

The proof is straightforward

$$\frac{d}{ds}\left(X_2^T\cdot\mathbf{S}\cdot X_1\right) = X_2^T\cdot\left((\mathbf{S}\cdot\mathbf{H})^T\cdot\mathbf{S} + \mathbf{S}\cdot(\mathbf{S}\cdot\mathbf{H})\right)\cdot X_1 \equiv 0$$

Phase space is defined as the 2n-dimentional space of canonical variable $\{q_i, P^i\}$, that is, the space where this Hamiltonian system evolves.

Proving that transport matrices for Hamiltonian system are symplectic is very similar:

$$\mathbf{M}^T \cdot \mathbf{S} \cdot \mathbf{M} = \mathbf{S} \tag{5-36}$$

Beginning from the simple fact that the unit matrix is symplectic: $\mathbf{I}^T \cdot \mathbf{S} \cdot \mathbf{I} \equiv \mathbf{S}$, i.e. $\mathbf{M}(s_o|s_o) = \mathbf{I}$is symplectic, and following with the proof of (5-36)

$$\frac{d}{ds}\left(\mathbf{M}^\mathbf{T} \cdot \mathbf{S} \cdot \mathbf{M}\right) = \mathbf{M}^\mathbf{T} \cdot \left((\mathbf{S} \cdot \mathbf{H})^T \cdot \mathbf{S} + \mathbf{S} \cdot (\mathbf{S} \cdot \mathbf{H})\right) \cdot \mathbf{M} \equiv 0$$

Symplectic square matrices of dimensions *2n x 2n*, which include unit matrix **I**, create a symplectic group, where the product of symplectic matrices also is a symplectic matrix.

The symplectic condition (5-36) is very powerful and should not be underappreciated. Before going further, we should ask ourselves several questions: How can the inverse matrix of **M** be found? Are there invariants of motion to hold-on to? Can something specific be said about a real accelerator wherein there are small but all-important perturbations beyond the linear equation of motions?

**Group G is defined as a set of elements, with a definition of a product of any two elements of th group;* $P = A \bullet B \in G$; $A, B \in G$. *The product must satisfy the associative law* $A \bullet (B \bullet C) = (A \bullet B) \bullet C$; *there is an unit element in the grou* $I \in G; I \bullet A = A \bullet I = A : \forall A \in G$; *and inverse elements* $\forall A \in G; \exists B (called\ A^{-1}) \in G : A^{-1}A = AA^{-1} = I$.

Just a reminder…

As you probably surmised, the Hamiltonian method yields many answers and this why it is so vital to research. We can count them: The general transport matrix **M** (solution of $\mathbf{M}' = \mathbf{D}(s)\cdot\mathbf{M}$ with arbitrary **D**) has *(2n)*^{2} independent elements. Because the symplectic condition $\mathbf{M}^T\cdot\mathbf{S}\cdot\mathbf{M}-\mathbf{S}=\mathbf{0}$ represents an asymmetric matrix with n-diagonal elements equivalently being zeros, and the conditions above and below the diagonal are identical – then only the *n(2n-1)* condition remains and only the *n(2n+1)* elements are independent. For *n=1 (1D)* there is only one condition, for *n=2* there are 6 conditions, and n=3 (3D) there are *15* conditions. Are these facts of any use in furthering this exploration?

First, symplecticity makes the matrix determinant to be unit:

$$\det\left[\mathbf{M}^T(s)\cdot\mathbf{S}\cdot\mathbf{M}(s)\right]=\det\mathbf{S}\rightarrow\ (\det\mathbf{M}(s))^2=1\rightarrow\det M=\pm 1;$$

$$but\quad \det\mathbf{M}(0)=1\rightarrow det\mathbf{M}(s)=1\,\#$$

i.e., as we discussed before, it preserves the 2n-D phase space volume occupied by the ensemble of particles (system). The other invariants preserved by symplectic transformations are called Poincaré invariants and are the sum of projections onto the appropriate over- manifold in two, four…. (2n-2) dimensions:

$$\sum_i\int\prod_{i=1}^n dq^i dP_i = inv;\quad \sum_i\iint dq^i dP_i = inv;\quad ,\sum_{i\neq j}\iint\iint dq^i dP_i dq^j dP_j = inv\ldots.$$

The most important consequence of symplecticity is that we simple explicit expression for inverse matrix – to appreciate this try to write explicit expression for an arbitrary 6 x 6 matrix (you can start from 4x4 to appreciate the difficulty and ugliness of the result!).

Simply by multiplying symplecticity condition (5-36) from left by –**S** we get

$$-\mathbf{S}\cdot\mathbf{M}^{\mathbf{T}}\cdot\mathbf{S}\cdot\mathbf{M} = -\mathbf{S}^2 = \mathbf{I} \Rightarrow \mathbf{M}^{-1} = \mathbf{S}\cdot\mathbf{M}^{\mathbf{T}}\cdot\mathbf{S} \tag{5-37}$$

As an easy exercise for 2x2 symplectic (i.e. with unit determinant – see note below) matrices, you can show that $\mathbf{M} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ (5-37) gives $\mathbf{M} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$. It is a much less trivial task to invert 6x6 matrix; hence, the power of symplecticity allows us enact many theoretical manipulations that otherwise would be impossible. Obviously, and easy to prove, transposed symplectic and inverse symplectic matrices also are also symplectic:

$$\mathbf{M}^{-1\mathbf{T}}\cdot\mathbf{S}\cdot\mathbf{M}^{-1} = \mathbf{S};\ \mathbf{M}\cdot\mathbf{S}\cdot\mathbf{M}^{\mathbf{T}} = \mathbf{S};$$

$$\mathbf{M}^{-1} = -\mathbf{S}\cdot\mathbf{M}^{\mathbf{T}}\cdot\mathbf{S} \rightarrow \mathbf{M}\cdot\mathbf{M}^{-1} = -\mathbf{M}\cdot\mathbf{S}\cdot\mathbf{M}^{\mathbf{T}}\cdot\mathbf{S} = \mathbf{I} \rightarrow \mathbf{M}\cdot\mathbf{S}\cdot\mathbf{M}^{\mathbf{T}} = \mathbf{S}\ \#$$

$$\mathbf{M}^{-1T} = -(\mathbf{S}\cdot\mathbf{M}^{\mathbf{T}}\cdot\mathbf{S})^{\mathbf{T}} \rightarrow \mathbf{M}^{-1\mathbf{T}}\cdot\mathbf{S}\cdot\mathbf{M}^{-1} = \mathbf{S}\cdot(\mathbf{M}\cdot\mathbf{S}^3\cdot\mathbf{M}^{\mathbf{T}})\cdot\mathbf{S} = \mathbf{S}\ \# \tag{5-38}$$

The easiest is inversion for matrix in Dragt's convention:

$$\overline{X} = \begin{bmatrix} q \\ P \end{bmatrix} \rightarrow \overline{\mathbf{M}} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix};\ \overline{\mathbf{S}} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{I} & \mathbf{0} \end{bmatrix} \Rightarrow$$

$$\overline{\mathbf{M}}^{-1} = -\overline{\mathbf{S}}\cdot\overline{\mathbf{M}}^{\mathrm{T}}\cdot\overline{\mathbf{S}} = -\begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{I} & \mathbf{0} \end{bmatrix}\cdot\begin{bmatrix} \mathbf{A}^{\mathbf{T}} & \mathbf{C}^{\mathbf{T}} \\ \mathbf{B}^{\mathbf{T}} & \mathbf{D}^{\mathbf{T}} \end{bmatrix}\cdot\begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{I} & \mathbf{0} \end{bmatrix} = \begin{bmatrix} \mathbf{D}^{\mathbf{T}} & -\mathbf{B}^{\mathbf{T}} \\ -\mathbf{C}^{\mathbf{T}} & \mathbf{A}^{\mathbf{T}} \end{bmatrix} \tag{5-39}$$

Inversion in traditional convention is slightly less elegant but it offers number of additional results and is worth perusing. For example, matrix **M** can be represented as n^2 combinations of 2x2 matrices M_{ij}:

$$\underline{\mathbf{M}} = \begin{bmatrix} M_{11} & \dots & M_{1n} \\ \dots & \dots & \dots \\ M_{n1} & \dots & M_{nn} \end{bmatrix}; \ \underline{\mathbf{S}} = \begin{bmatrix} \boldsymbol{\sigma} & \dots & 0 \\ \dots & \dots & \dots \\ 0 & \dots & \boldsymbol{\sigma} \end{bmatrix}; \ \boldsymbol{\sigma} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix};$$

$$\underline{\mathbf{M}}^T \cdot \underline{\mathbf{S}} \cdot \underline{\mathbf{M}} = \begin{bmatrix} \sum_{i=1}^{n} M_{1i}{}^T \boldsymbol{\sigma} M_{1i} & \dots & \dots \\ \dots & \dots & \dots \\ \dots & \dots & \sum_{i=1}^{n} M_{ni}{}^T \boldsymbol{\sigma} M_{ni} \end{bmatrix} = \begin{bmatrix} \boldsymbol{\sigma} & \dots & 0 \\ \dots & \dots & \dots \\ 0 & \dots & \boldsymbol{\sigma} \end{bmatrix}; \qquad (5\text{-}40)$$

$$\sum_{i=1}^{n} M_{1i}{}^T \boldsymbol{\sigma} M_{1i} = \boldsymbol{\sigma} \cdot \sum_{i=1}^{n} \det M_{1i}.$$

We can demonstrate the requirement for the symplectic condition (5-38) is that the sum of determinants in each row of these 2x2 matrices is equal to one; the same is true for the columns:

$$\sum_{i=1}^{n} det\left[M_{ij}\right] = \sum_{j=1}^{n} det\left[M_{ij}\right] = 1 \qquad (5\text{-}42)$$

with a specific prediction for decoupled matrices, which are block diagonal:

$$\underline{\mathbf{M}} = \begin{bmatrix} M_{11} & 0 \dots 0 & 0 \\ \dots & \dots & \dots \\ 0 & 0 \dots 0 & M_{nn} \end{bmatrix}; \ det M_{kk} = 1 \qquad (5\text{-}43)$$

Other trivial and useful features are: for the columns

$$\underline{\mathbf{M}} = \begin{bmatrix} C_1 & C_2 & \ldots. & C_{2n-1} C_{2n} \end{bmatrix} \Rightarrow$$

$$C_{2k-1}{}^T \cdot \underline{\mathbf{S}} \cdot C_{2k} = -C_{2k}{}^T \cdot \underline{\mathbf{S}} \cdot C_{2k-1} = 1,\ k = 1,..,n \tag{5-44}$$

$$\textit{others are } 0$$

or lines of the symplectic matrix:

$$\underline{\mathbf{M}} = \begin{bmatrix} L_1 \\ L_2 \\ \ldots. \\ L_{2n-1} \\ L_{2n} \end{bmatrix} \Rightarrow -L_{2k} \cdot \underline{\mathbf{S}} \cdot L_{2k-1}{}^T = L_{2k-1} \cdot \underline{\mathbf{S}} \cdot L_{2k}{}^T = 1, \textit{others are } 0 \tag{5-45}$$

Still the most powerful conclusion for us is that

$$\mathbf{M}^{-1} = -\mathbf{S}\cdot\mathbf{M}^{\mathrm{T}}\cdot\mathbf{S} = -\begin{bmatrix} 0 & -1 & \ldots & 0 & 0 \\ 1 & 0 & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & 0 & -1 \\ 0 & 0 & .. & 1 & 0 \end{bmatrix} \begin{bmatrix} m_{11} & m_{21} & \ldots & m_{2n-1,1} & m_{2n,1} \\ m_{12} & m_{22} & \ldots & m_{2n-1,2} & m_{2n,2} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ m_{1,2n-1} & m_{2,2n-1} & \ldots & m_{2n-1,2n-1} & m_{2n,2n-1} \\ m_{1,2n} & m_{2,2n} & .. & m_{2n-1,2n} & m_{2n,2n} \end{bmatrix}$$

$$\mathbf{M}^{-1} = \begin{bmatrix} m_{12} & m_{22} & \ldots & m_{2n-1,2} & m_{2n,2} \\ -m_{11} & -m_{21} & \ldots & -m_{2n-1,1} & -m_{2n,1} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ m_{1,2n} & m_{2,2n} & \ldots & m_{2n-1,2n} & m_{2n,2n} \\ -m_{1,2n-1} & -m_{2,2n-1} & .. & -m_{2n-1,2n-1} & -m_{2n,2n-1} \end{bmatrix} \cdot \begin{bmatrix} 0 & -1 & \ldots & 0 & 0 \\ 1 & 0 & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & 0 & -1 \\ 0 & 0 & .. & 1 & 0 \end{bmatrix} =$$

$$\mathbf{M}^{-1} = \begin{bmatrix} m_{22} & -m_{21} & \ldots & m_{2n,2} & -m_{2n-1,2} \\ -m_{12} & m_{11} & \ldots & -m_{2n,1} & m_{2n-1,1} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ m_{2,2n} & -m_{1,2n} & \ldots & m_{2n,2n} & -m_{2n,2n-1} \\ -m_{2,2n-1} & m_{1,2n-1} & .. & -m_{2n-1,2n} & m_{2n-1,2n-1} \end{bmatrix}$$

e.g. as simple as it can go:

$$\left(\mathbf{M}^{-1}\right)_{2k-1,2j-1} = \left(\mathbf{M}\right)_{2j,2k}; \left(\mathbf{M}^{-1}\right)_{2k,2j} = \left(\mathbf{M}\right)_{2j-1,2k-1};\ j,k = 1,\ldots,n$$

$$\left(\mathbf{M}^{-1}\right)_{2k,2j-1} = -\left(\mathbf{M}\right)_{2j,2k-1}; \left(\mathbf{M}^{-1}\right)_{2k-1,2j} = -\left(\mathbf{M}\right)_{2j-1,2k};\ j,k = 1,\ldots,n$$

Anybody who tried to write analytical expression for inverse of 6x6 matrix would really appreciate this wonderful simplicity

What we learned today?

- Using 2n dimensional coordinate-momentum phase space with symplectic metric is a natural way of studying Hamiltonian systems
- Trajectory originated at two different points of phase space will never cross
- Volume of the phase space occupied by ensemble of particles frequently called emittance
  - More rigorous definition of emittance will be given later in the course
- Instead of studying individual particles trajectories, we can introduce, generally a non-linear, map of the phase space on itself
- This map is locally symplectic, which immediately gives us *n(2n-1)* invariants of motion
- A second order Hamiltonian generates *2n* *s*-dependent ordinary linear differential equation, which have a solution in form of linear *2nx2n* matrix, which is (naturally) is symplectic
- Symplecticity of matrix provides *n(2n-1)* conditions on matrix element and allows easily to write inverse matrix – the late is the major advantage for any analytical studies

In next class we will find a very general way of calculating the transport matrices