Transition crossing

Transition crossing occurs when the energy of the particle reaches the so called transition energy, and therefore the slip factor changes sign:

Slip factor: $\eta = \frac{1}{\gamma_t^2} - \frac{1}{\gamma_t}$

origin of γ_t: $\frac{\Delta L}{L} = \frac{1}{\gamma_t^2}\left(\frac{\Delta P}{P}\right)$ $\quad$ γ_t is the factor that relates fractional difference in travel distance for a non-synchronous particle & the synchronous particle to the difference in their energies as they travel between two accelerating section

Sign of η is important for the phase stability condition.

Recall

for ν_s to be real, $\eta \cos\phi_s$ must be less than zero,

i.e. if $\eta < 0$, $\gamma < \gamma_t$, motion is stable for $\cos\phi_s > 0$

if $\eta > 0$, $\gamma > \gamma_t$, motion is stable for $\cos\phi_s < 0$

Therefore, for acceleration to remain stable, transition crossing will force a change in the sign of $\cos\phi_s$ (the synchronous phase.)

Let's start the investigation of the effect of transition crossing on phase stability by writing equation 2.61 for a circular accelerator with the following conditions:

$\omega_{rf} = \frac{hv}{R}$; $R = \frac{\text{cimcumfrance}}{2\pi}$; $h =$ harmonicnumber

$v\tau = 2\pi R$

(2.61): $\frac{d^2\Delta\Phi}{dt^2} - \frac{1}{\eta/\tau}\frac{d}{dt}\left(\frac{\eta}{\tau}\right)\frac{d}{dt}\Delta\phi + \Omega_s^2\,\Delta\phi = 0$

Where,

$$\lambda = \frac{\eta \omega_{rf} c^2}{v^2 E_s}$$

$$\Omega_s = \left[- \frac{\eta \omega_{rf} c^2 eV \cos\phi_s}{\tau v^2 E_s} \right]^{\frac{1}{2}}$$

$$\therefore \frac{d^2 \Delta\phi}{dt^2} - \frac{\tau v^2 E_s}{\eta \omega_{rf} \cancel{c^2}} \frac{d}{dt} \left(\frac{\eta \omega_{rf} \cancel{c^2}}{v^2 E_s \tau} \right) \frac{d}{dt} \Delta\phi$$

$$- \frac{\eta \omega_{rf} c^2 eV \cos\phi_s}{\tau v^2 E_s} \Delta\phi = 0$$

$$\therefore \frac{d^2 \Delta\phi}{dt^2} - \frac{\cancel{2\pi R} \cdot \cancel{v} E_s}{\eta \cancel{hv}/\cancel{R}} \frac{d}{dt} \left(\frac{\eta \cancel{hv}/\cancel{R}}{\cancel{2\pi R} \cdot \cancel{v} E_s} \right) \frac{d}{dt} \Delta\phi$$

$$- \frac{\eta \, h \cancel{v}/R \, c^2 eV \cos\phi_s}{2\pi R \cdot \cancel{v} \, E_s}$$

$$\Rightarrow \frac{d^2 \Delta\phi}{dt^2} - \frac{1}{\eta / E_s} \frac{d}{dt} \left(\frac{\eta}{E_s} \right) \frac{d\Delta\phi}{dt} - \left(\frac{h c^2 eV \cos\phi_s}{2\pi R^2} \right) \left(\frac{\eta}{E_s} \right) \Delta\phi = 0 \qquad (2.77)$$

Suppose, we cross transition at $t=0$ such that

$d(\eta/E_s)/dt = K$, constant; i.e. η & E_s shift synchronously such that the ratio is continuous

$\Rightarrow \eta/E_s = Kt$

$eV|\cos\phi_s| =$ constant, i.e. synchronous phase switches signs, but maintains its value. This would allow the synchronous particle to get accelerated at the same rate before and after the transition crossing

$$\frac{d^2\Delta\phi}{dt^2} - \frac{1}{\cancel{\kappa} t}\cdot \cancel{\kappa}\,\frac{d\Delta\phi}{dt} - \frac{h c^2 e V \cos\phi_s}{2\pi R^2}\,\kappa t\,\Delta\phi = 0 \qquad (2.78)$$

Under the two assumptions above the equation of motion is exact. At large t, where we are far from transition, we expect to recover the results of two previous subsections. This is Because with transition occurring at time t=0, at large time, we will have the case of slowly varying energy, I.e. Adiabatic damping. If we cross transition holding only dE_s/dt constant, equation 2.78 remains a good approximation; i.e. it would no longer be exact.

Let's consider the cases of $t>0$ & $t<0$ separately

for $t>0$, we have crossed the transition energy, $\gamma>\gamma_t$, $\eta = \frac{1}{\gamma_t^2} - \frac{1}{\gamma^2} > 0$

i.e. for stability, $\cos\phi_s < 0$

So, $$\frac{d^2\Delta\phi}{dt^2} - \frac{1}{t}\frac{d\Delta\phi}{dt} + \left(\frac{\kappa h c^2 e V|\cos\phi_s|}{2\pi R^2}\right) t\,\Delta\phi = 0$$

for $t<0$, $\cos\phi_s>0$ & we define $t_- \equiv -t$,

$$\frac{d^2\Delta\phi}{dt^2} - \frac{1}{t_-}\frac{d\Delta\phi}{dt_-} + \left(\frac{\kappa h c^2 e V|\cos\phi_s|}{2\pi R^2}\right) t_-\,\Delta\phi = 0$$

The two equations are identical in form. These equations are a special form of the generalized Bessel differential equations

$$u'' + \frac{1-2a}{z}u' + \left[(q g z^{g-1})^2 + \frac{a^2 - \nu^2 g^2}{z^2}\right]u = 0$$

where $u = u(z)$ & $u' \equiv \frac{du}{dz}$ and has solutions

$$u(z) = z^a J_\nu(q z^g)$$

$$u(z) = z^a N_\nu(q z^g)$$

J_ν & N_ν are Bessel & Neumann functions of order ν; in our case, $a=1$, $g=\frac{3}{2}$, $\nu=\frac{3}{2}$, and

$$q = \sqrt{\frac{4hc^2eV\,|\cos\phi_s|\,k}{9\times 2\pi R^2}} \qquad (2.84)$$

Thus, the solution is

$$\Delta\phi(t) = \underset{\uparrow}{A}\,|t|\,J_{2/3}\left(q\,|t|^{3/2}\right) + \underset{\uparrow}{B}\,|t|\,N_{2/3}\left(q\,|t|^{3/2}\right)$$

constants of integration

These are the solutions that incorporate the response at transition. One can show that the asymptotic behavior of the solutions presented here will result in the same scaling laws as derived previously; specifically,

- Amplitude can be shown to scale as $\left(\eta/E_s\right)^{1/4}$
- Phase can be shown to scale as $\int \Omega_s\,dt$

Let's look at the behavior of the solution at transition, I.e., t=0

The J term goes to zero: $t\,J_{2/3}\left(q\,t^{2/3}\right) \longrightarrow t\,\dfrac{(q/2)^{2/3}\,t}{\Gamma(5/3)} \longrightarrow 0$

N term

$$t\,N_{2/3}\left(q\,t^{3/2}\right) \longrightarrow -t\,\frac{\Gamma(2/3)}{\pi\,(q/2)^{2/3}\,t} \longrightarrow -\frac{2^{2/3}\,\Gamma(2/3)}{\pi\,q^{2/3}}$$

$\therefore$ Transition solution:

$$\Delta\phi(0) = -\frac{2^{2/3}\,\Gamma\left(\frac{2}{3}\right)B}{\pi\,q^{2/3}} \qquad (2.95)$$

what about the behavior of ΔE

$$\tau \Delta E = \frac{2\pi R^2}{hc^2} \frac{1}{\eta/E_s} \frac{d\Delta\phi}{dt} \quad (2.96)$$

use identity $\quad J'_{2/3}(z) = -J_{5/3}(z) + \frac{2}{3}\frac{1}{z} J_{2/3}(z)$

we get
$$\frac{d\Delta\phi}{dt} = A\left[2 J_{2/3}(qt^{3/2}) - \frac{3}{2} q t^{3/2} J_{5/3}(qt^{3/2})\right] + B\left[2 N_{2/3}(qt^{3/2}) - \frac{3}{2} q t^{3/2} N_{5/3}(qt^{3/2})\right]$$

as it turns out, for $t \to 0$, only the first term will contribute and becomes
$$\tau \Delta E(0) = \frac{2^{1/3} A (2\pi R^2) q^{2/3}}{hc^2 \Gamma(\frac{5}{3}) K} \quad (2.104)$$

Notes

1. By selection of the initial condition, (A or B=0), a particular particle may reach $\Delta\phi = 0$ or $\Delta E = 0$.
2. $A = B = 0$ describe the synchronous particle, i.e. $\Delta\phi = 0$ forall time.
3. What about other particles that describe the ellipse in the $\Delta\phi - E$ phase space? It can be shown that the energy excursion (i.e. ΔE) for the particles in the phase space remains bounded, and the fractional energy difference $\Delta E/E$ can be expressed as

$$\frac{\Delta E_t / E_t}{\Delta E_i / E_i} = \frac{1}{\Gamma(5/3)} \left(\frac{v_t}{v_i}\right) \left[\frac{2^{11}\pi^7}{3^{14}} \frac{hc^2}{V_t^2} \frac{\cos\phi_s}{\sin^2\phi_s} \frac{E_i^9}{E_0^4 E_t^4 eV} |\eta_i|^3 \right]^{1/2}$$

subscript $t \to$ values at transition

subscript $i \to$ initial values, e.g. at injection.

$E_0 = mc^2$ is the rest energy of particle

voltage V } evaluated at transition.
synchronous phase ϕ_s }

* The procedure for arriving at this result is presented on the last pg of the lecture as an appendix

So $\frac{\Delta E}{E} \not\to \infty$, but how large does it get?

put in the numbers for main ring at Fermilab:

Take $h = 1113$, $\phi_s = 45°$, $\gamma_t = 18$, $eV = 1\,\text{MeV}$, $E_i = 9\,\text{GeV}$,

$$\frac{\Delta E_t / E_t}{\Delta E_i / E_i} = 1.11$$

This means that the energy spread at transition is about 11% larger than the energy spread at injection under these assumptions.

Need for transverse focusing

In this chapter, we saw how an RF field can be used to accelerate charged particles in such a way that stable oscillations about the design energy will be maintained.

But if these were the only sources, motion in at least one transverse dimension would be unstable. This because the cavity includes an azimuthal magnetic field:

$\bar{E} = E_0 J_0(\frac{\omega}{c}\rho)\hat{z} \Rightarrow$ accelerates electrons $\to$ increasing v_z

$\bar{B} = B_0 J_1(\frac{\omega}{c}\rho)\hat{\phi} \Rightarrow$ creates a transverse force on beam

$$\Rightarrow \bar{F}_\perp = q(\underset{0}{\cancel{E_\perp}} + \underbrace{v_z \times B_\phi}_{})$$

direction: $\hat{z} \times \hat{\phi} = \hat{r}$

The radial force on the particles during acceleration will need to be compensated for to ensure stability

Appendix: Procedure for finding maximum energy excursions at transition:

This value is calculated by studying an ensemble of particle on an ellipse in the longitudinal phase space in the adiabatic limit. For this ensemble, each particle will have a different initial conditions characterized by different values of A and B. But for all particles,

$$A^2 + B^2 = \text{same value},$$

This is in $\Delta\phi - E$ space at transition. $\Delta E \propto A$ and $\Delta\phi \propto B$

$\frac{\Delta\phi^2}{a^2} + \frac{E^2}{b^2} = 1$ is the ellipse eqn. Maximum excursion will occur therefore when $B=0$, i.e. amplitude of ΔE^2 is largest in the ellipse eqn.

Recall as $t \to \infty$

$$J_\nu \to \sqrt{\frac{2}{\pi z}} \cos\left(z - \frac{1}{2}\nu\pi - \frac{1}{4}\pi\right)$$

$$N_\nu \to \sqrt{\frac{2}{\pi z}} \sin\left(z - \frac{1}{2}\nu\pi - \frac{1}{4}\pi\right)$$

Therefore, for large t,

$$\tau \Delta E = \frac{3\pi R^2}{hc^2} \sqrt{\frac{2}{\pi}} \left[\frac{4hc^2 eV |\cos\phi_s| E_s}{9 \times 2\pi R^2 k^2 |\eta|}\right]^{1/4} \left[-A \sin\theta_{2/3} + B\cos\theta_{2/3}\right] \quad (2.105)$$

from solving eq. 96 & using $t \to \infty$ approximation

where, $\theta_{2/3} = \int \Omega_s \, dt - \frac{2}{3}\frac{\pi}{2} - \pi/4$

$$\Delta\phi = \sqrt{\frac{2}{\pi}} \left[\frac{9 \times 2\pi R^2 |\eta|}{4hc^2 eV |\cos\phi_s| E_s k^2}\right]^{1/4} \left[A\cos\theta_{2/3} + B \sin\theta_{2/3}\right]$$

For this particular particle (with the largest energy excursions), constant 'B' is set to zero. Therefore, variable 'A' characterizes the amplitude of energy oscillation for large values of 't' as well as for transition (where t=0). So we can compare maximum energy excursions at transition with maximum energy excursions at large t (both excursion are maximized for the same particle).

Set B to zero & you get $A_{Max} = \frac{(\tau \Delta E)_i}{f_i} = \frac{(\tau \Delta E)_t}{f_t}$

initial values $t \to -\infty$

transition values $t = 0$

$$f_i = \frac{3\pi R^2}{hc^2} \sqrt{\frac{2}{\pi}} \left[\frac{4 hc^2 eV |\cos\phi_s| E_s}{9 \times 2\pi R^2 K^2 \eta} \right]_i^{1/4}$$

← from $\lim_{t \to -\infty} (\tau \Delta E)$

$$f_t = \frac{2^{1/3} (2\pi R^2)\, q^{2/3}}{hc^2 \Gamma(\frac{5}{3}) K}$$

← from $\tau \Delta E(0)$

K at transition: $K = \frac{2eV \sin\phi_s}{\tau_t \gamma_t^2 E_t^2}$

Fractional energy values ($\Delta E / E$) is derived using these results:

$$\frac{\Delta E_t / E_t}{\Delta E_i / E_i} = \frac{1}{\Gamma(\frac{5}{3})} \left(\frac{v_t}{v_i} \right) \left[\frac{2^{11} \pi^7}{3^{14}} \frac{hc^2}{v_t^2} \frac{\cos\phi_s}{\sin^2\phi_s} \frac{E_i^9}{E_s^4 E_t^4 eV} |\eta_i|^3 \right]^{1/12}$$