

Problem 1. 3 x 5 points. Function of a Jordan block

(a) Show that powers of $m \times m$ Jordan block

$$\mathbf{G} = \begin{bmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \lambda \end{bmatrix}$$

are

$$\mathbf{G}^n = \begin{bmatrix} \lambda^n & C_1^n \lambda^{n-1} & C_2^n \lambda^{n-2} & \dots & C_k^n \lambda^{n-k} & C_{k+1}^n \lambda^{n-k-1} & \dots \\ 0 & \lambda^n & C_1^n \lambda^{n-1} & \dots & C_{k-1}^n \lambda^{n+1-k} & C_k^n \lambda^{n-k} & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & \lambda^n \end{bmatrix}; C_k^n = \frac{n!}{(n-k)!k!} \quad (1)$$

Suggestion: use $\mathbf{G}^0 = \mathbf{I}; \mathbf{G}^1 = \mathbf{G}$ - as first step, they satisfy (1). Then use induction assuming that (1) is correct for n and show that $\mathbf{G}^{n+1} = \mathbf{G} \cdot \mathbf{G}^n$ satisfy (1) for $n+1$. Use a well know ratio $C_k^{n+1} = C_k^n + C_{k-1}^n$.

(b) For a polynomial function $f(x) = \sum_{n=0}^N f_n x^n$ show that

$$f(\mathbf{G}) = \sum_{n=0}^N f_n \mathbf{G}^n = \begin{bmatrix} \sum_{n=0}^{\infty} f_n \lambda^n & \dots & \sum_{n=0}^{\infty} f_n C_k^n \lambda^{n-k} & \dots & \dots \\ 0 & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \sum_{n=0}^{\infty} f_n \lambda^n & \dots \end{bmatrix}$$

and $\sum_{n=0}^N f_n C_k^n \lambda^{n-k} = \frac{1}{k!} \frac{d^k f}{d\lambda^k} \#$

(c) Prove that for an arbitrary (well behaved function!) $f(x) = \sum_{n=0}^{\infty} f_n x^n$

$$(d) f(\mathbf{G}) = \begin{bmatrix} f(\lambda) & f'(\lambda)/1! & \dots & f^{(k)}(\lambda)/k! & \dots & f^{(n-1)}(\lambda)/(n-1)! \\ 0 & f(\lambda) & \dots & \dots & \dots & f^{(n-2)}(\lambda)/(n-2)! \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \dots & \dots & f'(\lambda)/1! \\ 0 & 0 & \dots & \dots & \dots & f(\lambda) \end{bmatrix}$$

Solution:

$$\mathbf{G}^0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \dots & 0 \\ 0 & 0 & & 1 \end{bmatrix}; \mathbf{G}^1 = \begin{bmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & \dots & 1 \\ 0 & 0 & & \lambda \end{bmatrix}$$

$$\mathbf{G}^2 = \begin{bmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & \dots & 1 \\ 0 & 0 & & \lambda \end{bmatrix} \cdot \begin{bmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & \dots & 1 \\ 0 & 0 & & \lambda \end{bmatrix} = \begin{bmatrix} \lambda^2 & \lambda & 1\dots & 0 \\ 0 & \lambda^2 & \lambda\dots & 0 \\ 0 & 0 & \dots & \lambda \\ 0 & 0 & 0 & \lambda^2 \end{bmatrix}$$

Induction: let assume it is correct for n:

$$\mathbf{G}^n = \begin{bmatrix} \lambda^n & n\lambda^{n-1}/1! & n(n-1)\lambda^{n-2}/2!\dots & \dots \\ 0 & \lambda^n & n\lambda^{n-1}/1! & \dots \\ 0 & 0 & \dots & n\lambda^{n-1}/1! \\ 0 & 0 & 0 & \lambda^n \end{bmatrix}$$

$$\mathbf{G}^{n+1} = \begin{bmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & \dots & 1 \\ 0 & 0 & & \lambda \end{bmatrix} \cdot \begin{bmatrix} \lambda^n & n\lambda^{n-1}/1! & n(n-1)\lambda^{n-2}/2!\dots & \dots \\ 0 & \lambda^n & n\lambda^{n-1}/1! & \dots \\ 0 & 0 & \dots & n\lambda^{n-1}/1! \\ 0 & 0 & 0 & \lambda^n \end{bmatrix} =$$

$$\begin{bmatrix} \lambda^{n+1} & (n+1)\lambda^n/1! & (n(n-1)+2n)\lambda^{n-1}/2!\dots & \dots \\ 0 & \lambda^{n+1} & (n+1)\lambda^n/1! & \dots \\ 0 & 0 & \dots & (n+1)\lambda^n/1! \\ 0 & 0 & 0 & \lambda^{n+1} \end{bmatrix}$$

$$\begin{bmatrix} \lambda^n & C_1^n \lambda^{n-1} & C_2^n \lambda^{n-2} & \dots & C_k^n \lambda^{n-k} & C_{k+1}^n \lambda^{n-k-1} & \dots \\ 0 & \lambda^n & C_1^n \lambda^{n-1} & \dots & C_{k-1}^n \lambda^{n+1-k} & C_k^n \lambda^{n-k} & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & \lambda^n \end{bmatrix}$$

$$\begin{bmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & \dots & 1 \\ 0 & 0 & & \lambda \end{bmatrix} \cdot \begin{bmatrix} \lambda^n & C_1^n \lambda^{n-1} & C_2^n \lambda^{n-2} & \dots & C_k^n \lambda^{n-k} & C_{k+1}^n \lambda^{n-k-1} & \dots \\ 0 & \lambda^n & C_1^n \lambda^{n-1} & \dots & C_{k-1}^n \lambda^{n+1-k} & C_k^n \lambda^{n-k} & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & \lambda^n \end{bmatrix} =$$

$$\begin{bmatrix} \lambda^{n+1} & (C_1^n + 1)\lambda^n & (C_2^n + C_1^n)\lambda^{n-2} & \dots & (C_k^n + C_{k-1}^n)\lambda^{n-k+1} & (C_{k+1}^n + C_k^n)\lambda^{n-k} & \dots \\ 0 & \lambda^{n+1} & (C_1^n + 1)\lambda^n & \dots & \dots & (C_k^n + C_{k-1}^n)\lambda^{n-k+1} & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & \lambda^{n+1} \end{bmatrix}$$

Using relation for polynomial coefficients: $C_k^{n+1} = C_k^n + C_{k-1}^n$; $C_k^n = n!/k!/(n-k)!$ proves the point for $n+1$. Hence, it is correct for all orders.

Hence, we can now calculate a polynomial functions or any function expandable into a Taylor series:

$$f(\mathbf{G}) = \sum_{n=0}^{\infty} f_n \mathbf{G}^n = \sum_{n=0}^{\infty} f_n \begin{bmatrix} \lambda^n & C_1^n \lambda^{n-1} & \dots & C_k^n \lambda^{n-k} & \dots & \dots \\ 0 & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & \dots & \lambda^n & \dots \end{bmatrix} = \begin{bmatrix} \sum_{n=0}^{\infty} f_n \lambda^n & \dots & \sum_{n=0}^{\infty} f_n C_k^n \lambda^{n-k} & \dots & \dots \\ 0 & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \dots & \sum_{n=0}^{\infty} f_n \lambda^n \end{bmatrix}$$

The final stroke is noting that

$$\begin{aligned} \sum_{n=0}^{\infty} f_n C_k^n \lambda^{n-k} &= \sum_{n=0}^{\infty} f_n \cdot \frac{n! \lambda^{n-k}}{k! (n-k)!} = \frac{1}{k!} \sum_{n=0}^{\infty} f_n \cdot \frac{n! \lambda^{n-k}}{(n-k)!} = \frac{1}{k!} \sum_{n=0}^{\infty} f_n \cdot \lambda^{n-k} \prod_{j=0}^{k-1} (n-j) \\ &= \frac{1}{k!} \frac{d^k}{d\lambda^k} \sum_{n=0}^{\infty} f_n \cdot \lambda^n = \frac{1}{k!} \frac{d^k f}{d\lambda^k} \# \end{aligned}$$

which gives final:

$$f(\mathbf{G}) = \begin{bmatrix} f(\lambda) & f'(\lambda)/1! & \dots & f^{(k)}(\lambda)/k! & f^{(n-1)}(\lambda)/(n-1)! \\ 0 & f(\lambda) & \dots & f^{(n-2)}(\lambda)/(n-2)! & \dots \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & f'(\lambda)/1! & \dots \\ 0 & 0 & \dots & f(\lambda) & \dots \end{bmatrix}$$