Problem 1. 3 x 5 points. Function of a Jordan block

(a) Show that powers of m x m Jordan block

$$\mathbf{G} = \begin{bmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & ... & 1 \\ 0 & 0 & 0 & \lambda \end{bmatrix}$$

are

$$\mathbf{G}^n = \begin{bmatrix} \lambda^n & C_1^n \lambda^{n-1} & C_2^n \lambda^{n-2} & ... & C_k^n \lambda^{n-k} & C_{k+1}^n \lambda^{n-k-1} & .. \\ 0 & \lambda^n & C_1^n \lambda^{n-1} & ... & C_{k-1}^n \lambda^{n+1-k} & C_k^n \lambda^{n-k} & \\ .... & .... & .... & ... & ... & ... & ... \\ 0 & 0 & 0 & 0 & 0 & 0 & \lambda^n \end{bmatrix}; C_k^n = \frac{n!}{(n-k)!k!} \quad (1)$$

Suggestion: use $\mathbf{G}^0 = \mathbf{I}; \mathbf{G}^1 = \mathbf{G}$ - as first step, they satisfy 1. Than use induction assuming that (1) is correct for ***n*** and show that $\mathbf{G}^{n+1} = \mathbf{G} \cdot \mathbf{G}^n$ satisfy (1) for **n+1**. Use a well know ratio $C_k^{n+1} = C_k^n + C_{k-1}^n$.

(b) For a polynomial function $f(x) = \sum_{n=0}^{N} f_n x^n$ show that

$$f(\mathbf{G}) = \sum_{n=0}^{N} f_n \mathbf{G}^n = \begin{bmatrix} \sum_{n=0}^{\infty} f_n \lambda^n ... & \sum_{n=0}^{\infty} f_n C_k^n \lambda^{n-k} & .... \\ 0 & ... & .... \\ 0 & 0 & \sum_{n=0}^{\infty} f_n \lambda^n \end{bmatrix}$$

and $\sum_{n=0}^{N} f_n C_k^n \lambda^{n-k} = \frac{1}{k!} \frac{d^k f}{d\lambda^k}$ #

(c) Prove that for an arbitrary (well behaved function!) $f(x) = \sum_{n=0}^{\infty} f_n x^n$

(d) $$f(\mathbf{G}) = \begin{bmatrix} f(\lambda) & f'(\lambda)/1! & ...f^{(k)}(\lambda)/k! & f^{(n-1)}(\lambda)/(n-1)! \\ 0 & f(\lambda) & ... & f^{(n-2)}(\lambda)/(n-2)! \\ ... & .... & ... & ... \\ 0 & 0 & ... & f'(\lambda)/1! \\ 0 & 0 & ... & f(\lambda) \end{bmatrix}$$