Nonlinear dynamics

Outline

- Examples for nonlinearities in particle accelerator
- Approaches to study nonlinear resonances
- Chromaticity, resonance driving terms and dynamic aperture

Nonlinearities in accelerator

In accelerator, to the lowest order of δ(the relative energy deviation), particles' motion is governed by transversely, the Hill's equations

$$x'' + K_x(s)x = \pm\frac{\Delta B_y}{B\rho}, \quad y'' + K_y(s)y = \mp\frac{\Delta B_x}{B\rho}$$

$$K_x(s) = \frac{1}{\rho^2} \mp \frac{B_1}{B\rho}, \qquad K_y(s) = \pm\frac{B_1}{B\rho}$$

and longitudinally, the pendulum's equation

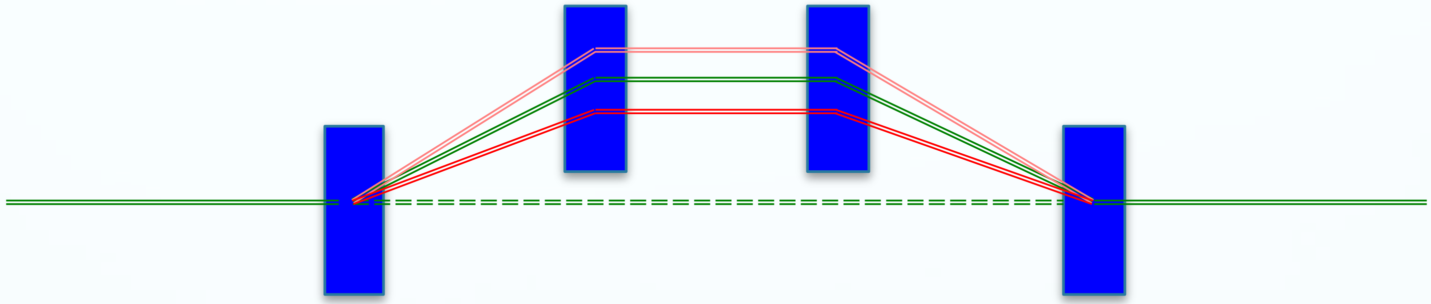
$$\dot{\delta} = \frac{\omega_0}{2\pi\beta^2 E} eV(\sin\phi - \sin\phi_s), \quad \dot{\phi} = h\omega_0\eta\delta$$

Both equations are nonlinear. In a modern accelerator, the particles' motions (both transverse and longitudinal) are highly nonlinear!

The nonlinearity may arise from nonlinear field error (usually resides in high field magnets),usage of higher order magnets (sextupoles, octupoles,etc), RF cavities etc. We will see their effects in the following examples.

Example 1: bunch compressor

Many modern light sources utilize a bunch compression system(chicane), composed of bending magnets, to perform bunch rotation in longitudinal phase space and reduce bunch length to achieve higher peak current.

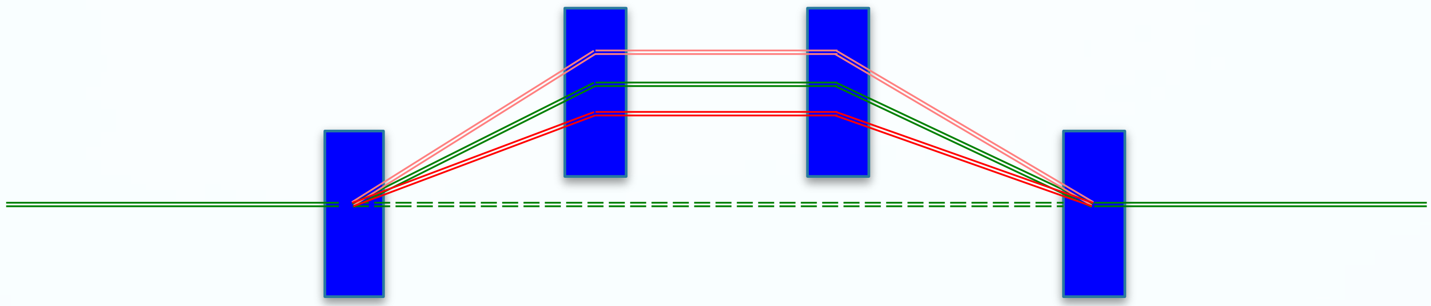


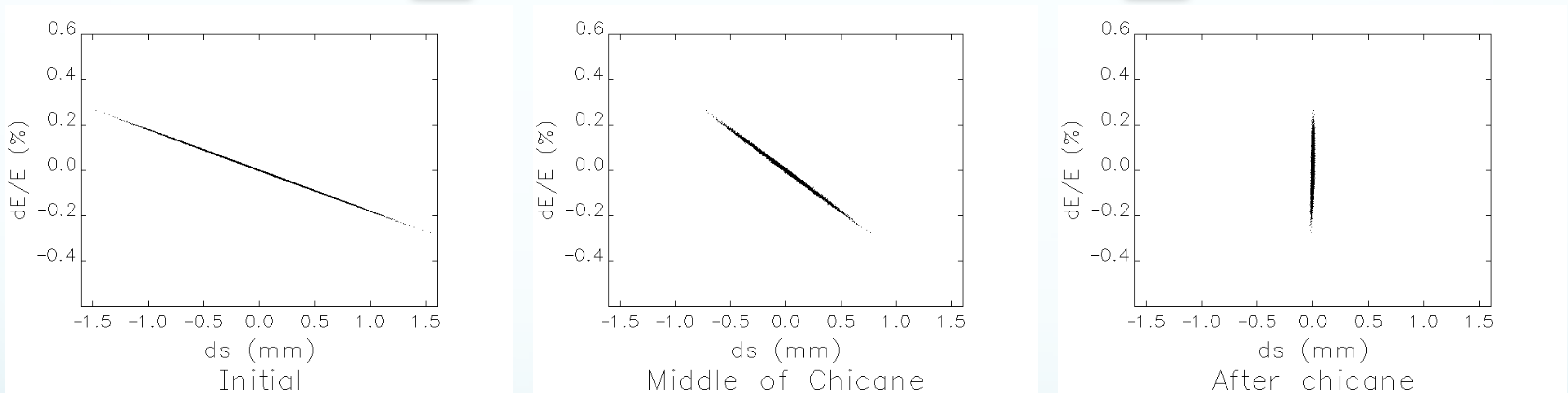
the strength of such system can be described by R_{56} (proportional to the contraction of bunch length)

$$R_{56} = -L\theta^2 - L_{dip}\theta^2 + O(\theta^4)$$

the system is composed of pure linear magnets and one would expect to see clean linear rotation in phase space

Example 1: bunch compressor



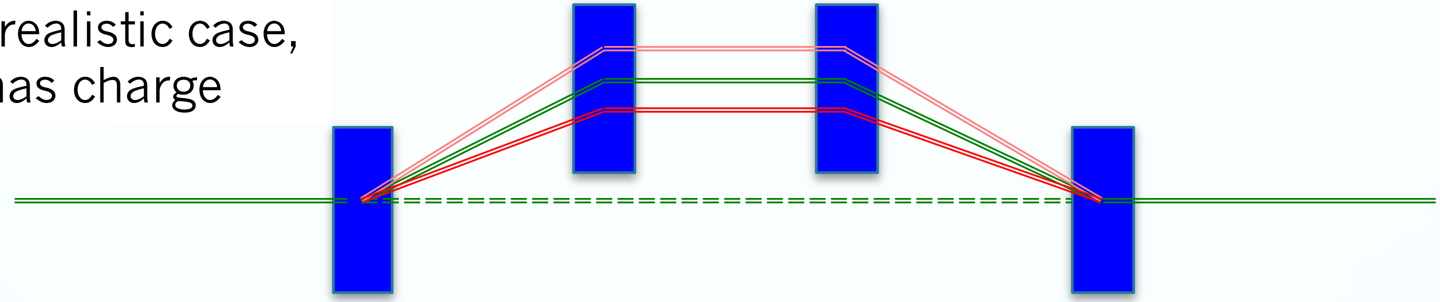


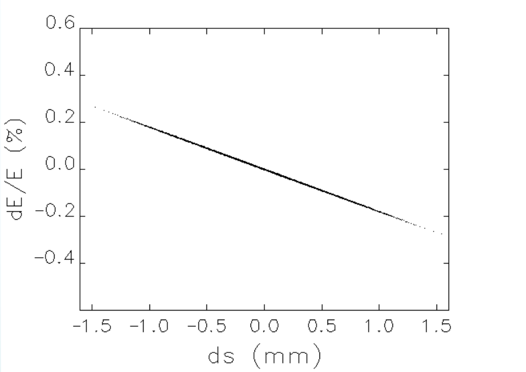
Without considering any nonlinear effects, an initially chirped bunch experiences linear rotation in chicane, resulting in a shorter bunch length at the exit of the chicane.

Is this real?

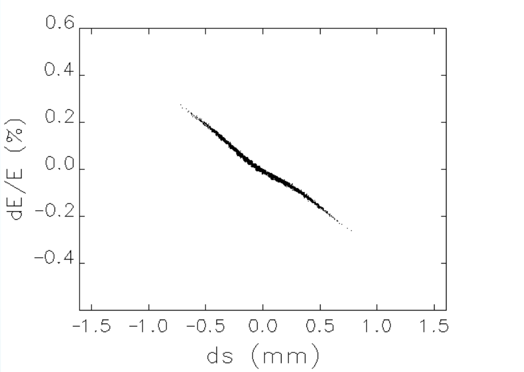
Example 1: bunch compressor

A more realistic case, bunch has charge

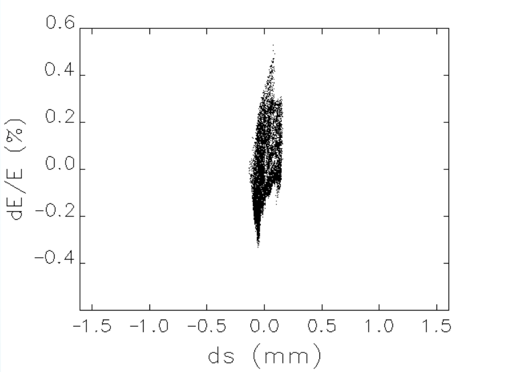




Initial



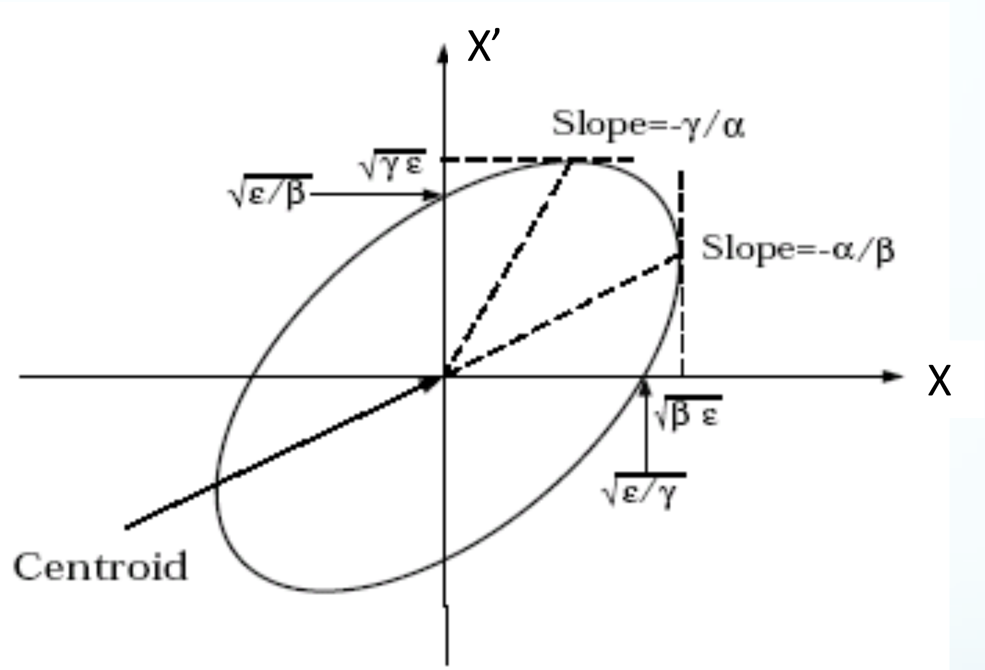
Middle of Chicane



After chicane

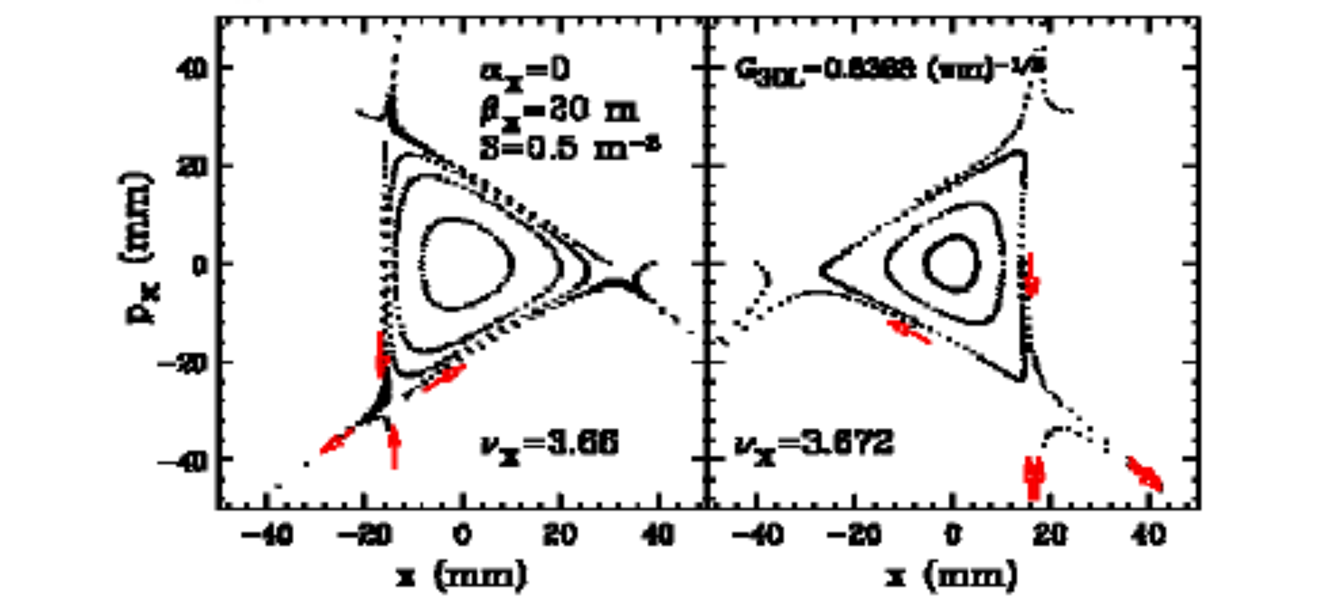
After add the wakefield into consideration, the nice linear bunch distribution becomes a birdie shape, which deteriorates the beam quality as well as results in making the chicane less efficient in bunch compression .

Example 2: storage ring



We know in a storage ring, a particle with action J possesses an elliptical motion in the phase space. Its tune determines how many turns it travels along the ellipse during one beam revolution. If we plot the phase space with normalized coordinates (x, P_x), it is a circle.

Example 2: storage ring



Tracking results show that with the existence of nonlinear magnets (sextupoles, for example), the ellipses in phase space deforms into a triangular shape. A stable region also forms where the particles inside are stable (confined in phase space)and particles outside are unstable and drifting in phase space (may cause real beam loss).

Nonlinearities in accelerator can't be avoided

From above examples, we can see:

① Nonlinear effects are important in many diverse accelerator systems, and can arise even in systems comprising elements that are often considered "linear".

② Nonlinear effects can occur in the longitudinal or transverse motion of particles moving along an accelerator beam line.

③ To understand nonlinear dynamics in accelerators we need to be able to construct dynamical maps for individual elements and complete systems and analyze these maps to understand the impact of nonlinearities on the performance of the system.

① If we have an accurate and thorough understanding of nonlinear dynamics in accelerators, we can attempt to mitigate adverse effects from nonlinearities.

Canonical transformation

Canonical transformation is a transformation from a set of canonical variables to another. For example, the new set of variables X is transformed by an existing set of canonical variables x by:

$$X = X(x) \qquad\qquad \frac{\partial X}{\partial x} = A, \quad and \quad A^T J A = J$$

the new set of variables obeys Hamilton's equations

$$\dot{X} = J \frac{\partial H}{\partial X}$$

and we call X canonical variables. Please note that from the definition of canonical transformation, it is naturally symplectic.

In accelerator physics, it is often convenient to transform the cartesian coordinates (x, px, y, py) into the action-angle variables (J, Φ).

Generating function

How to construct this canonical transformation?

The generating functions (e.g. 1st kind) are used to transform the coordinates q_i to Qi:

$$F_1 = F_1(q_i, Q_i, t)$$

thus the momenta conjugates read: $$p_i = \frac{\partial F_1}{\partial q_i}, \quad P_i = -\frac{\partial F_1}{\partial Q_i}$$

and the Hamiltonian becomes: $$\tilde{H} = H + \frac{\partial F_1}{\partial t}$$

We expect by applying this transformation, the Hamiltonian has simpler form as it is easier to solve. For example:

$H = p^2 + q^2 - 4pq^2 + 4q^2$ with $F_1 = qQ - 2q^3$ becomes $\tilde{H} = P^2 + Q^2$

A simple harmonic oscillator!!

Action-angle variables

The action angle variable (J, Φ) is defined as:

$$2J_z = \gamma_z z^2 + 2\alpha_z z z' + \beta_z z'^2,$$

$$\tan\phi_z = -\alpha_z - \beta_z \frac{z'}{z}$$

where (α,β,γ) are Twiss parameters.
The action angle variable is very important for linear beam dynamics. As we all know, for linear dynamics, it has properties

$$\frac{dJ_z}{ds} = 0, \quad \frac{d\phi_z}{s} = \frac{1}{\beta_z}$$

using a generating function $F_1(z,\phi_z) = -\frac{z^2}{2\beta_z}(\tan\phi_x + \alpha_x)$

and the Hamiltonian reduces to $H = \frac{J_z}{\beta_z}$ note this H is s dependent!

Action-angle variables

To study nonlinear dynamics, it is more useful to further construct a Hamiltonian that is s independent with canonical transformation. Consider a generating function of 2nd kind

$$F_2(\phi,\bar{J}) = (\phi - \int_0^s \frac{ds}{\beta} + \nu\theta)\bar{J}$$

where θ is the angle of reference orbit. The conjugate coordinates can be expressed as

$$\bar{\phi} = \phi - \int_0^s \frac{ds}{\beta} + \nu\theta, \quad \bar{J} = J$$

The new Hamiltonian becomes $\tilde{H} = H + \frac{\partial F_2}{\partial s} = \frac{\nu \bar{J}}{R}$

Further changing the coordinate from s to θ reduces the Hamiltonian to

$$\bar{H} = R\tilde{H} = \nu\bar{J} \qquad z = \sqrt{2\beta\bar{J}}\cos\Phi \qquad \Phi = \bar{\phi} + \int_0^s \frac{ds}{\beta} - \nu\theta = \bar{\phi} + \chi - \nu\theta$$

Treatments of nonlinearities

A number of powerful tools for analysis of nonlinear systems can be developed from Hamiltonian mechanics to describe the motion for a particle moving through a component in an accelerator beamline:(truncated) power series; Lie transform; (implicit) generating function.

Hamiltonian is usually not integrable. However, if the Hamiltonian can be written as a sum of integrable terms, an explicit symplectic integrator that is accurate to some specified order can be constructed to solve the system.

For a storage ring, We mainly discuss two approaches to analyze nonlinear dynamics:

1. Canonical perturbation method where nonlinear terms are treated as perturbation to the linear Hamiltonian (may not give correct pictures when nonlinear magnets are strong)
2. Normal form analysis, based on Lie transformation of the one-turn map (especially useful when dealing with resonance driving terms and dynamic aperture problems)

Perturbation treatment

The Hamiltonian for a linear system in action angle variable (J, Φ):

$$H = \nu J$$

the nonlinear elements' contribution can be written as

$$H = \nu J + \varepsilon V(\phi, J, s) = H_0 + \varepsilon V(\phi, J, s)$$

where ε is a small parameter. Please note that the perturbation V from nonlinear element is also a periodic function of the circumference L. Thus it is usually convenient to express it in terms of a sum over different orders:

$$V(\phi, J, s) = \sum_m V_m(J, s) e^{im\phi}$$

and treat them order by order (m being the order of nonlinear term).

Perturbation treatment for quadrupole error

Lets first apply it to the linear case (taking a quadrupole error as an example). Assume we have a small quadrupole field error k(s), the Hamiltonian (for horizontal motion) reads:

$$H = \frac{1}{2}\left(x'^2 + K_x x^2\right) + \frac{k(s)x^2}{2}$$

If transformed into action angle variables, it reads: $x = \sqrt{2\beta(s)J}\cos\Phi$

$$H = \frac{J}{\beta(s)} + \frac{1}{2}k(s)\beta(s)J(1+\cos 2\Phi) = H_0 + \frac{1}{2}k(s)\beta(s)J\cos 2\Phi$$

thus the term H_0 (independent of Φ) is $H_0 = \frac{J}{\beta(s)} + \frac{1}{2}k(s)\beta(s)J$

and the tune becomes $\nu = \frac{1}{2\pi}\int \frac{dH}{dJ}ds = \frac{1}{2\pi}\int\left(\frac{1}{\beta(s)} + \frac{1}{2}k(s)\beta(s)\right)ds$

The change of tune $\Delta\nu = \frac{1}{4\pi}\int k(s)\beta(s)\,ds$ ✔

Perturbation treatment for sextupole

We can follow the same procedure to deal with the Hamiltonian for sextupoles:

$$H = \frac{1}{2}\left(x'^2 + K_x x^2 + y'^2 + K_y y^2\right) + \frac{1}{6}k_2(s)\left(x^3 - 3xy^2\right)$$

where k2(s) is the sextupole gradient. Transform it into action-angle variables, we have

$$x = \sqrt{2\beta_x J_x}\cos\Phi_x \qquad y = \sqrt{2\beta_y J_y}\cos\Phi_y \qquad \Phi = \phi + \chi - \nu\theta$$

$$V = \frac{1}{6}k_2(s)\left(2\sqrt{2}\beta_x^{3/2}J_x^{3/2}\cos^3\Phi_x - 6\sqrt{2}\beta_x^{1/2}\beta_y J_x^{1/2}J_y\cos\Phi_x\cos^2\Phi_y\right)$$

Using trigonometry

$$\cos^3\phi = \frac{\cos 3\phi + 3\cos\phi}{4}, \quad \cos^2\phi = \frac{\cos 2\phi + 1}{2}$$

It becomes

$$V = \frac{\sqrt{2}}{12}k_2(s)\beta_x^{3/2}J_x^{3/2}(\cos 3\Phi_x + 3\cos\Phi_x)$$
$$-\frac{\sqrt{2}}{4}k_2(s)\beta_x^{1/2}\beta_y J_x^{1/2}J_y\left(2\cos\Phi_x + \cos(\Phi_x + 2\Phi_y) + \cos(\Phi_x - 2\Phi_y)\right)$$

Perturbation treatment for sextupole

From here we already see the contribution of a sextupole to different frequencies (tunes). To see sextupole's different modes' contribution, it is convenient to expand the perturbed potential in Fourier series as stated earlier

$$G = \frac{1}{2\pi}\sum_{l}\int V_m(J,s')e^{i(m\chi - m\nu\theta + l\theta)}\,ds'$$

where G is the Fourier transform of the perturbed potential induced by sextupoles. Note that this integral take out the χ and ν from the expression of Hamiltonian.

This G can be evaluated order by order, e.g. $G_{3,0,l}$ (which is correspondent to $3\nu_x = l$ resonance) reads:

$$G_{3,0,l} = \frac{\sqrt{2}}{24\pi}\oint k_2(s)\beta_x^{3/2}e^{i(3\chi_x - 3\nu_x\theta + l\theta)}ds$$

Perturbation treatment for sextupole

The Hamiltonian (in orbit angle θ) can be written as

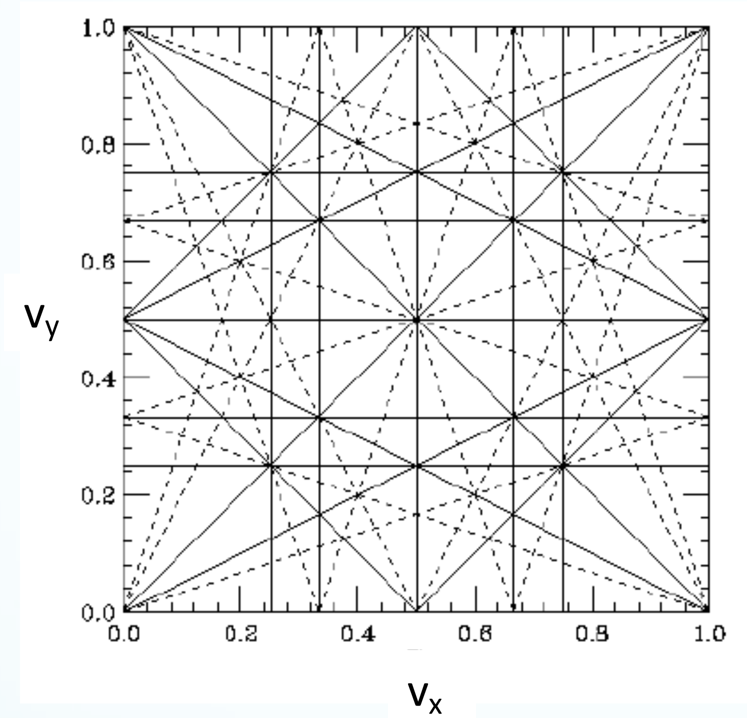
$$H = \nu_x J_x + \nu_y J_y + \sum_l G_{3,0,l} J_x^{3/2} \cos(3\phi_x - l\theta)$$
$$+ \sum_l G_{1,2,l} J_x^{1/2} J_y \cos(\phi_x + 2\phi_y - l\theta) + \sum_l G_{1,-2,l} J_x J_y^{1/2} \cos(\phi_x - 2\phi_y - l\theta) + \dots$$

where G's drive the correspondent resonances and … drives parametric resonance $\nu_x = l$

Table 2.3: Resonances due to sextupoles and their driving terms

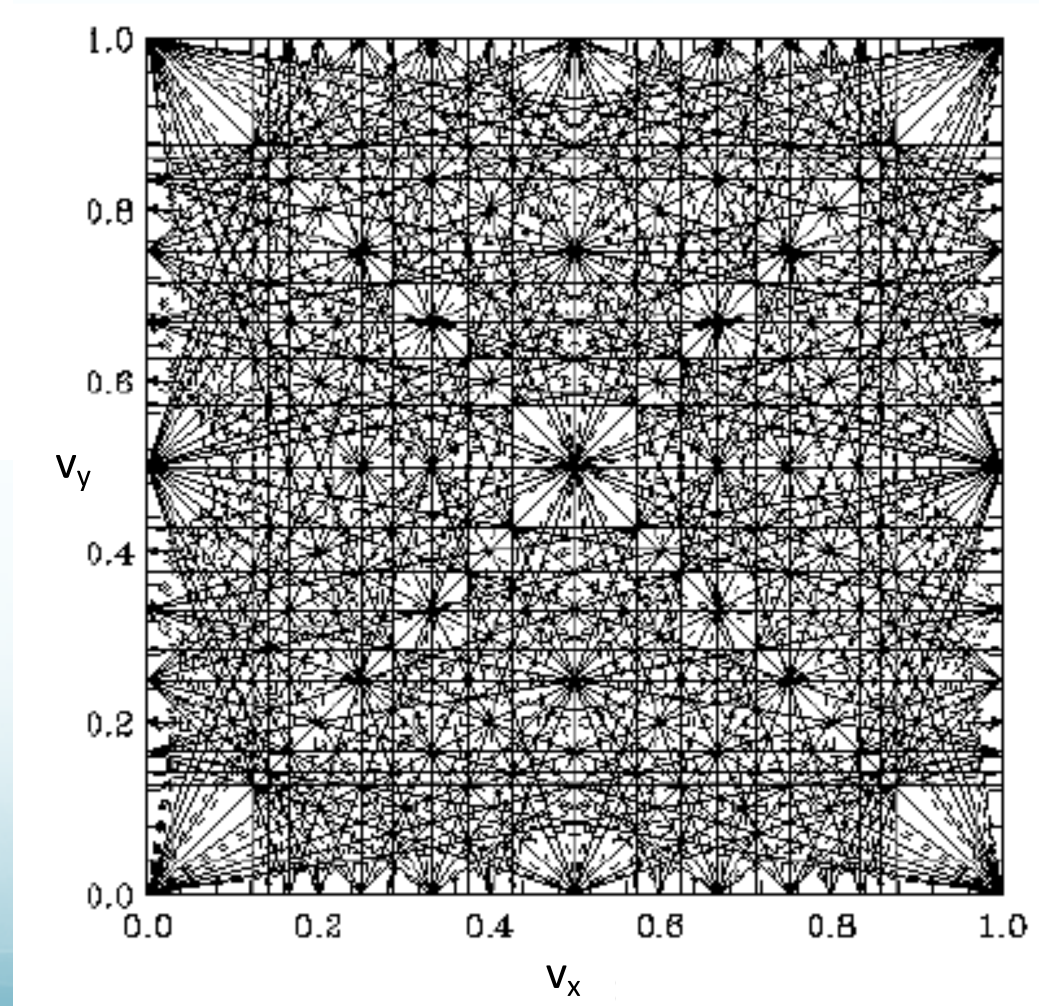
Resonance	Driving term	Lattice	Amplitude	Classification
$\nu_x + 2\nu_z = \ell$	$\cos(\Phi_x + 2\Phi_z)$	$\beta_x^{1/2}\beta_z$	$J_x^{1/2}J_z$	sum resonance
$\nu_x - 2\nu_z = \ell$	$\cos(\Phi_x - 2\Phi_z)$	$\beta_x^{1/2}\beta_z$	$J_x^{1/2}J_z$	difference resonance
$\nu_x = \ell$	$\cos\Phi_x$	$\beta_x^{1/2}\beta_z;\ \beta_x^{3/2}$	$J_x^{1/2}J_z,\ J_x^{3/2}$	parametric resonance
$3\nu_x = \ell$	$\cos 3\Phi_x$	$\beta_x^{3/2}$	$J_x^{3/2}$	parametric resonance

Resonance lines in tune space



Up to 4th order

Up to 8th order



Fixed points and separatrix

Stable and unstal pace where
particle can stay on). Considering
the mode

$= \phi_x - \frac{l}{3}\theta, \quad J = J_x$

The Hamiltonian

proximity

Solve for unstable

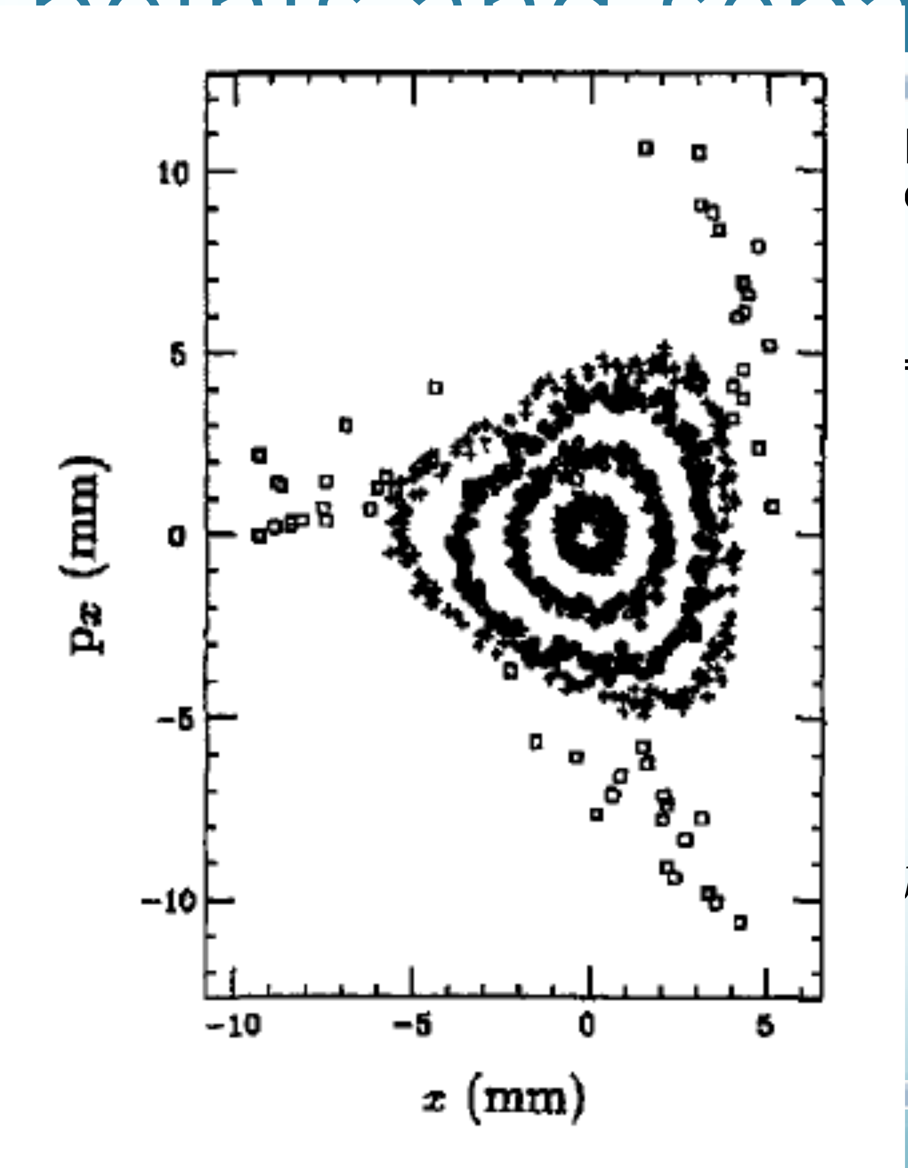
Gives 3 solutions

$if \quad \delta / G < 0$

$f \quad \delta / G > 0$

UFPs define sepa

Triangle changes direction w
at different sides of resonan



Tracking of sextupole

If sextupole can be treated as thin length (usually true with large radius R), the tracking of a particle dynamics in existence of sextupole magnets can be treated as a one turn map and an instantaneous kick. Starting from Hill's equation

$$x'' + K_x(s)x = \frac{1}{2}S(s)(x^2 - y^2), \qquad y'' + K_y(s)y = -S(s)xy$$

The change in the derivatives of coordinates can be written as

$$\Delta x' = \frac{1}{2}\int S(s)(x^2 - y^2)ds = \frac{1}{2}\bar{S}(x^2 - y^2), \qquad \Delta y' = -\int S(s)xyds = -\bar{S}xy$$

Given the initial particle distribution, the Poincare maps can be obtained by long term tracking applying the one turn map and instant kick in x',y'.

Normal form treatment

Instead of describing the dynamics in a beam line using an s-dependent Hamiltonian, we can construct a map, for example, in the form of a Lie transformation. Such a map may be constructed by concatenating the maps for individual elements. The beam dynamics (for example, the strengths of different resonances) may then be extracted from the transformation.

To better understand the concept of map (transformation), we take a look at the well-known linear transport matrix for a periodic accelerator (say, a storage ring)

$$\mathrm{M}=\begin{pmatrix}\cos\Phi+\alpha\sin\Phi & \beta\sin\Phi\\ -\gamma\sin\Phi & \cos\Phi-\alpha\sin\Phi\end{pmatrix},\ \beta\gamma=1+\alpha^2$$

the matrix is symplectic.

Normal form analysis of a linear system involves finding a transformation to variables in which the map appears as a pure rotation.

Normal form treatment

Consider matrix

$$N = \begin{pmatrix} \frac{1}{\sqrt{\beta}} & 0 \\ \frac{\alpha}{\sqrt{\beta}} & \sqrt{\beta} \end{pmatrix}$$

We find that NMN^{-1}

$$= \begin{pmatrix} \frac{1}{\sqrt{\beta}} & 0 \\ \frac{\alpha}{\sqrt{\beta}} & \sqrt{\beta} \end{pmatrix} \begin{pmatrix} \cos\Phi + \alpha\sin\Phi & \beta\sin\Phi \\ -\gamma\sin\Phi & \cos\Phi - \alpha\sin\Phi \end{pmatrix} \begin{pmatrix} \sqrt{\beta} & 0 \\ \frac{\alpha}{\sqrt{\beta}} & \frac{1}{\sqrt{\beta}} \end{pmatrix}$$

$$= \begin{pmatrix} \cos\mu & \sin\mu \\ -\sin\mu & \cos\mu \end{pmatrix} = R$$

Becomes a pure rotation in phase space.

Normal form treatment

The coordinates are "normalized" $\vec{x}_N = N\vec{x}$

And the normalized coordinates transform in one revolution as

$$\vec{x}_N \rightarrow NM\vec{x} = NMN^{-1}N\vec{x} = RN\vec{x} = R\vec{x}_N$$

Is simply a rotation in phase space.

Note that since the transformation N is symplectic, the normalized variables are canonical variables.

Normal form treatment

The treatment of nonlinear dynamics follows the same procedure however more complicated.

We can assume the map can be represented by a Lie transformation and factorized as

$$\mathbf{M} = R e^{:f_3:} e^{:f_4:} \cdots$$

Where f3 is a homogeneous polynomial of order 3 of the phase space coordinates and f4 is a homogeneous polynomial of order 4. The detailed order depends on the truncation.

The linear part of the map can be written in action angle variables as

$$R = e^{:-\mu J:}$$

Normal form treatment

To simplify this map, i.e., separate the contribution from different orders, we can construct a map M3

$$U = e^{:F_3:} M e^{:-F_3:}$$

Where F3 is a generator that removes resonance driving terms from $e^{:f_3:}$

So we have

$$U = e^{:F_3:} R e^{:f_3:} e^{:f_4:} e^{:-F_3:} = R R^{-1} e^{:F_3:} R e^{:f_3:} e^{:-F_3:} e^{:F_3:} e^{:f_4:} e^{:-F_3:}$$

Using relation $$e^{:h:} e^{:g:} e^{:-h:} = e^{:e^{:h:} g:}$$

$$U = R e^{:R^{-1} F_3:} e^{:f_3:} e^{:-F_3:} e^{:e^{:F_3:} f_4:}$$

Normal form treatment

Using Baker-Campbell-Hausdorff formula

$$e^{:A:}e^{:B:} = e^{:C:}, \quad where \quad C = A + B + \frac{1}{2}[A,B] + \cdots$$

The map now becomes

$$U = Re^{:R^{-1}F_3 + f_3 - F_3 + O(4):} e^{:e^{:F_3:} f_4:}$$

We can further reduce it to (non-trivial)

$$U = Re^{:f_3^{(1)}:} e^{:f_4^{(1)}:} = Re^{:R^{-1}F_3 + f_3 - F_3:} e^{:f_4^{(1)}:}$$

Where $f_3^{(1)} = R^{-1}F_3 + f_3 - F_3$ contains all the 3rd order contribution.

Normal form treatment

Thus the solution is

$$F_3 = \frac{f_3 - f_3^{(1)}}{I - R^{-1}}$$

Since f3 is periodic in the angle variable Φ, we can write

$$f_3 = \sum_m \bar{f}_{3,m}(J) e^{im\phi}$$

We can construct a f3(1) that does not have phase dependence, i.e., we can write it as

$$f_3^{(1)} = \bar{f}_{3,0}(J)$$

Thus now the generation function F3 reads

$$F_3 = \sum_{m \neq 0} \frac{\bar{f}_{3,m}(J) e^{im\phi}}{1 - e^{-im\mu}}$$

Normal form treatment

Taking Octupole as an example (assume it is the only nonlinear element in the beam line), we can write the map as

$$\mathbf{M} = Re^{:f_4:}$$

where f4 is

$$f_4 = -\frac{1}{24}k_3 l x^4$$

Rewrite it in action-angle variables $x = \sqrt{2\beta J}\cos\Phi$

$$f_4 = -\frac{1}{6}k_3 l\beta^2 J^2\cos^4\Phi = -\frac{1}{48}k_3 l\beta^2 J^2(3 + 4\cos 2\Phi + \cos 4\Phi)$$

Thus the generation function for normalized map $f_{4,0}$ reads $f_{4,0} = -\frac{1}{16}k_3 l\beta^2 J^2$

And the normalized map becomes (with BCH theorem)

$$\mathbf{M}_4 = Re^{:f_{4,0}:} = e^{:-\mu J - \frac{1}{16}k_3 l\beta^2 J^2:}$$

Normal form treatment

Thus the mapping of action-angle variables becomes

$$J \to J$$
$$\Phi \to \Phi + \mu + \frac{1}{8} k_3 l \beta^2 J$$

In other words, we see the tune shift with amplitude right away.

Similar to previous case for sextupole, we have

$$\mathbf{M}_4 = Re^{:f_{4,0}:} = e^{:-\mu J - \frac{1}{16} k_3 l \beta^2 J^2:} \doteq e^{:F_4:} M e^{:-F_4:}$$

Last equation is valid if we keep the normalization up to 4th order.

We can obtain the normalization generator F_4 easily

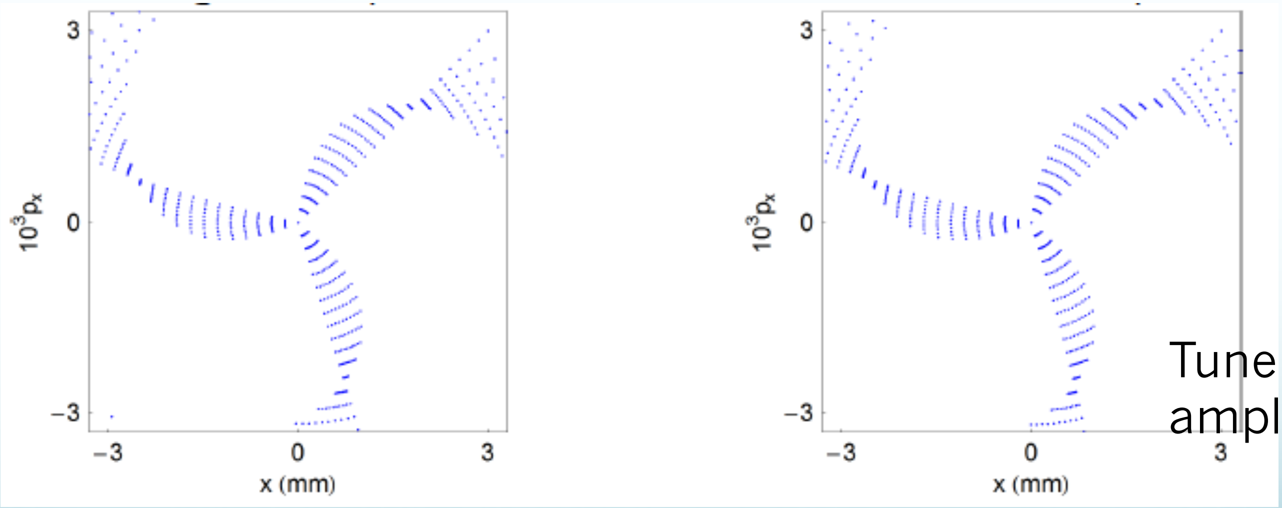
$$F_4 = \sum_{m \neq 0} \frac{f_{4,m}(J) e^{im\phi}}{1 - e^{-im\mu}}$$

$$F_4 = -\frac{1}{96} k_3 l \beta^2 J^2 \left(\frac{4[\cos 2\Phi - \cos 2(\Phi + \mu)]}{1 - \cos 2\mu} + \frac{\cos 4\Phi - \cos 4(\Phi + \mu)}{1 - \cos 4\mu} \right)$$

Normal form treatment

The normalized map now contains only action variable (easy to integrate) while all the phase information has been pushed to higher order.

From the generator F_4, we see the octupole drives half integer and quarter integer resonances. We can track the Poincare map using exact map and the normalized map respectively (assum $k_3 l = 4800\ m^{-3}$ and $\beta=1$ m). Assuming the tune μ is $0.33\times2\pi$ far from resonances



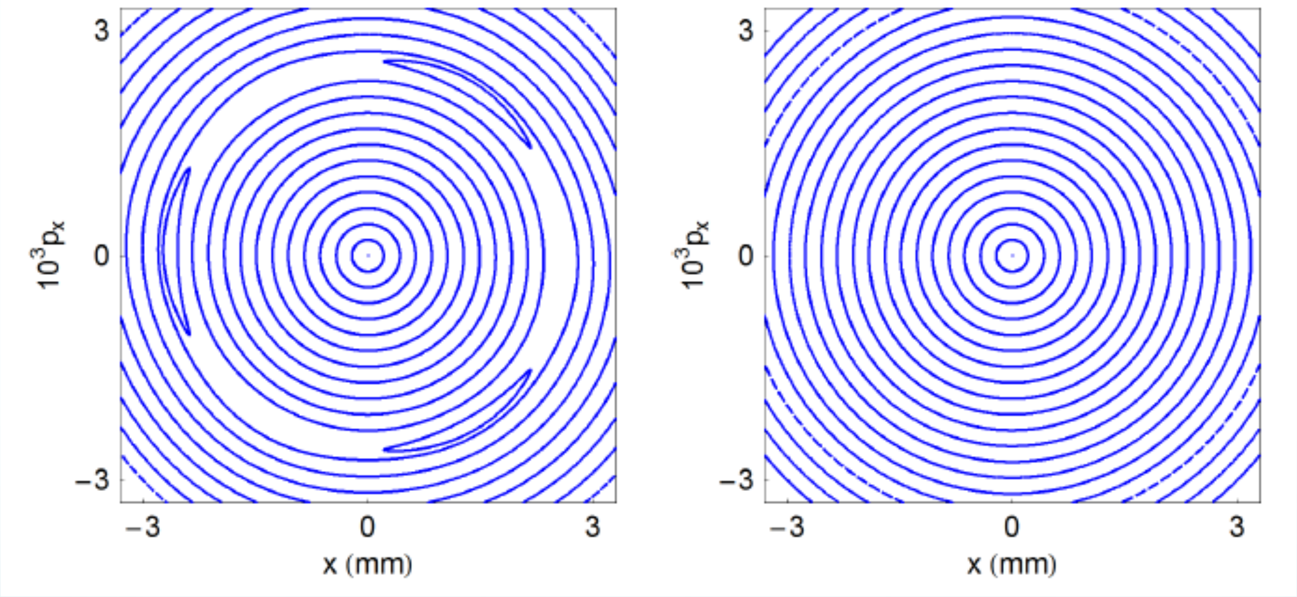
exact map normalized map

30 turns

Tune shift with amplitude!!

Normal form treatment

Tracking for longer turns results in different feature where we pay the price of the simplified (normalized) map. Some of the phase information (3rd order resonance island) is lost during this process.



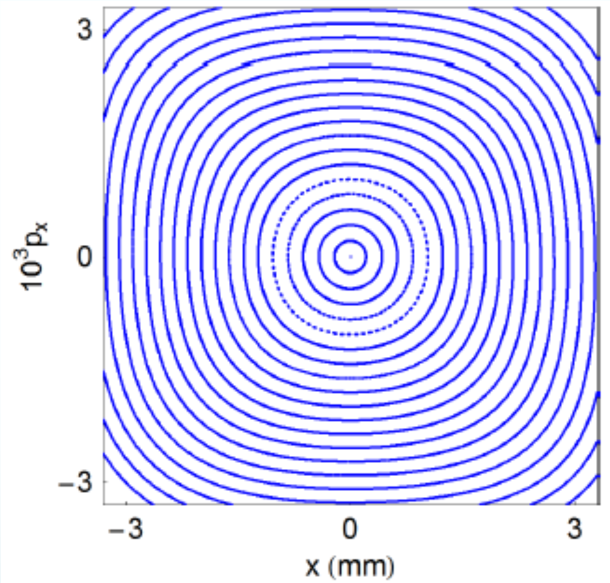
2500 turns

exact map

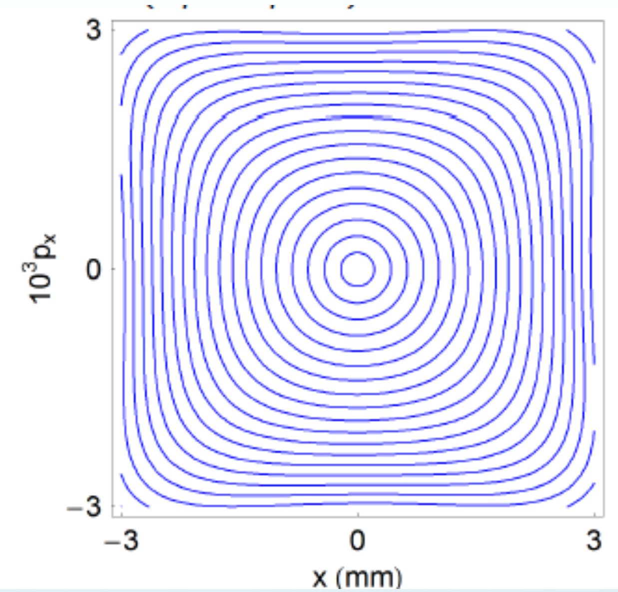
normalized map

Normal form treatment

Tracking for tunes near 4th order resonance is a bit tricky. Since the k_3l is positive, the tune shift with amplitude drives the tune up. Thus if the tune μ is $0.252\times2\pi$, we barely see resonances. The two tracking results resemble



exact map

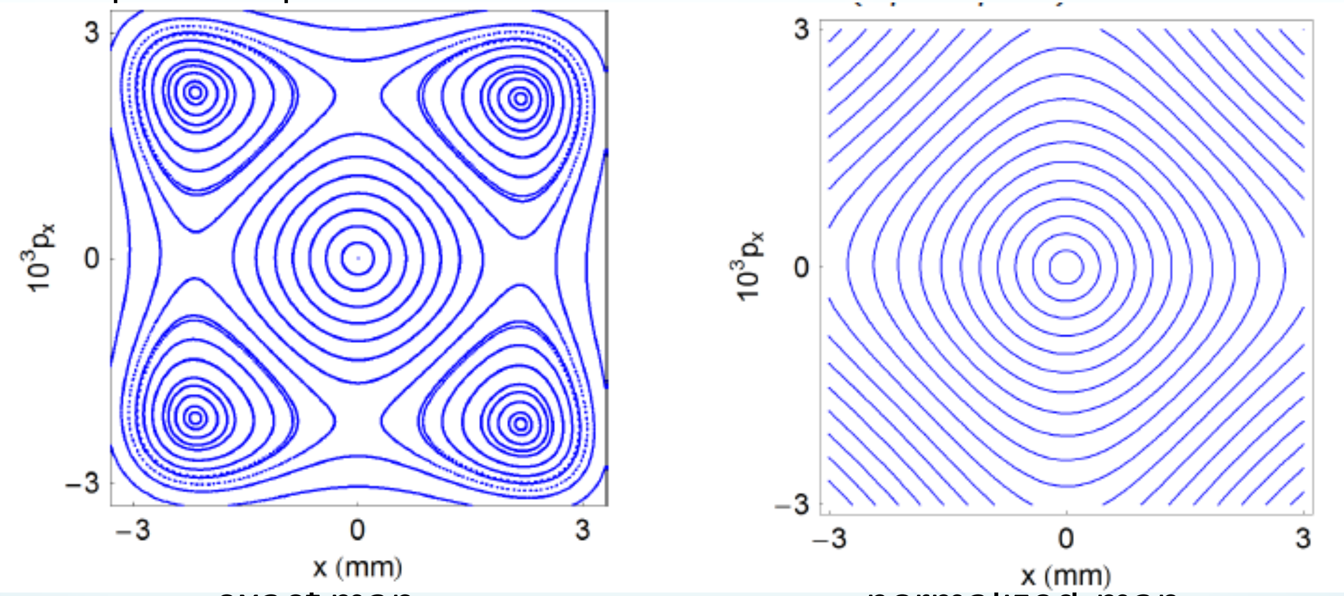


normalized map

2500 turns

Normal form treatment

For a tune less than quarter integer, i.e.,μ is 0.248×2π, we see strong resonances from exact tracking while for the normalized map, we only see a rotation in phase space.



exact map normalized map

2500 turns

Normal form of a one turn map preserves the information on tune amplitude dependence while loses the key phase information (when close to resonances). Need to retain higher order terms!

Resonance driving terms(RDTs)

We can interpret the Fourier coefficients $\overline{f}_{3,m}(J)$ as resonance strengths. And the generating function diverges when resonance condition mμ=2π is satisfied, meaning such driving term has large effect. Put it into polynomial expression, the generating function can be written as

$$F = \sum_{jklm} f_{jklm} \varsigma_x^+ \varsigma_x^- \varsigma_y^+ \varsigma_y^- = F_3 + F_4 + \cdots$$

where

$$f_{jklm} = \frac{h_{jklm}}{1 - e^{i2\pi[(j-k)\nu_x + (l-m)\nu_y]}}$$

hjklm are called resonance driving terms in many accelerator tracking codes. The entire process of the normal form the one turn map can be visualized as

$$\begin{tikzcd}
\overline{x(n)} \arrow[r, "M(J,\phi)"] \arrow[d, "\Phi"'] & \overline{x(n+1)} \arrow[d, "\Phi = e^{:F:}"] \\
\overline{\varsigma(n)} \arrow[r, "U(J) = e^{:H:}"] & \overline{\varsigma(n+1)}
\end{tikzcd}$$

Resonance driving terms(RDTs)

Incorporating the optics of a lattice, the resonance driving term (RDT) coefficients h_{jklm} (1st order RDT) are usually calculated as

$$h_{jklm} = c\sum_{i=1}^{N} S_2 \beta_{xi}^{(j+k)/2} \beta_{yi}^{(l+m)/2} e^{i[(j-k)\mu_{xi} + (l-m)\mu_{yi}]}$$

It is very sensitive to linear lattice thus a carefully designed linear lattice with proper phase advance per periodic structure benefits greatly in reducing the RDTs (we will talk about a few tactics later).

Chromatic aberration

Sextupoles (and even higher order magnets) are necessary in an accelerator design (not only existing as the field error of strong linear magnets).

Sextupoles are used to correct the chromatic aberration, i.e., tune shift, that resides in linear lattice (in comparison to the aberration that exists in optics).

We can define chromaticities

$$\Delta\nu_x = \left[-\frac{1}{4\pi}\oint \beta_x(s)K_x(s)ds\right]\delta \equiv C_x\delta, \quad C_x = d\nu_x / d\delta$$

$$\Delta\nu_y = \left[-\frac{1}{4\pi}\oint \beta_y(s)K_y(s)ds\right]\delta \equiv C_y\delta, \quad C_y = d\nu_y / d\delta$$

The chromaticity induced by quadrupole field is called natural chromaticity.

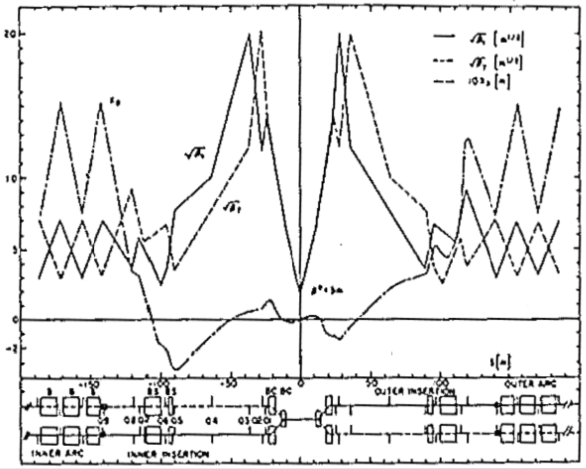
Chromaticity

Chromaticity can be very large. Taking FODO lattice as an example,

$$C_{X,\mathrm{nat}}^{\mathrm{FODO}} = -\frac{1}{4\pi} N\left(\frac{\beta_{\max}}{f} - \frac{\beta_{\min}}{f}\right) = -\frac{\tan(\Phi/2)}{\Phi/2}\nu_X \approx -\nu_X$$

Natural chromaticity per cell is approaching cell tune (when phase advance per cell is not large).

Chromaticities from interaction region (for colliders) can be huge due to the low beta.



$$C_{total} = N_{IR} C_{IR} + C_{ARCs}$$

$$C_{IR} = -\frac{2\Delta s}{4\pi\beta^*} \approx -\frac{1}{2\pi}\sqrt{\frac{\beta_{\max}}{\beta^*}}$$

For a ring with C~1000 m, the chromaticities can easily exceed negative few hundred units, which cause severe instablities.

Chromaticity correction

In existence of a sextupole element, the Hill's equation becomes

$$x''_\beta + (K_x(s) + K_2 D\delta)x_\beta = 0, \quad y''_\beta + (K_y(s) - K_2 D\delta)y_\beta = 0$$

$$x = x_\beta + D\delta$$

$$\Delta K_x(s) = K_2(s)D(s)\delta, \quad \Delta K_y(s) = -K_2(s)D(s)\delta$$

thus

$$C_x = -\tfrac{1}{4\pi}\oint \beta_x(s)[K_x(s) - K_2(s)D(s)]ds$$

$$C_y = -\tfrac{1}{4\pi}\oint \beta_y(s)[K_y(s) + K_2(s)D(s)]ds$$

In order to minimize their strength, the chromatic sextupoles should be located near quadrupoles, where $\beta_x D_x$ and $\beta_y D_x$ are maximum.
A large ratio of β_x/β_y for the focusing sextupole and a large ratio of β_y/β_x for the defocussing sextupole are needed for optimal independent chromaticity control.

Chromaticity correction 2nd

To avoid head-tail instability, we need to satisfy:

$$C_x/\eta > 0, \quad \eta = \frac{1}{\gamma_T^2} - \frac{1}{\gamma^2} > 0$$

The 2nd order chromaticity can be expressed as $C_x^{(2)} = -C_x^{(1)} - \frac{|J_{p,x}|^2}{4(\nu_x - p/2)\delta^2}$

By pairing adjacent sextupole families

$$S_{F1} \to S_{F1} + (\Delta S)_F, \qquad S_{D1} \to S_{D1} + (\Delta S)_D,$$
$$S_{F2} \to S_{F2} - (\Delta S)_F \qquad S_{D2} \to S_{D2} - (\Delta S)_D$$

$$\Delta J_{p,x} = \frac{\delta}{2\pi} N[\beta_F (\Delta S)_F D_F + \beta_D (\Delta S)_D D_D e^{i\pi/4}]$$

Under conditions $\quad p \approx 2\nu \quad \Phi \approx \pi/2$

We design the linear lattice to have 90 deg phase advance per FODO cell to remove the potential cancellation between sextupoles and the change in stopband integral linearly depends on the change in sextupole strengths.

Dynamic aperture (DA)

Dynamic aperture determines the stable region in 2d real space (x-y) while particles travel along the accelerator. It is very important for particle dynamic study especially in effects that requires tracking over many revolutions (decided by system's damping time, could range from 1000 (light sources) to 1,000,000 (proton/heavy ion storage rings).

Dynamic aperture is a clear indication of nonlinear resonances that reside in an accelerator. Its size is limited by the utilize of nonlinear magnets to correct chromatic aberration. Thus designing the lattice with the nonlinear magnets' strengths reduced is crucial in improving DA.

Careful tuning of multipole nonlinear elements can also result in reducing the resonance driving terms thus improving the DA.

There are many ways of determining the DA of a specific lattice. Mostly commonly used techniques include line search mode (single-line, n-line,etc…) and frequency map analysis.

Line search analysis

Line search mode requires tracking particles with different initial positions (or gradually increasing the particle offset till it is lost) to determine the boundary of the stable region. Itself is machine expensive however can be easily parallelized.

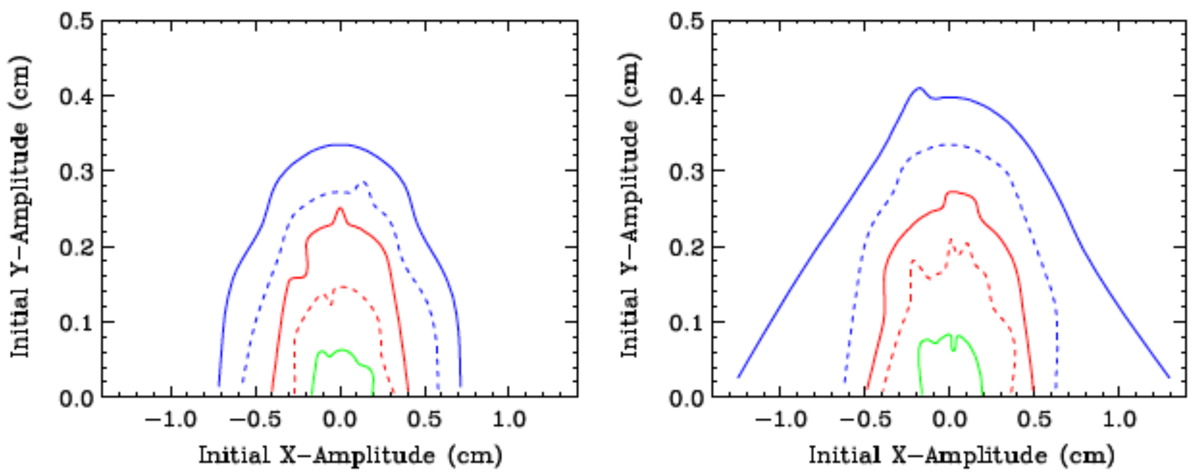


Figure 10: *Momentum dependent dynamic aperture without errors for OPA (left) and 4th-order geometric achromat (right) solutions with chromaticity set to zero, where:* $\delta = 0$ *(blue solid), 0.5% (blue dash), 1% (red solid), 1.5% (red dash), 2% (green).*

Frequency map analysis(FMA)

If we perform a discrete Fourier transform on the tracking data with initial position. We can obtain the betatron tunes (for N turn tracking, the precision is merely 1/N). If we repeat this process with different initial positions, we can obtain a tune map. To indicate the variation of the tunes over different turns of the ring, we can define a diffusion or regularity which describes the difference between the tunes over various periods (usually the first half of the tracking (Q_{x1}, Q_{y1}) and the second half(Q_{x2}, Q_{y2})). In other words, we define a diffusion constant D

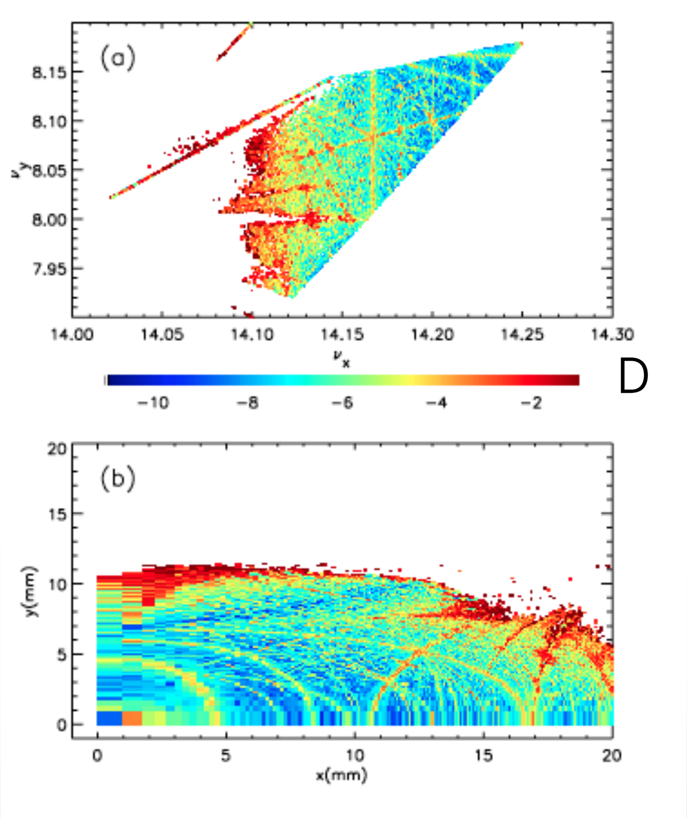
$$D = \log_{10}\sqrt{(Q_{y2}-Q_{y1})^2+(Q_{x2}-Q_{x1})^2}$$

The rule of thumb is when D is small, the variation is low (or regular) and particle motion is stable. On the other hand, when D is large, the variation is high (or irregular) and particle motion is unstable (chaotic). The points in tune space with large variation (chaotic) usually lies on the crossing of different resonance lines.

Frequency map analysis(FMA)

The obtained resonance feature in frequency space (tune space) can then be easily related into 2 dimension x-y real space and used as an indicator of the size of stable region. It may discover some resonance islands that line search is not capable of finding as well as the important tune shifts and strong resonances that we need to avoid. FMA is often used in accelerator design to identify the dynamical behavior.

Experimental construction of FM requires very high precision measurements and some data mining techniques to further improve the precision, e.g., Hanning filter, data interpolation, NAFF, etc…



A plot showing the FM for an ideal lattice for ALS in tune space (a) and real space (b).