

2403 **Chapter 9**
2404 **Weak Focusing Synchrotron**

2405 **Abstract** This Chapter is a brief introduction to the weak focusing synchrotron,
2406 and to the theoretical material needed for the simulation exercises. It relies on basic
2407 charged particle optics and acceleration concepts introduced in the previous Chap-
2408 ters, and further addresses

- 2409 - fixed closed orbit,
- 2410 - periodic structures,
- 2411 - periodic stability,
- 2412 - optical functions,
- 2413 - synchrotron motion,
- 2414 - depolarizing resonances.

2415 The simulation of weak synchrotrons only require a very limited number of opti-
2416 cal elements; actually two are enough: DIPOLE or BEND to simulate combined
2417 function dipoles, and DRIFT. Particle monitoring requires keywords introduced in
2418 the previous Chapters, including FAISCEAU, FAISTORE, possibly PICKUPS. Spin
2419 motion computation and monitoring resort to SPNTRK, SPNPRT, FAISTORE,

2420 Notations used in the Text

$B; \mathbf{B}; B_x, B_y, B_z$	field value; field vector; components
$B\rho = p/q; B\rho_0$	particle rigidity; reference rigidity
C	closed orbit length, $C = 2\pi R$
E	particle energy
EFB	Effective Field Boundary
f_{rf}	RF frequency
h	RF harmonic number
$m; m_0; M$	mass; rest mass; in units of MeV/c^2
$n = \frac{\rho}{B} \frac{dB}{dp}$	focusing index, a local quantity
$\mathbf{p}; p_0$	particle momentum vector; reference momentum
P_i, P_f	initial, final asymptotic polarization at traversal of a spin resonance
q	particle charge
r	orbital radius
R	average radius, $R = C/2\pi$
2421 s	path variable
v	particle velocity
$V(t); \hat{V}$	oscillating voltage; its peak value
x, x', y, y'	radial and axial coordinates in Serret-Frénet frame
$\beta = v/c; \beta_0; \beta_s$	normalized particle velocity; reference; synchronous
$\gamma = E/m_0$	Lorentz relativistic factor
$\Delta p, \delta p$	momentum offset
ϵ_u	Courant-Snyder invariant (u: x, r, y, l, Y, Z, s, etc.)
ϵ_R	strength of a depolarizing resonance
$\phi; \phi_s$	particle phase at voltage gap; synchronous phase
ϕ_z	betatron phase advance, z stands for x or y
φ	spin angle to the vertical axis
$\langle A \rangle; \langle A \rangle _u$	average of A; over variable u

2422 Introduction

2423 The synchrotron is an outcome the phase focusing concept [1, 2], combined with
 2424 constant orbit acceleration [3]. Phase focusing states that off-crest acceleration with
 2425 proper phase of the voltage oscillation at arrival of a particle at the accelerating
 2426 gap results in stable longitudinal motion, “longitudinal focusing”, around a stable,
 2427 fixed, “synchronous phase”. The reference orbit in a synchrotron on the other hand,
 2428 is maintained at constant radius by ramping the guide field in synchronism with the
 2429 acceleration, a concept already familiar at the time with the betatron [4].

2430 Phase focusing was demonstrated experimentally in 1946 using a cyclotron
 2431 dipole [5]. Demonstration of phase stability at constant orbital radius followed in

2432 1946, using an existing betatron ring [6]. Due to the cycling of the acceleration, a
 2433 synchrotron accelerates particle bunches, comprised of particles that have proper
 2434 relationship, in both frequency and phase, with the oscillating voltage at the acceler-
 2435 ating gap, or gaps around the ring. The concept allowed greatest energy reach, it led
 2436 to the construction of a series of proton rings with increasing energy: 1 GeV at Birm-
 2437 ingham (1953), 3.3 GeV at the Cosmotron (BNL, 1953), 6.2 GeV at the Bevatron
 2438 (1954), 10 GeV at the Synchro-Phasotron (Dubna, 1957), and a few additional ones
 2439 beyond 1952~1953, into the era of the technology which would essentially dethrone
 2440 it: the strong focusing synchrotron. The general layout of these first synchrotrons
 2441 included straight sections (often 4, Fig. 9.1), which allowed insertion of injection
 2442 (Fig. 9.2) and extraction systems, accelerating cavities, orbit correction and beam
 2443 monitoring equipment.

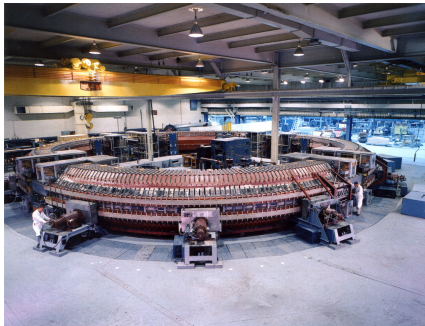


Fig. 9.1 The Cosmotron at BNL, reached its full design energy of 3.3 GeV in 1953. It was used until 1968 [8]

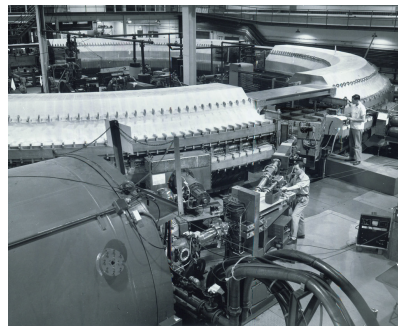


Fig. 9.2 Details of the low energy injection line and injection straight section at the Cosmotron [9]

2444 9.1 Basic Concepts and Formulae

2445 The synchrotron is based on two key principles: a varying magnetic field to maintain
 2446 the accelerated bunch on a constant orbit, with constant transverse focusing, namely,

$$B(t) = p(t)/q\rho, \quad \rho = \text{constant} \quad (9.1)$$

2447 and longitudinal phase stability, possibly including modulation of the accelerating
 2448 voltage frequency in order to follow the velocity change of the bunch [1, 2]. The
 2449 modulation of the oscillating voltage frequency is maintained in synchronism with
 2450 the bunch revolution motion, of which the period varies with time following

$$f_{RF}(t) = hf_{rev}(t) \quad (9.2)$$

2451 with h an integer, the “RF harmonic”. Since the orbit is maintained unchanged
 2452 turn after turn, the revolution frequency varies, in inverse proportion to particle
 2453 velocity. These are two major evolutions compared to the cyclotron, where, instead,
 2454 the magnetic field and the oscillating voltage frequency are fixed.

2455 The synchronism between RF voltage frequency and revolution time (Eq. 9.2)
 2456 allows maintaining the bunch at an appropriate phase, the “synchronous phase”, with
 2457 respect to the oscillating voltage when passing the accelerating gap (this is discussed
 2458 in a next Section).

2459 Synchronous acceleration is technologically simpler in the case of electrons, as
 2460 frequency modulation is unnecessary beyond a few MeV of particle energy. For
 2461 instance, from $v/c = 0.9987$ at 10 MeV to $v/c \rightarrow 1$ at very high energy, the relative
 2462 change in revolution frequency amounts to $\delta f_{\text{rev}}/f_{\text{rev}} = \delta\beta/\beta < 0.0013$.

2463 Constant closed orbit reduces the radial extent of individual guiding magnets
 2464 compared to a cyclotron dipole which must encompass a spiraling orbit, and leads
 2465 to a circular string of dipoles, a ring structure. An archetype of a weak focusing
 2466 synchrotron ring is shown in Fig. 9.3, Saturne I, a 3 GeV, 4-period, 68.9 m circum-
 2467 ference, transverse index focusing synchrotron at Saclay [10]. Operation at Saturne I
 2468 started in 1957, plans for the acceleration of polarized beams at the time motivated
 2469 theoretical investigation of resonant depolarization [11]. The four dipoles of the
 2470 squared ring are 1150 tons each; the straight sections are 4 m long; injection is in
 2471 the north one (foreground), from a 3.6 MeV Van de Graaff (not visible); the south
 2472 section houses the extraction system; a beam detection system is located in the east
 2473 straight; the RF cavity is in the west one and provides a peak voltage of a few kW,
 2474 whereas the peak power requested from the RF system for acceleration does not
 2475 exceed 2 kW.

2476 For the sake of comparison: a synchro-cyclotron dipole is a pair of full, massive
 2477 cylindrical poles; greater energy requires greater radial extent of the magnet to allow
 2478 the necessary increase of the bend field integral (namely, $\oint B dl = 2\pi R_{\text{max}} \hat{B} =$
 2479 p_{max}/q - note that \hat{B} can be pushed to ~ 2 T as the field is fixed) and accordingly
 2480 of the diameter of the bulky cylinder, thus the volume of iron increases more than
 2481 quadratically with bunch rigidity.

2482 A second example of a weak focusing synchrotron is shown in Figure 9.4, the ZGS
 2483 at Argonne, a 12 GeV, 4-period, 172 m circumference, zero-gradient synchrotron:
 2484 ZGS had the particularity of using wedge focusing to ensure transverse beam sta-
 2485 bility. ZGS was operated over 1964-1979, polarized beam acceleration happened in
 2486 July 1973, to 8.5 GeV/c, and up to 12 GeV/c in the following years [12]. Pulsed
 2487 quadrupoles were used to pass through several depolarizing intrinsic resonances,
 2488 a method known as resonance crossing by fast “tune-jump”. ZGS proton injector
 2489 was comprised of a 20 keV source, followed by a 750 keV Cockcroft-Walton and a
 2490 50 MeV linac.

2491 The acceleration is cycled in a synchrotron, from injection to top energy, repeat-
 2492 edly. The cycling of the magnetic field, in synchronism with the acceleration voltage,
 2493 maintains a constant orbit; the field law $B(t)$ depends on the type of power supply. If
 2494 the ramping uses a constant electromotive force, then

Fig. 9.3 Saturne I at Saclay [10], a 3 GeV, 4-period, 68.9 m circumference, weak focusing synchrotron, field index $n \approx 0.6$ [13]

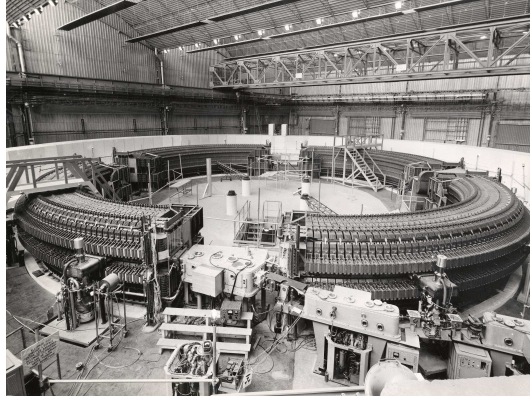
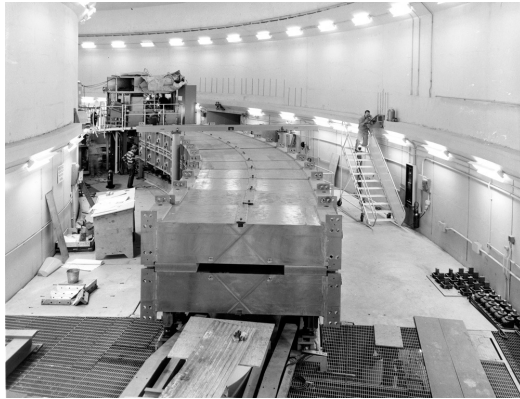


Fig. 9.4 The ZGS at Argonne during construction. A 12 GeV, 4-period, 172 m circumference, wedge focusing synchrotron. Two persons can be seen standing on the left and on the right of the ring, in the background, giving an idea of the size of the magnets



$$B(t) \propto (1 - e^{-\frac{t}{\tau}}) = 1 - \left[1 - \left(\frac{t}{\tau}\right) + \left(\frac{t}{\tau}\right)^2 - \dots \right] \approx \frac{t}{\tau} \quad (9.3)$$

2495 essentially linear. In that case $\dot{B} = dB/dt$ does not exceed a few Tesla/second, thus
2496 the repetition rate of the acceleration cycle is of the order of a Hertz.

2497 If the magnet winding is part of a resonant circuit the field law has the form

$$B(t) = B_0 + \frac{\hat{B}}{2}(1 - \cos \omega t) \quad (9.4)$$

2498 so that, in the interval of half a voltage repetition period, namely $t : 0 \rightarrow \pi/\omega$,
2499 the field increases from an injection threshold value to a maximum value at highest
2500 rigidity, $B(t) : B_0 \rightarrow B_0 + \hat{B}$. The latter determines the highest achievable energy:
2501 $\hat{E} = pc/\beta = q\hat{B}\rho c/\beta$. The repetition rate with resonant magnet cycling can reach a
2502 few tens of Hertz.

2503 In both cases anyway B imposes its law and the other quantities comprising the
2504 acceleration cycle (RF frequency in particular) will follow B(t).

2505 For the sake of comparison again: in a synchrocyclotron the field is constant,
 2506 acceleration can be cycled as fast as the voltage system allows; assume a conservative
 2507 10 kVolts per turn, thus of the order of 10,000 turns to 100 MeV, with velocity
 2508 $0.046 < v/c < 0.43$ from 1 to 100 MeV, proton. Take $v \approx 0.5c$ to make it simple,
 2509 an orbit circumference below 30 meter, thus the acceleration takes of the order of
 2510 $10^4 \times C/0.5c \approx \text{ms}$ range, potentially a repetition rate in kHz range, more than an
 2511 order of magnitude beyond what a rapid-cycling pulsed synchrotron allows.

2512 The next decades following the invention of the synchrotron saw an all-out break-
 2513 through, with applications in many fields of science, in medicine, industry. The
 2514 weak focusing synchrotron allowed colliding particle beams of highest energies on
 2515 fixed targets in nucleus fission and particle production experiments, leading to the
 2516 discovery of several fundamental particles. Its technological simplicity still makes
 2517 it an appropriate technology today in low energy beam application when relatively
 2518 low beam current is not a concern: it essentially requires a single type of a simple
 2519 dipole magnet, an accelerating gap, some command-control instrumentation, and
 2520 that's it! whereas it procures greater beam manipulation flexibilities compared to
 2521 (synchro-)cyclotrons.

2522 Transverse beam stability in a weak focusing synchrotron ring inherits from the
 2523 cyclotron techniques, focusing in the dipoles results from the presence of a transverse
 2524 field gradient $0 < n < 1$ and/or from wedge focusing, as in the aforementioned two
 2525 examples, Saturne 1 synchrotron [14] and the ZGS [7].

2526 A weak focusing synchrotron is comprised of a string of dipoles separated by
 2527 field free drift spaces, forming a $\frac{2\pi}{N}$ -symmetric, N-periodic structure. Each period
 2528 ensures a $\frac{2\pi}{N}$ fraction of the 2π bending, N=4 for instance in Saturne I (Fig. 9.3) and
 2529 in ZGS (Fig. 9.4). In the ZGS a period is comprised of a pair of 45 degree dipoles,
 2530 a total of 8 dipoles around the ring, whereas Saturne I features a single 90 degree
 2531 dipole per period, 4 dipoles in total.

2532 Introducing straight sections in the magnetic structure of the ring allows room
 2533 for inserting the various devices that garnish a synchrotron and contribute beam ma-
 2534 nipulation flexibility: an accelerating cavity, injection and extraction systems, beam
 2535 diagnostics equipment, special optical elements, tune jump quadrupoles possibly for
 2536 polarized beam handling, etc.

2537 9.1.1 Transverse Stability

2538 The introduction to transverse stability in this Section leans on the weak focusing
 2539 concepts introduced in the Classical Cyclotron Chapter (Chap. 4). Radial motion
 2540 stability around a reference closed orbit in an axially symmetric dipole field requires
 2541 the geometrical configuration of particle orbits sketched in Fig. 9.6, resulting from
 2542 magnetic rigidity $B \times \rho$ an increasing function of radius, which, on the closed orbit
 2543 (radius = ρ_0), expresses as $\frac{\partial B\rho}{\partial \rho} \geq 0$, viz. $1 + \frac{\rho}{B_0} \frac{\partial B}{\partial \rho} \geq 0$. Vertical stability requires
 2544 the gap height to increase with radius, thus field decreases with radius, $\frac{\partial B_y}{\partial \rho} < 0$

2545 (Fig. 4.9). This is the focusing method which was used in the classical cyclotron
 2546 (Sec. 4.2.2). Introduce the field index

$$n = -\frac{\rho_0}{B_0} \left. \frac{\partial B_y}{\partial \rho} \right|_{x=0, y=0} \quad (9.5)$$

This results in the typical magnet segment shown in Fig. 9.5.

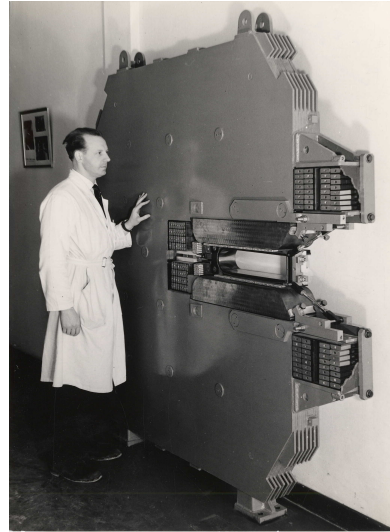
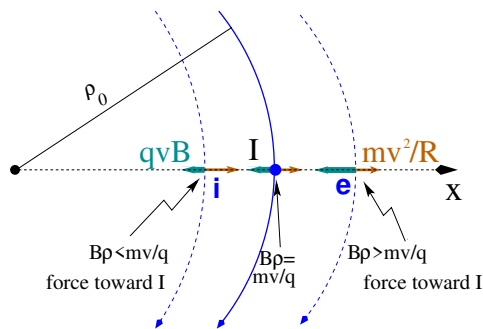


Fig. 9.5 A segment of Saturne I weak focusing synchrotron magnet, with its hardly visible gap tapering (greater outward) to satisfy the weak index condition $0 < n < 1$ [13]

2547 Transverse motion stability in an axially symmetric structure, with or without
 2548 drift spaces, thus summarizes in
 2549

$$0 < n < 1 \quad (9.6)$$

Fig. 9.6 Radial motion stability in an axially symmetric structure. The resultant $F_r = -qvB + mv^2/r$, is zero at I : $B_0\rho_0 = mv/q$. The resultant at i is toward I if $qvB_i < mv^2/\rho_i$, i.e. $B_i\rho_i < mv/q$; the resultant at e is toward I if $qvB_e > mv^2/\rho_e$, i.e. $B_e\rho_e > mv/q$



2550 Adding drift spaces requires defining two radii, namely,
 2551 (i) the magnet curvature radius ρ_0 (Fig. 9.7),
 2552 (ii) an average radius $R = C/2\pi = \rho_0 + Nl/\pi$ (with C the length of the reference
 2553 closed orbit and $2l$ the drift length) (Fig. 9.8) which also writes

$$R = \rho_0(1 + k), \quad k = \frac{Nl}{\pi\rho_0} \quad (9.7)$$

2554 The reference orbit is comprised of arcs of radius ρ_0 in the magnets, and straight
 2555 segments along the drift spaces that connect these arcs. Adding drift spaces decreases
 2556 the average focusing around the ring. Trajectories of different momenta are parallel.

Fig. 9.7 In a sector dipole with radial index $n \neq 0$, closed orbits follow arcs of constant B . A closed orbit at $p_0 + \Delta p$ follows an arc of radius $\rho_0 + \Delta\rho$, $\Delta\rho = \Delta p/(1+n)qB_0$

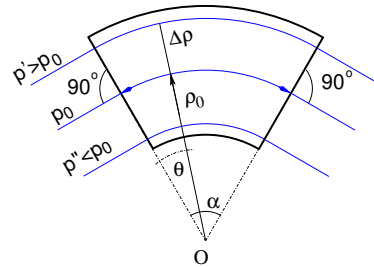
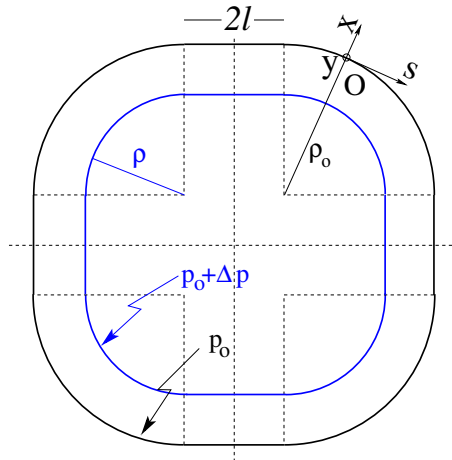


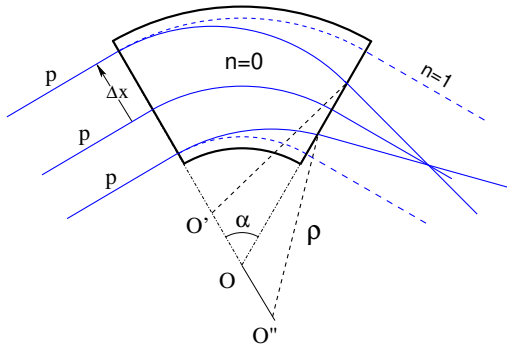
Fig. 9.8 A $2\pi/4$ axially symmetric structure with four drift spaces. Orbit length on reference momentum p_0 is $C = 2\pi\rho_0 + 8l$. $(O; s, x, y)$ is the moving frame, along the reference orbit. The orbit for momentum $p = p_0 + \Delta p$ ($\Delta p < 0$, here) is at constant distance $\Delta x = \frac{\rho_0}{1-n} \frac{\Delta p}{p_0} = \frac{R}{(1+k)(1-n)} \frac{\Delta p}{p_0}$ from the reference orbit



2557 *Geometrical focusing:*

2558 In a constant field dipole (radial field index $n=0$), the longer (respectively shorter)
 2559 path in the magnetic field for parallel trajectories entering the magnet at greater
 2560 (respectively smaller) radius results in geometrical focusing. Referring to Fig. 9.9,
 2561 this effect can be cancelled, *i.e.*, the deviation made the same whatever the entrance
 2562 radius, if the curvature center is made independent of the entrance radius: $OO' = 0$,
 2563 $O''O = 0$. This requires trajectories at an outer (inner) radius to experience a smaller
 2564 (greater) field so to satisfy $BL = B\rho\alpha = C^{st}$. Differentiating $B\rho = C^{st}$ yields
 2565 $\frac{\Delta B}{B} + \frac{\Delta\rho}{\rho} = 0$, with $\Delta\rho = \Delta x$. Thus the field $B(x)$ must satisfy $n = -\frac{\rho_0}{B_0} \frac{\Delta B}{\Delta x} = 1$ in
 2566 order to cancel the geometrical focusing resulting from the curvature.

Fig. 9.9 Geometrical focusing: in a sector dipole with focusing index $n = 0$, parallel incoming rays of equal momenta experience the same curvature radius ρ , they exit converging, as a result of the longer path of outer trajectories in the field, compared to inner ones. An index value $n=1$ cancels that effect: rays exit parallel



2567 *Focal distance associated with the curvature:*

2568 Assume $n=0$, reference radius $\rho = \rho_0$, reference arc length $\mathcal{L} = \rho_0\alpha$. From $\frac{d^2x}{ds^2} +$
 2569 $\frac{1}{\rho_0^2}x = 0$ one gets

$$\Delta x' = \int \frac{d^2x}{ds^2} ds \approx -\frac{x}{\rho_0^2} \int ds = -\frac{x}{\rho_0^2} \mathcal{L} \stackrel{\text{def.}}{=} -\frac{x}{f} \Rightarrow f = \frac{\rho_0^2}{\mathcal{L}} \quad (9.8)$$

2570 Optical drawbacks of the weak focusing method are, the weakness of the focusing
 2571 and the absence of independent radial and axial focusing.

2572 **Wedge Focusing**

2573 This is the focusing method in the ZGS. Profiling the magnet gap in order to adjust
 2574 the focal distance complicates the magnet; $n=0$, a parallel gap, makes it simpler.
 2575 In the ZGS the focal distance is designed based on proper entrance and exit wedge

2576 angles (Fig. 9.10): opening the magnetic sector increases the horizontal focusing
 2577 (and decreases the vertical focusing); closing the magnetic sector has the reverse
 2578 effect.

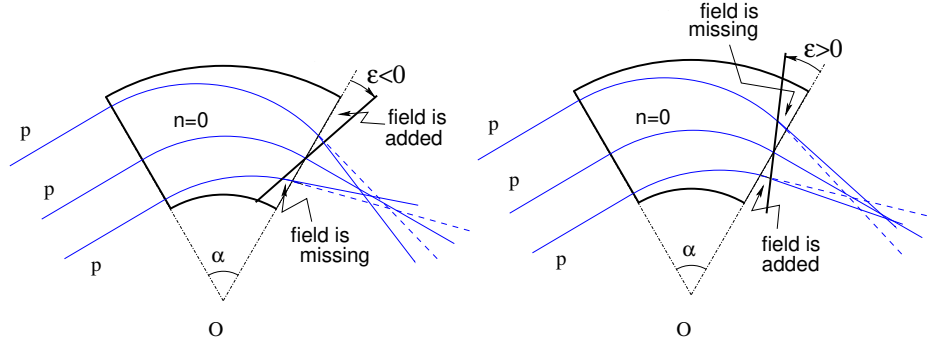


Fig. 9.10 Left: a focusing wedge ($\epsilon < 0$ by convention); opening the sector increases the horizontal focusing. Right: a defocusing wedge ($\epsilon > 0$ by convention); closing the sector decreases the horizontal focusing. The focal distance of the bend plane respectively decreases, increases. The effect is the opposite in the vertical plane, opening/closing the sector decreases/increases the vertical focusing.

2579 *Vertical focusing at the EFB*

2580 The magnetic field falls off smoothly in the fringe field region at the ends of a magnet,
 2581 from its value in the body to zero at some distance from the iron. The extent of the
 2582 fall-off is commensurate with the gap size, its shape depends on such factors as the
 2583 profiling of the iron at the EFB (Fig. 9.11) or the positioning and shape of the coils.

2584 From an optics standpoint, the main effect of the fringe field is the existence of a
 2585 longitudinal component of the field, $\mathbf{B}_s(s)$. In a mid-plane symmetry dipole, $\mathbf{B}_s(s)$
 2586 is non-zero off the median plane, and normal to the iron (Fig. 9.11).

2587 The focal distance f associated with a wedge angle ϵ (Fig. 9.10) satisfies

$$\frac{1}{f} = \tan \frac{\epsilon}{\rho_0} \quad (9.9)$$

2588 with $\epsilon > 0$ if the sector is closing, by convention. In a point transform approximation,
 2589 at the wedge the trajectory undergoes a local deviation proportional to the distance
 2590 to the optical axis, namely,

$$\Delta x' = \frac{\tan \epsilon}{\rho_0} \Delta x, \quad \Delta y' = -\frac{\tan \epsilon}{\rho_0} \Delta y \quad (9.10)$$

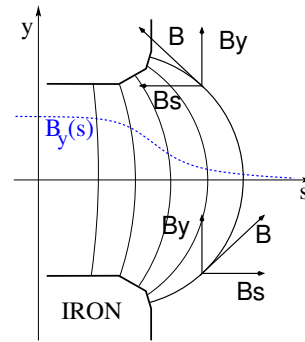


Fig. 9.11 Field components in the $B_y(s)$ fringe field region at a dipole EFB

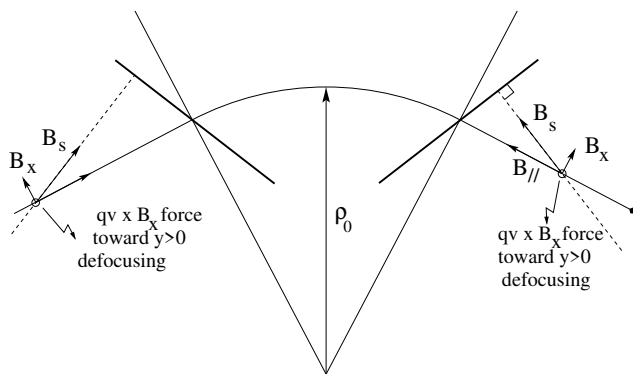


Fig. 9.12 Field components in the fringe field region at the end of a dipole ($y > 0$, here, referring to Fig. 9.11). $B_{//}$ is parallel to the particle velocity. This configuration is vertically defocusing: a charged particle traveling off mid-plane is pulled away from the the latter under the effect of $\mathbf{v} \times \mathbf{B}_x$ force component. Inspection of the $y < 0$ region gives the same result: the charge is pulled away from the median plane

2591 Wedge vertical focusing in the ZGS ($\epsilon > 0$) was at the expense of horizontal
 2592 geometrical focusing (Fig. 9.7). This was an advantage though for the acceleration
 2593 of polarized beams, as radial field components (which are responsible for depolar-
 2594 ization) were only met at the EFBs of the eight main dipoles [12]. Preserving beam
 2595 polarization at high energy required tight control of the tunes, and this was achieved
 2596 by, in addition, pole face winding at the ends of the dipoles [15, 16]; these coils
 2597 where pulsed to control amplitude detuning, resulting in tune control at 0.01 level,
 2598 they also compensated eddy currents induced sextupole perturbations affecting the
 2599 vertical tune.

2600 *Fringe field extent*

2601 The fringe field extent, say λ , may be taken into account in the thin lens approximation
 2602 of the wedge focusing. It only modifies the horizontal focusing to the second order
 2603 in the coordinates, but changes the vertical focusing to the first order, namely

$$\Delta x' = \frac{\tan \epsilon}{\rho_0} \Delta x, \quad \Delta y' = -\frac{\tan(\epsilon - \psi)}{\rho_0} \Delta y \quad (9.11)$$

2604 wherein

$$\psi = I_1 \frac{\lambda}{\rho_0} \frac{1 + \sin^2 \epsilon}{\cos \epsilon}, \quad \text{with } I_1 = \int_{s(B=0)}^{s(B=B_0)} \frac{B(s)(B_0 - B(s))}{B_0^2} \frac{ds}{\lambda} \quad (9.12)$$

2605 and the integral I_1 extends over the field fall-off where B evolves between 0 to a
 2606 plateau value B_0 inside the magnet.

2607 **Off-momentum orbits**

In a dipole with field index $n = -\frac{\rho_0}{B_0} \frac{\partial B_y}{\partial \rho}$, concentric orbits with different momenta
 $p = p_0 + \Delta p$ (Fig. 9.7) are distant

$$\Delta x = \frac{\rho_0}{1-n} \frac{\Delta p}{p_0}$$

2608 from the reference orbit at $p = p_0$. Given that $n < 1$,

- 2609 - higher momentum orbits, $p > p_0$, have a greater radius,
- 2610 - lower momentum orbits, $p < p_0$, have a smaller radius.

2611 In a structure with axial symmetry, with drift sections (Fig. 9.8) or without
 2612 (classical and AVF cyclotrons for instance), the ratio $\frac{\Delta x}{\rho_0 \frac{dp}{p_0}} = \frac{1}{1-n}$ is independent
 2613 of the azimuth s . Equilibrium trajectories enter and exit parallel to the optical axis
 2614 of the bending dipoles. Introduce the geometrical radius $R = (1+k)\rho_0$ (Eq. 9.7) to
 2615 account for the added drifts, the chromatic dispersion of the orbits thus amounts to

$$\frac{\Delta x}{\Delta p/p_0} \equiv \frac{\Delta R}{\Delta p/p_0} = \frac{R}{(1-n)(1+k)} \quad (9.13)$$

2616 Thus the dispersion function

$$D(s) = \frac{R}{(1-n)(1+k)} = D, \quad \text{constant} \quad (9.14)$$

2617 is s -independent, the distance of a chromatic orbit to the reference orbit is constant
 2618 around the ring.

2619 *Chromatic orbit length*

2620 In an axially symmetric structure the difference in closed orbit length $\Delta C = 2\pi\Delta R$
 2621 resulting from the difference in momentum arises in the dipoles, as all orbits are
 2622 parallel in the drifts (Fig. 9.8). Hence, from Eq. 9.13, the relative closed orbit
 2623 lengthening factor, “momentum compaction”

$$\alpha = \frac{\Delta C}{C} / \frac{\Delta p}{p_0} \equiv \frac{\Delta R}{R} / \frac{\Delta p}{p_0} = \frac{1}{(1-n)(1+k)} \approx \frac{1}{v_x^2} \quad (9.15)$$

with $k = Nl/\pi\rho_0$ (Eq. 9.7). A note regarding the relationship $\alpha \approx 1/v_x^2$ between momentum compaction and horizontal wave number (it will be addressed quantitatively, below): this approximation was established in the case of a cylindrically symmetric structure, for which $v_x = \sqrt{1-n}$ (Eq. 4.19, ‘Classical Cyclotron’ Chapter). Adding short drifts such that $k \rightarrow 0$ (i.e., $Nl \ll \pi\rho_0$), the relation still holds, thus leading to

$$v_x \approx \sqrt{(1-n)(1+k)} \approx \sqrt{(1-n)}\left(1 + \frac{k}{2}\right)$$

2624 **9.1.2 Betatron motion in a periodic structure, periodic stability**2625 **Equations of motion**

2626 The first order differential equations of motion in the Serret-Frénet frame (Fig. 9.8)
 2627 derive from the Lorentz equation,

$$\frac{d\mathbf{m}\mathbf{v}}{dt} = q\mathbf{v} \times \mathbf{B} \Rightarrow m \frac{d}{dt} \begin{pmatrix} \frac{ds}{dt} \mathbf{s} \\ \frac{dx}{dt} \mathbf{x} \\ \frac{dy}{dt} \mathbf{y} \end{pmatrix} = q \begin{pmatrix} \left(\frac{dx}{dt} B_y - \frac{dy}{dt} B_x \right) \mathbf{s} \\ -\frac{ds}{dt} B_y \mathbf{x} \\ \frac{ds}{dt} B_x \mathbf{y} \end{pmatrix} \quad (9.16)$$

2628 Introduce the field index $n = -\frac{\rho_0}{B_0} \frac{\partial B_y}{\partial x}$ evaluated on the reference orbit, with $B_0 =$
 2629 $B_y(\rho_0, y = 0)$; assume transverse stability: $0 < n < 1$. Taylor expansion of the field
 2630 components in the moving frame write

$$\begin{aligned} B_y(\rho) &= B_y(\rho_0) + x \left. \frac{\partial B_y}{\partial x} \right|_{\rho_0} + O(x^2) \approx B_y(\rho_0) - n \frac{B_y}{\rho_0} \Big|_{\rho_0} x = B_0 \left(1 - n \frac{x}{\rho_0}\right) \\ B_x(0 + y) &= \underbrace{B_x(0)}_{=0} + y \underbrace{\left. \frac{\partial B_x}{\partial y} \right|_{\rho_0}}_{= \frac{\partial B_y}{\partial x}} + \text{higher order in } y \approx -n \frac{B_0}{\rho_0} y \end{aligned} \quad (9.17)$$

2631 Introduce in addition $ds \approx v dt$, Eqs. 9.16, 9.17 lead to the differential equations of
 2632 motion in a dipole field

$$\frac{d^2x}{ds^2} + \frac{1-n}{\rho_0^2}x = 0, \quad \frac{d^2y}{ds^2} + \frac{n}{\rho_0^2}y = 0 \quad (0 < n = \frac{\rho_0}{B_0} \frac{\partial B_y}{\partial x} < 1) \quad (9.18)$$

2633 It results that, in an S-periodic structure comprised of dipoles, wedges and drift
 2634 spaces, the differential equation of motion takes the general form of Hill's equation, a
 2635 second order differential equation with periodic coefficient, namely (with z standing
 2636 for x or y),

$$\begin{cases} \frac{d^2z}{ds^2} + K_z(s)z = 0 \\ K_z(s+S) = K_z(s) \end{cases} \quad \text{with} \quad \begin{cases} \text{in dipoles : } \begin{cases} K_x = (1-n)/\rho_0^2 \\ K_y = n/\rho_0^2 \end{cases} \\ \text{at a wedge : } K_y = \pm(\tan \epsilon)/\rho_0 \\ \text{in drift spaces : } K_x = K_y = 0 \end{cases} \quad (9.19)$$

2637 $K_z(s)$ is S-periodic, $S = 2\pi R/N$ ($S = C/4$, for instance, in the 4-periodic ring
 2638 Saturne 1 (Figs. 9.3, 9.8)). G. Floquet has established [17] that the two independent
 2639 solutions of Hill's second order differential equation have the form

$$\begin{cases} z_1(s) = \sqrt{\beta_z(s)} e^{i \int_0^s \frac{ds}{\beta_z(s)}} \\ dz_1(s)/ds = \frac{i - \alpha_z(s)}{\beta_z(s)} z_1(s) \end{cases} \quad \text{and} \quad \begin{cases} z_2(s) = z_1^*(s) \\ dz_2(s)/ds = dz_1^*(s)/ds \end{cases} \quad (9.20)$$

2640 wherein $\beta_z(s)$ and $\alpha_z(s) = -\beta'_z(s)/2$ are S-periodic functions, from what it results
 2641 that

$$z_{\frac{1}{2}}(s+S) = z_{\frac{1}{2}}(s) e^{\pm i \mu_z} \quad (9.21)$$

2642 wherein

$$\mu_z = \int_{s_0}^{s_0+S} \frac{ds}{\beta_z(s)} \quad (9.22)$$

2643 is the betatron phase advance over a period. A real solution of Hill's equation is
 2644 the linear combination $A z_1(s) + A^* z_2^*(s)$. Take A of the form $A = \frac{1}{2} \sqrt{\varepsilon/\pi} e^{i\phi}$
 2645 (the introduction of the constant multiplicative factor $\sqrt{\varepsilon/\pi}$ is justified below), the
 2646 general solution of Eq. 9.19 then takes the form (noting $(*)' = d(*)/ds$)

$$\begin{cases} z(s) = \sqrt{\beta_z(s)\varepsilon/\pi} \cos\left(\int \frac{ds}{\beta_z} + \phi\right) \\ z'(s) = -\sqrt{\frac{\varepsilon/\pi}{\beta_z(s)}} \sin\left(\int \frac{ds}{\beta_z} + \phi\right) + \alpha_z(s) \cos\left(\int \frac{ds}{\beta_z} + \phi\right) \end{cases} \quad (9.23)$$

2647 The motion coordinates satisfy the following ellipse equation, Courant-Snyder in-
 2648 variant,

$$\frac{1}{\beta_z(s)} [z^2 + (\alpha_z(s)z + \beta_z(s)z')^2] = \frac{\varepsilon}{\pi} \quad (9.24)$$

2649 At a given azimuth s of the periodic structure the observed turn-by-turn motion lies
 2650 on that ellipse (Fig. 9.13). The form of the ellipse depends on the observation azimuth
 2651 s via the respective local values of $\alpha_z(s)$ and $\beta_z(s)$, but its surface ε is invariant.
 2652 Motion along the ellipse is clockwise, as can be figured from Eq. 9.23 considering
 2653 an observation azimuth s where the ellipse is upright, $\alpha_z(s) = 0$.

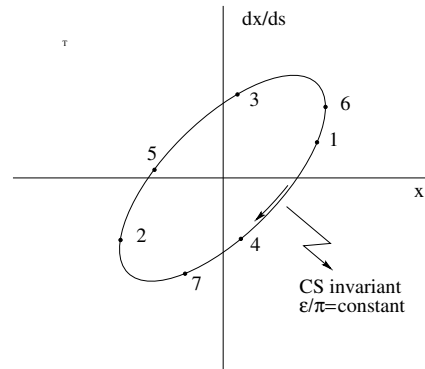


Fig. 9.13 Courant-Snyder invariant and turn-by-turn harmonic motion. The form of the ellipse depends on the observation azimuth s but its surface ε is invariant

2654 If a turn is comprised of N periods, the phase advance over a turn (from one
 2655 location to the next on the ellipse in Fig. 9.13) is

$$\int_{s_0}^{s_0+NS} \frac{ds}{\beta_z(s)} = N \int_{s_0}^{s_0+S} \frac{ds}{\beta_z(s)} = N\mu_z \quad (9.25)$$

2656 *Weak focusing approximation*

In the case of a cylindrically symmetric structure, a sinusoidal motion (Eqs. 4.13, 4.14, “Classical Cyclotron” Chapter) is the exact solution of the first order differential equations of motion. In that case the latter have a constant (s -independent) coefficient, $K_x = (1 - n)/R_0^2$ and $K_y = n/R_0^2$, respectively. Adding drift spaces results in Hill’s differential equation with periodic coefficient $K(s + S) = K(s)$ (Eq. 9.19) and to a pseudo harmonic solution (Eq. 9.23). Due to the weak focusing the beam envelope (Eq. 9.30) is only weakly modulated, thus so is $\beta_z(s)$. In a practical manner, the modulation of $\beta_z(s)$ does not exceed a few percent, this justifies introducing the average value $\bar{\beta}_z$ to approximate the phase advance by

$$\int_0^s \frac{ds}{\beta_z(s)} \approx \frac{s}{\bar{\beta}_z} = v_z \frac{s}{R}$$

2657 The right equality is obtained by applying this approximation to the the phase advance
 2658 per period (Eq. 9.32), namely $\mu_z = \int_{s_0}^{s_0+S} \frac{ds}{\beta_z(s)} \approx S/\bar{\beta}_z$, and introducing the wave
 2659 number of the N -period optical structure

$$\nu_z = \frac{N\mu_z}{2\pi} = \frac{\text{phase advance over a turn}}{2\pi} \quad (9.26)$$

2660 so that

$$\bar{\beta}_z = \frac{R}{\nu_z} \quad (9.27)$$

2661 Substituting in Eq. 9.23 results in the approximate solution

$$\begin{cases} z(s) \approx \sqrt{\beta_z(s)\varepsilon/\pi} \cos\left(\nu_z \frac{s}{R} + \phi\right) \\ z'(s) = -\sqrt{\frac{\varepsilon/\pi}{\beta_z(s)}} \sin\left(\nu_z \frac{s}{R} + \phi\right) + \alpha_z(s) \cos\left(\nu_z \frac{s}{R} + \phi\right) \end{cases} \quad (9.28)$$

2662 In this approximation, the differential equations of motion (Eq. 9.19) can be
2663 expressed under the form

$$\frac{d^2x}{ds^2} + \frac{\nu_x^2}{R^2}x = 0, \quad \frac{d^2y}{ds^2} + \frac{\nu_y^2}{R^2}y = 0 \quad (9.29)$$

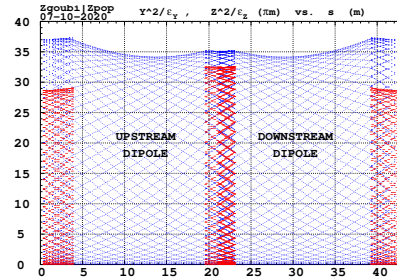
2664 *Beam envelopes*

2665 The beam envelope $\hat{z}(s)$ (with z standing for x or y) is determined by the particle of
2666 maximum invariant ε/π , it is given by

$$\pm \hat{z}(s) = \pm \sqrt{\beta_z(s)\varepsilon/\pi} \quad (9.30)$$

As $\beta_z(s)$ is S -periodic, so is the envelope, $\hat{z}(s+S) = \hat{z}(s)$. In a cell with symmetries

Fig. 9.14 ***** rem-
place par envelope in saturnel
***** Beam envelope
along Saturne I four cells,
generated by a single particle
over many turns. The extreme
excursion at any azimuth s
tangents the envelope. En-
velopes along a cell feature
central symmetry, as does the
cell



2667 (for instance symmetry with respect to the center of the cell), the envelope features
2668 the same symmetries. Envelope extrema are at azimuth s where $\beta_z(s)$ is minimum,
2669 or maximum, *i.e.*, where $\alpha_z = 0$ as $\beta'_z = -2\alpha_z$. This is illustrated in Fig. 9.14. No
2670 particular hypothesis regarding the amplitude of the motion is required here, it does
2671

2672 not have to be paraxial and can be arbitrarily large (as long as transverse stability
2673 still holds).

2674 In the paraxial approximation, envelopes along the optical structure can be deter-
2675 mined by resorting to matrix transport (*cf.* reminders in Section 19.3.2). An initial
2676 beam matrix at some azimuth s , as well as the phase advance over a period, can
2677 be obtained using the stability criterion (Eq. 19.3.3). This is a simple exercise in
2678 the case of Saturne I type of structure (Figs. 9.3, 9.8). The transport matrix of the
2679 symmetric drift-dipole-drift cell satisfies

$$\begin{aligned}
 [T_{\text{per.}}] &= \begin{bmatrix} 1 & l \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \cos(\sqrt{K_z}\rho_0\alpha) & \frac{1}{\sqrt{K_z}} \sin(\sqrt{K_z}\rho_0\alpha) \\ -\sqrt{K_z} \sin(\sqrt{K_z}\rho_0\alpha) & \cos(\sqrt{K_z}\rho_0\alpha) \end{bmatrix} \begin{bmatrix} 1 & l \\ 0 & 1 \end{bmatrix} \\
 &= \begin{bmatrix} \cos(\sqrt{K_z}\rho_0\alpha) - \sqrt{K_z}l \sin(\sqrt{K_z}\rho_0\alpha) & 2l \cos(\sqrt{K_z}\rho_0\alpha) + \frac{1}{\sqrt{K_z}} \sin(\sqrt{K_z}\rho_0\alpha)(1 - K_z l^2) \\ -\sqrt{K_z} \sin(\sqrt{K_z}\rho_0\alpha) & \cos(\sqrt{K_z}\rho_0\alpha) - \sqrt{K_z}l \sin(\sqrt{K_z}\rho_0\alpha) \end{bmatrix} \\
 &\approx \begin{bmatrix} \cos \sqrt{K_z}(\rho_0\alpha + l) & 2l \cos(\sqrt{K_z}\rho_0\alpha) + \frac{1}{\sqrt{K_z}} \sin(\sqrt{K_z}\rho_0\alpha) \\ -\sqrt{K_z} \sin(\sqrt{K_z}\rho_0\alpha) & \cos \sqrt{K_z}(\rho_0\alpha + l) \end{bmatrix} \quad (9.31)
 \end{aligned}$$

2680 The approximation is obtained by assuming that the drift length $2l$ is small compared
2681 to the arc length $\rho_0\alpha$. From the stability criterion $[T_{\text{per.}}] = I \cos \mu_z + J \sin \mu_z$ it results
2682 that $\frac{1}{2} \text{Tr}[T_{\text{per.}}] = \cos \mu_z$, which yields the phase advance

$$\mu_z = \sqrt{K_z}(\rho_0\alpha + l) = \sqrt{K_z}\rho_0\alpha(1 + k/2) \quad (9.32)$$

2683 With $\nu_z = N\mu_z/2\pi$ and (Eq. 9.19) $K_x = (1 - n)/\rho_0^2$, $K_y = n/\rho_0^2$, $N\alpha = 2\pi$,
2684 $k = 2l/\rho_0\alpha \ll 1$, this yields for the horizontal and vertical tunes

$$\nu_x \approx \sqrt{1 - n}(1 + \frac{k}{2}) \approx \sqrt{(1 - n)\frac{R}{\rho_0}}, \quad \nu_y \approx \sqrt{n}(1 + \frac{k}{2}) \approx \sqrt{n\frac{R}{\rho_0}} \quad (9.33)$$

2685 The identification $[T_{\text{per.}}] = I \cos \mu_z + J \sin \mu_z$ allows writing $[T_{\text{per.}}]$ under the
2686 form

$$[T_{\text{per.}}] = \begin{bmatrix} \cos \sqrt{K_z}(\rho_0\alpha + l) & \frac{1 + \sqrt{K_z}l \cot(\sqrt{K_z}\rho_0\alpha)}{\sqrt{K_z}} \sin \sqrt{K_z}(\rho_0\alpha + l) \\ -\frac{\sqrt{K_z}}{1 + \sqrt{K_z}l \cot(\sqrt{K_z}\rho_0\alpha)} \sin \sqrt{K_z}(\rho_0\alpha + l) & \cos \sqrt{K_z}(\rho_0\alpha + l) \end{bmatrix} \quad (9.34)$$

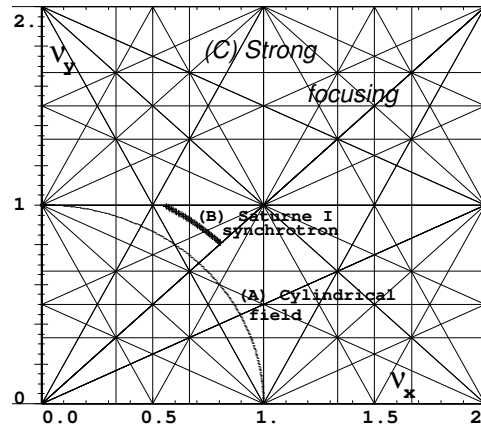
2687 so leading to the optical functions at the center of the drift,

$$\alpha_z = 0, \quad \beta_z = \frac{1}{\sqrt{K_z}} \left[1 + \sqrt{K_z}l \cot(\sqrt{K_z}\rho_0\alpha) \right] \quad (9.35)$$

2688 *Stability diagram*

2689 The “working point” of the synchrotron is the couple (ν_x, ν_y) at which the acceler-
 2690 ator is operated, it fully characterizes the focusing. In a structure with cylindrical
 2691 symmetry (cf. Eq. 4.15) $\nu_x = \sqrt{1-n}$ and $\nu_y = \sqrt{n}$ so that $\nu_x^2 + \nu_y^2 = 1$: when the
 2692 radial field index n is changed the working point stays on a circle of radius 1 in the
 stability diagram (or “tune diagram”, Fig. 9.15). If drift spaces are added, in a first

Fig. 9.15 Location of the working point in the tune diagram, in case of (A) field with revolution symmetry, on a circle of radius 1; (B) sector field with index + drift spaces, on a circle of radius $(\sqrt{R/\rho_0})$; (C) strong focusing, $(|n| \gg 1)$, in large ν_x, ν_y regions.



2693 approximation (Eq. 9.33)
 2694

$$\nu_x = \sqrt{(1-n)\frac{R}{\rho_0}}, \quad \nu_y = \sqrt{n\frac{R}{\rho_0}}, \quad \nu_x^2 + \nu_y^2 = \frac{R}{\rho_0} \quad (9.36)$$

2695 the working point is located on the circle of radius $\sqrt{R/\rho_0} > 1$.

Horizontal and vertical focusing are not independent: if ν_x increases then ν_y decreases and reciprocally; none can exceed the limits

$$0 < \nu_{x,y} < \sqrt{R/\rho_0}$$

2696 This is a lack of flexibility which strong focusing will overcome by providing two
 2697 knobs so allowing adjustment of both tunes separately.

2698 **9.1.3 Longitudinal Motion**2699 **Acceleration of the Ideal Particle**

2700 In a synchrotron, the field B is varied (a function performed by the power supply)
 2701 as well as the bunch momentum p (a function performed by the accelerating cavity)
 2702 in such a way that at any time $B(t)\rho = p(t)/q$ (ρ is the curvature radius of the
 2703 central trajectory in the bending magnets). If this condition is fulfilled, then at all
 2704 times during the acceleration cycle the central trajectory remains on the design
 2705 optical axis, which is comprised of the reference arc in the dipoles, of the axis of
 2706 the vacuum pipe in the straight section, of the accelerating cavities, of the beam
 2707 position monitors, etc. Given the energies involved, the magnet supply imposes its
 2708 law and the cavity follows $B(t)$, the best in can. A schematic $B(t)$ law is represented
 in Fig. 9.16.

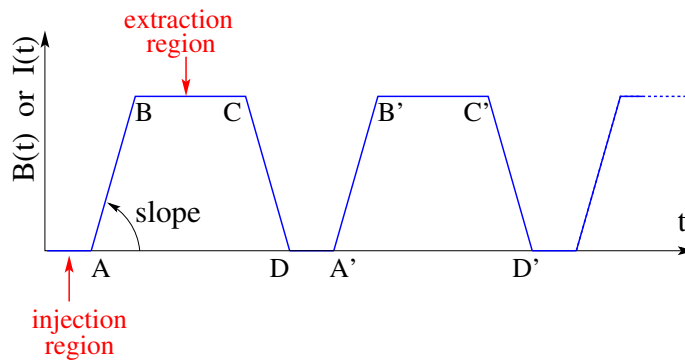


Fig. 9.16 Cycling $B(t)$ in a pulsed synchrotron. Ignoring saturation, $B(t)$ is proportional to the magnet power supply current $I(t)$. Bunch injection occurs at low field, in the region of A, extraction occurs at top energy, on the high field plateau. (AB): field ramp up (acceleration); (BC): flat top (includes beam extraction period); (CD): field ramp down; (DA'): thermal relaxation. (AA'): repetition period; $(1/AA')$: repetition rate; $slope$: ramp velocity $\dot{B} = dB/dt$ (Tesla/s).

2709

Typical values from Saturne I synchrotron are given in Tab. 9.1. As the central

Table 9.1 Saturne I field parameters

\dot{B}	1.8 T/s
B_{max}	1.5 T
ρ	8.42 m
$B_{max}\rho$	13 T m

trajectory length is fixed ($2\pi R \approx 68.9$ m, see Tab. 9.2) whereas particle velocity increases turn after turn, thus the revolution time T_{rev} varies.

$$T_{rev} = \frac{\text{duration of a turn}}{\text{velocity}} = \frac{2\pi R}{\beta c}$$

$$R_{\text{Sat.I}} = 10.97 \text{ m}, \begin{cases} \text{initial } E = 3.6 \text{ MeV} \\ \text{final } E = 2.94 \text{ GeV} \end{cases} \Rightarrow \begin{cases} T_{\text{rev}} = \frac{2\pi R}{0.09 \times 3 \times 10^8} = 16.5 \mu\text{s}; f = 0.06 \text{ MHz} \\ T_{\text{rev}} = \frac{2\pi R}{0.97 \times 3 \times 10^8} = 0.24 \mu\text{s}; f = 4.2 \text{ MHz} \end{cases}$$

The accelerating voltage $\hat{V}(t) = \sin \omega_{\text{rf}} t$ is maintained in synchronism with the revolution motion, thus its angular frequency ω_{rf} follows hf_{rev} ,

$$\omega_{\text{rf}} = h\omega_{\text{rev}} = h \frac{c}{R} \frac{B(t)}{\sqrt{\left(\frac{m_0}{q\rho}\right)^2 + B^2(t)}}$$

2710 Energy gain

2711 The variation of the particle energy over a turn amounts to the work of the force
2712 $F = dp/dt$ on the charge at the cavity, namely

$$\Delta W = F \times 2\pi R = 2\pi q R \rho \dot{B} \quad (9.37)$$

Over most of the acceleration cycle in a slow-cycling synchrotron \dot{B} is usually constant (Eq. 9.3), thus so is ΔW . At Saturne I for instance

$$\frac{\Delta W}{q} = 2\pi R \rho \dot{B} = 68.9 \times 8.42 \times 1.8 = 1044 \text{ volts}$$

The field ramp lasts

$$\Delta t = (B_{\text{max}} - B_{\text{min}})/\dot{B} \approx B_{\text{max}}/\dot{B} = 0.8 \text{ s}$$

The number of turns to the top energy ($W_{\text{max}} \approx 3 \text{ GeV}$) is

$$N = \frac{W_{\text{max}}}{\Delta W} = \frac{3 \times 10^9 \text{ eV}}{1044 \text{ eV}} \approx 3 \times 10^6$$

2713 Adiabatic damping of betatron oscillations

During acceleration, focusing strengths follow the increase of particle rigidity, so to maintain the tunes ν_x and ν_y constant. As a result of the longitudinal acceleration at the cavity though, the longitudinal energy of the particles is modified. This results in a decrease of the amplitude of betatron oscillations (an increase if the cavity is decelerating). The mechanism is sketched in Fig. 9.17: the slope, respectively before (index 1) and after (index 2) the cavity is

$$\frac{dx}{ds} = \frac{m \frac{dx}{dt}}{m \frac{ds}{dt}} = \frac{p_x}{p_s}, \quad \frac{dx}{ds} \Big|_2 = \frac{m \frac{dx}{dt} \Big|_2}{m \frac{ds}{dt} \Big|_2} = \frac{p_{x,2}}{p_{s,2}}$$

Particle mass and velocity are modified at the traversal of the cavity but, as the

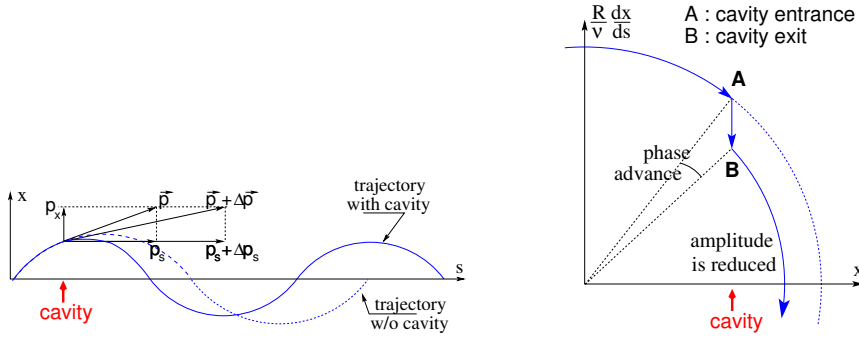


Fig. 9.17 Adiabatic damping of betatron oscillations, here from $x' = p_x/p_s$ before the cavity, to $x'_2 = p_x/(p_s + \Delta p_s)$ after the cavity. In the horizontal phase space, to the right, decrease of $\Delta \left(\frac{dx}{ds} \right)$ if $\frac{dx}{ds} > 0$, increase of $\Delta \left(\frac{dx}{ds} \right)$ if $\frac{dx}{ds} < 0$

force is longitudinal, $dp_x/dt = 0$ thus $p'_x = p_x$, the increase in momentum is purely longitudinal, $p'_s = p_s + \Delta p$. Thus

$$\left. \frac{dx}{ds} \right|_2 = \frac{p_x}{p_s + \Delta p} \approx \frac{p_x}{p_s} \left(1 - \frac{\Delta p}{p_s} \right)$$

and as a consequence the slope dx/ds varies across the cavity,

$$\Delta \left(\frac{dx}{ds} \right) = \left. \frac{dx}{ds} \right|_2 - \frac{dx}{ds} = - \frac{dx}{ds} \frac{\Delta p_s}{p_s}$$

2714 The slope varies in proportion to the slope, with opposite sign if $\Delta p/p > 0$ (acceleration) thus a decrease of the slope. This variation has two consequences on the
 2715 betatron oscillation (Fig. 9.17):
 2716 - a change of the betatron phase,
 2717 - a modification of the betatron amplitude.
 2718

2719 *In matrix form*

2720 Coordinate transport through the cavity writes $\begin{cases} x_2 = x \\ x'_2 \approx \frac{p_x}{p_s} \left(1 - \frac{dp}{p} \right) = x' \left(1 - \frac{dp}{p} \right) \end{cases}$,
 2721 hence the transfer matrix of the cavity,

$$[C] = \begin{bmatrix} 1 & 0 \\ 0 & 1 - \frac{dp}{p} \end{bmatrix} \tag{9.38}$$

2722 its determinant is $1 - dp/p \neq 1$: the system is non-conservative (the surface in phase
 2723 space is not conserved). Assume one cavity in the ring and not $[T] \times [C]$ the one-turn
 2724 matrix with origin at entrance of the cavity. Its determinant is $det[T] \times det[C] =$
 2725 $det[C] = 1 - \frac{dp}{p}$. Over N turns the coordinate transport matrix is $([T][C])^N$, its
 2726 determinant is $(1 - \frac{dp}{p})^N \approx 1 - N \frac{dp}{p}$. The surface of the beam ellipse is $\varepsilon \times$
 2727 $det[T]_{turn} = \varepsilon_0 - \varepsilon \frac{dp}{p}$ thus $\frac{d\varepsilon}{\varepsilon} = -\frac{dp}{p}$, the solution of which is

$$\varepsilon \times p = \text{constant, or } \beta\gamma\varepsilon = \text{constant} \quad (9.39)$$

2728 **Synchrotron motion; the synchronous particle**

2729 By “synchrotron motion”, or “phase oscillations”, it is meant a mechanism that
 2730 stabilizes the longitudinal motion of a particle around a synchronous phase, in virtue
 2731 of

2732 (i) the presence of an accelerating cavity with its frequency indexed on the
 2733 revolution time,

2734 (ii) with the bunch centroid positioned either on the rising slope of the oscillating
 2735 voltage (low energy regime), or on the falling slope (high energy regime).

The synchronous (or “ideal”) particle follows the equilibrium trajectory around
 the ring (the reference closed orbit, about which all other particles will undergo a
 betatron oscillation) and its velocity satisfies

$$B\rho = \frac{p}{q} = \frac{mv}{p} \rightarrow v = \frac{qB\rho}{m}$$

2736 - the revolution time is $T_{rev} = \frac{2\pi R}{v} = \frac{2\pi R}{\beta c} = \frac{2\pi R}{qB\rho/m}$
 - the angular revolution frequency follows the increase of B:

$$\omega_{rev} = \frac{2\pi}{T_{rev}} = \frac{qB\rho}{mR}$$

2737 - during the acceleration $B(t)$ increases at a $\frac{dB}{dt} = \dot{B}$ rate normally of the order of a
 2738 Tesla/second.

- in order for the ideal particle to stay on the closed orbit during the acceleration, its
 changing momentum must at all time satisfy $B(t)\rho = p(t)/q$. This defines $p(t)$ as a
 function of $B(t)$, and the following B dependence of mass and angular frequency:

$$m(t) = \gamma(t)m_0 = \frac{q\rho}{c} \sqrt{\left(\frac{m_0}{qc\rho}\right)^2 + B(t)^2}$$

$$\omega_{rev}(t) = \frac{c}{R} \frac{B(t)}{\sqrt{\left(\frac{m_0}{qc\rho}\right)^2 + B(t)^2}}$$

2739 - the RF voltage frequency $\omega_{RF}(t) = h\omega_{rev}(t)$ follows $B(t)$, this maintains the
 2740 synchronous phase at a fixed value
 2741 - over a turn the gain in energy is $\Delta W = 2\pi qR\rho\dot{B}$, the reference particle experiences
 2742 a voltage $V = \Delta W/q = 2\pi R\rho\dot{B}$.
 2743 Simulation wise, the ramping of the guide field can be assumed to follow a step
 2744 function in correlation with the step increase of particle momentum at the RF cavity.
 2745 In that manner, the synchronous particle is maintained on the design orbit, at radius
 2746 $\rho = p(t)/qB(t)=\text{constant}$ in the guide magnets.

2747 *Phase Stability*

2748 The mechanism of phase stability has, first experient in the synchrocyclotron [18]
 2749 has been introduced in the eponym Chapter (Chap. 8). It is re-visited here accounting
 2750 for specificities of the operation of a synchrotron, such as the constant radius orbit, of
 2751 the concept of transition energy.

Note ϕ_s the RF phase at arrival of the synchronous particle at the aforementioned accelerating cavity, its energy gain is

$$\Delta W = q\hat{V} \sin \phi_s = 2\pi qR\rho\dot{B}$$

The condition $|\sin \phi_s| < 1$ imposes a lower limit to the cavity voltage for acceleration to happen, namely

$$\hat{V} > 2\pi R\rho\dot{B}$$

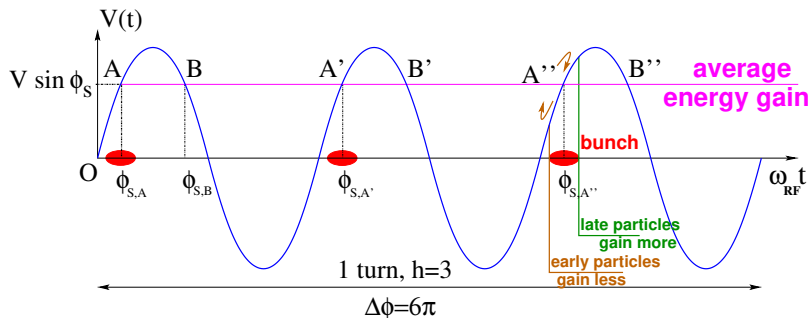


Fig. 9.18 Mechanism of phase stability, “longitudinal focusing”. Below transition ($\gamma < \gamma_{tr}$) phase stability occurs for a synchronous phase taken at either of the $h=3$ stable locations A, A', A'': a particle with higher energy goes around the ring more rapidly than the synchronous particle, it arrives earlier at the voltage gap (at $\phi < \phi_{s,A}$) and experiences a lower voltage; at lower energy the particle is slower, it arrives at the gap later compared to the synchronous particle, at $\phi > \phi_{s,A}$, and experiences a greater voltage; this results overall in a stable oscillatory motion around the synchronous phase. Beyond transition ($\gamma > \gamma_{tr}$) the stable phase is at either of the $h=3$ stable locations B, B', B'': a particle which is less energetic than the synchronous particle arrives earlier, $\phi < \phi_{s,B}$, it experiences a greater voltage, and inversely when it eventually gets more energetic than the synchronous particle

2752 Referring to Fig. 9.18, the synchronous phase can be placed on the left (A A' A''...
 2753 series in the Figure, or on the right (B B' B''... series) of the oscillating voltage crest.
 2754 One and only one of these two possibilities, and which one depends on the optical
 2755 lattice and on particle energy, ensures that particles in a bunch remain grouped in the
 2756 vicinity of the synchronous particles. The transition between these two regimes (A
 2757 series or B series) occurs at the transition γ , γ_{tr} , a property of the lattice. If the bunch
 2758 energy is below transition energy, $E_{bunch} < m\gamma_{tr}$, the bunch has to present itself on
 2759 the left of the crest (A series), if the bunch energy is greater than transition energy,
 2760 $E_{bunch} > m\gamma_{tr}$, the bunch has to present itself on the right of the crest (B series).

2761 Transition energy

2762 The transition between the two regimes occurs at $\frac{dT_{rev}}{T_{rev}} = 0$. With $T = 2\pi/\omega = C/v$,
 2763 this can be written $\frac{d\omega_{rev}}{\omega_{rev}} = -\frac{dT_{rev}}{T_{rev}} = \frac{dv}{v} - \frac{dC}{C}$. With $\frac{dv}{v} = \frac{1}{\gamma^2} \frac{dp}{p}$ and momentum
 2764 compaction $\alpha = \frac{dC}{C} / \frac{dp}{p}$, (Eq. 9.15), this can be written

$$\frac{d\omega_{rev}}{\omega_{rev}} = -\frac{dT_{rev}}{T_{rev}} = \left(\frac{1}{\gamma^2} - \alpha\right) \frac{dp}{p} = \eta \frac{dp}{p} \quad (9.40)$$

2765 wherein the phase-slip factor has been introduced,

$$\eta = \overbrace{\frac{1}{\gamma^2}}^{\text{kinematics}} - \underbrace{\alpha}_{\text{lattice}} \quad (9.41)$$

2766 In a weak focusing structure $\alpha \approx 1/\nu_x^2$ (Eqs. 4.19, 9.15), thus the phase stability
 2767 regime is

$$\text{below transition, i.e. } \phi_s < \pi/2, \quad \text{if } \gamma < \nu_x \quad (9.42)$$

$$\text{above transition, i.e. } \phi_s > \pi/2, \quad \text{if } \gamma > \nu_x \quad (9.43)$$

$$(9.44)$$

2768 In weak focusing synchrotrons the horizontal tune $\nu_x = \sqrt{(1-n)R/\rho_0}$ (Eq. 9.33)
 2769 may be ≥ 1 , and subsequently $\gamma_{tr} \approx \nu_x \geq 1$ depending on the horizontal tune value.
 2770 Saturne I for instance, with $\nu_x \approx 0.7$ (Tab. 9.2), operated above transition energy.

2771 9.1.4 Spin Motion, Depolarizing Resonances

2772 The availability of polarized proton sources allowed the acceleration of polarized
 2773 beams to high energy. The possibility was considered from the early times of the
 2774 ZGS [19], up to 70% polarization transmission through the synchrotron was fore-
 2775 seen, polarization manipulation concepts included harmonic orbit correction, tune
 2776 jump at strongest depolarizing resonances (Fig. 9.19). Acceleration of a polarized
 2777 proton beam happened for the first time in a synchrotron and to multi-GeV energy in
 2778 1973, four years after the ZGS startup. Beams were accelerated up to 17 GeV with
 2779 substantial polarization maintained [12]. Experiments were performed to assess the
 2780 possibility of polarization transmission in strong focusing synchrotrons, and polar-
 2781 ization lifetime in colliders [20]. Acceleration of polarized deuteron was achieved in
 2782 the late 1970s, when sources were made available [21].

2783 The field index is essentially zero in the ZGS, transverse focusing is ensured
 2784 by wedge angles at the ends of the height dipoles, which is thus the only location
 2785 where non-zero horizontal field components are found. The vertical wave number
 2786 is small in addition, less than 1. This results in depolarizing resonance strengths
 2787 on the weak side, “As we can see from the table, the transition probability [from
 2788 spin state $\psi_{1/2}$ to spin state $\psi_{-1/2}$] is reasonably small up to $\gamma = 7.1$ ” [12], i.e.
 2789 $G\gamma = 12.73$, $p = 6.6$ GeV/c; the table referred to stipulates a transition probability
 2790 $P_{\frac{1}{2}, -\frac{1}{2}} < 0.042$, whereas resonances beyond that energy range feature $P_{\frac{1}{2}, -\frac{1}{2}} > 0.36$.
 2791 Beam depolarization up to 6 GeV/c, under the effect of these resonances, is illustrated
 2792 in Fig. 9.19.

2793 In weak focusing synchrotron particles experience radial fields all along the
 2794 bend dipoles as an effect of the radial field index, as they undergo vertical betatron
 2795 oscillations. However these radial field components are weak, and so is their effect
 2796 on spin motion, as long as the particle energy (the γ factor in the spin precession
 2797 equation) is not too high.

Assuming a defect-free ring, the vertical betatron motion excites “intrinsic” spin
 resonances, located at

$$G\gamma_R = kP \pm \nu_y$$

with k an integer and P the period of the ring. In the ZGS for instance, $\nu_y \approx 0.8$
 (Tab. 9.3), the ring $P=4$ -periodic, thus $G\gamma_R = 4k \pm 0.8$. Strongest resonances are
 located at

$$G\gamma_R = MPk \pm \nu_y$$

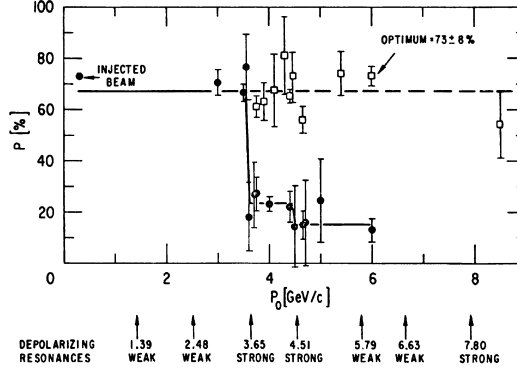
2798 with M the number of cells per superperiod [22, Sec. 3.II]. In the ZGS, $M=2$ thus
 2799 strongest resonances occur at $G\gamma_R = 2 \times 4k \pm 0.8$.

In the presence of vertical orbit defects, non-zero periodic transverse fields are ex-
 perimented along the closed orbit, they excite “imperfection” depolarizing resonances,
 located at

$$G\gamma_R = k$$

with k an integer. In the case of systematic defects the periodicity of the orbit is
 that of the lattice, P , imperfection resonances are located at $G\gamma_R = kP$. Strongest

Fig. 9.19 Depolarizing intrinsic resonance landscape up to 6 GeV/c at the ZGS (solid circles). Systematic resonances are located at $G\gamma_R = 4 \times \text{integer} \pm \nu_y$, stronger ones at $G\gamma_R = 8 \times \text{integer} \pm \nu_y$. Tune jump was used to preserve polarization when crossing strong resonances (empty circles) [23]



imperfection resonances are located at

$$G\gamma_R = MPk$$

2800 with M the number of cells per superperiod [22, Sec. 3.II]. Crossing a depolarizing
 2801 resonance, during acceleration, causes a loss of polarization given by (Froissart-Stora
 2802 formula [11])

$$\frac{P_f}{P_i} = 2e^{-\frac{\pi}{2} \frac{|\epsilon_R|^2}{\alpha}} - 1 \quad (9.45)$$

2803 from a value P_i upstream to an asymptotic value P_f downstream of the resonance.
 2804 This assumes an isolated resonance, passed with a crossing speed

$$\alpha = G \frac{d\gamma}{d\theta} = \frac{1}{2\pi} \frac{\Delta E}{M} \quad (9.46)$$

2805 with ΔE the energy gain per turn and M the mass. ϵ_R is the resonance strength.

2806 *Spin precession axis. Resonance width*

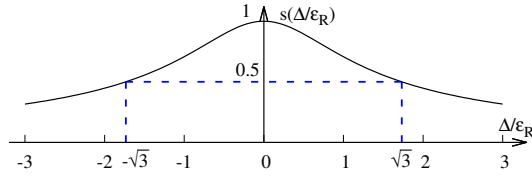
2807 Consider the spin vector $\mathbf{S}(\theta) = (S_\eta, S_\xi, S_y)$ of a particle in the laboratory frame,
 2808 with θ the orbital angle around the accelerator. Introduce the projection $s(\theta)$ of \mathbf{S} in
 2809 the median plane

$$s(\theta) = S_\eta(\theta) + jS_\xi(\theta) \quad (\text{and } S_y^2 = 1 - s^2) \quad (9.47)$$

2810

2811 It can be shown that in the case of a stationary solution of the spin motion (*i.e.*,
 2812 the spin precession axis) s satisfies [24] (Fig. 9.20)

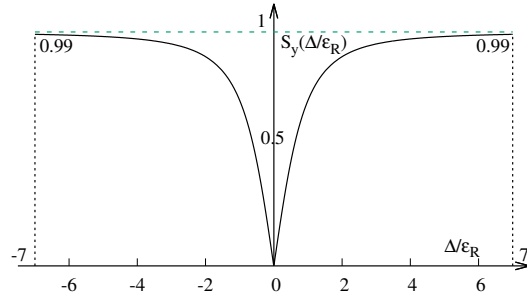
Fig. 9.20 Modulus of the horizontal spin component. $s = 1/2$ at distance $\Delta = \pm\sqrt{3}\epsilon_R$ from $G\gamma_R$



$$s^2 = \frac{1}{1 + \frac{\Delta^2}{|\epsilon_R|^2}} \quad (9.48)$$

wherein $\Delta = G\gamma - G\gamma_R$ is the distance to the resonance. The resonance width is a

Fig. 9.21 Dependence of polarization on the distance to the resonance. For instance $S_y = 0.99$, 1% depolarization, corresponds to $\Delta = 7|\epsilon_R|$. On the resonance, $\Delta = 0$, the precession axis lies in the median plane, $S_y = 0$



2813
2814 measure of its strength (Fig. 9.21). The quantity of interest is the angle, ϕ , of the
2815 spin precession direction to the vertical axis, given by (Fig. 9.21)

$$\cos \phi(\Delta) \equiv S_y(\Delta) = \sqrt{1 - s^2} = \frac{\Delta/|\epsilon_R|}{\sqrt{1 + \Delta^2/|\epsilon_R|^2}} \quad (9.49)$$

2816 On the resonance, $\Delta = 0$, the spin precession axis lies in the bend plane: $\phi = \pm\pi/2$.
2817 $S_y = 0.99$ (1% depolarization) corresponds to a distance to the resonance $\Delta = 7|\epsilon_R|$,
2818 and spin precession axis at an angle $\phi = \arccos(0.99) = 8^\circ$ from the vertical.

2819 Conversely,

$$\frac{\Delta^2}{|\epsilon_R|^2} = \frac{S_y^2}{1 - S_y^2} \quad (9.50)$$

The precession axis is common to all spins, S_y is a measure of the polarization along the vertical axis,

$$S_y = \frac{N^+ - N^-}{N^+ + N^-}$$

2820 wherein N^+ and N^- denote the number of particles in spin states $\frac{1}{2}$ and $-\frac{1}{2}$ respectively.
2821

2822 *Spin motion through weak resonances*

Depolarizing resonances are weak up to several GeV in a weak focusing synchrotron, as the radial and/or longitudinal fields, which stem from a small radial field index and from dipole fringe fields, are weak. Spin motion $S_y(\theta)$ through a resonance in that case (*i.e.*, assuming $S_{y,f} \approx S_{y,i}$, with $S_{y,f}$ and $S_{y,i}$ the asymptotic vertical spin component values respectively upstream and downstream of the resonance) can be calculated in terms of the Fresnel integrals

$$C(x) = \int_0^x \cos\left(\frac{\pi}{2}t^2\right) dt, \quad S(x) = \int_0^x \sin\left(\frac{\pi}{2}t^2\right) dt$$

namely, with the origin of the orbital angle taken at the resonance [24] (Fig. 9.22)

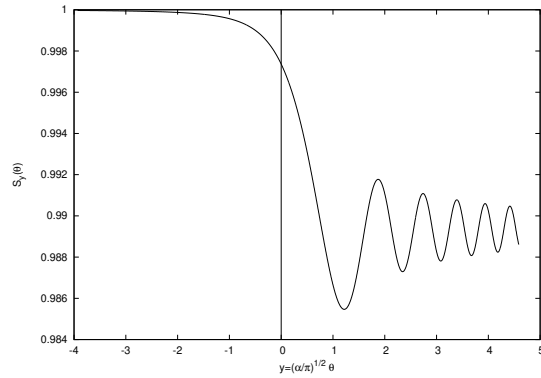


Fig. 9.22 Vertical component of spin motion $S_y(\theta)$ through a weak depolarizing resonance (after Eq. 9.51). The vertical bar is at the location of the resonance, which coincides with the origin of the orbital angle

2823

$$\begin{aligned} \text{if } \theta < 0 : \left(\frac{S_y(\theta)}{S_{y,i}} \right)^2 &= 1 - \frac{\pi}{\alpha} |\epsilon_R|^2 \left\{ \left[0.5 - C\left(-\theta\sqrt{\frac{\alpha}{\pi}}\right) \right]^2 + \left[0.5 - S\left(-\theta\sqrt{\frac{\alpha}{\pi}}\right) \right]^2 \right\} \\ \text{if } \theta > 0 : \left(\frac{S_y(\theta)}{S_{y,i}} \right)^2 &= 1 - \frac{\pi}{\alpha} |\epsilon_R|^2 \left\{ \left[0.5 + C\left(\theta\sqrt{\frac{\alpha}{\pi}}\right) \right]^2 + \left[0.5 + S\left(\theta\sqrt{\frac{\alpha}{\pi}}\right) \right]^2 \right\} \end{aligned} \quad (9.51)$$

2824 In the asymptotic limit,

$$\frac{S_y(\theta)}{S_{y,i}} \xrightarrow{\theta \rightarrow \infty} 1 - \frac{\pi}{\alpha} |\epsilon_R|^2 \quad (9.52)$$

2825 which identifies with the development of Froissart-Stora formula $P_f/P_i = 2 \exp(-\frac{\pi}{2} \frac{|\epsilon_R|^2}{\alpha}) -$
2826 1, to first order in $|\epsilon_R|^2/\alpha$. This approximation holds in the limit that higher order
2827 terms can be neglected, viz. $|\epsilon_R|^2/\alpha \ll 1$.

2828 **9.2 Exercises**

2829 **9.1 Construct Saturne I synchrotron. Spin Resonances**

2830 Solution: page 346

2831 In this exercise, Saturne I synchrotron is modeled in `zgoubi`, and spin resonances
2832 in a weak focusing gradient synchrotron are studied.

2833 (a) Construct a model of the Saturne I synchrotron, using `DIPOLE`. Use Fig. 9.23
2834 as a guidance, and parameters given in Tab. 9.2. Assume that the reference orbit is
2835 the same at all energies, on nominal radius, 841.93 cm. It is judicious (although in
2836 no way an obligation) to take $RM=841.93$ in `DIPOLE`.

2837 Check the correctness of the model by producing the lattice parameters of the
2838 ring. `TWISS` can be used for that. Compare with the lattice parameters given in
2839 Tab. 9.2.

2840 Produce a tune scan of the wave numbers over the radial field index $0.5 \leq n \leq$
2841 0.757 operation range. The `REBELOTE` do loop can be used for that, to repeatedly
2842 change n and compute a `MATRIX`. Compare with theoretical expectations.

2843 (b) Produce a graph of the betatron functions along the Saturne I cell. Provide
2844 checks of the correctness of the computation.

2845 Check the theoretical periodic dispersion (Eq. 9.14) against the radial distance
2846 between on- and off-momentum closed orbits obtained from tracking. Provide a plot
2847 of the dispersion function.

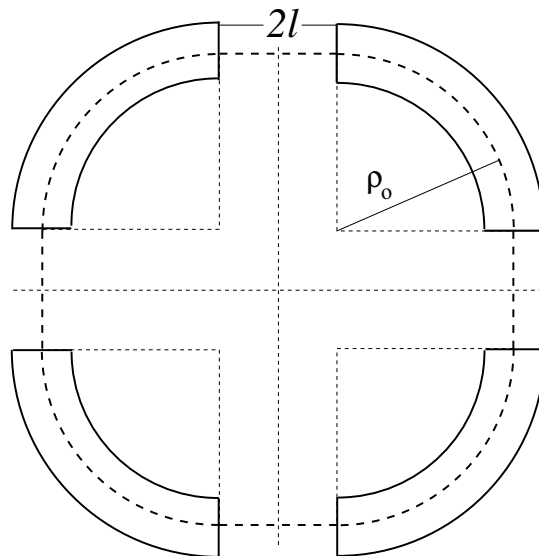


Fig. 9.23 A schematic layout of Saturne I, a $2\pi/4$ axial symmetry structure, comprised of 4 radial field index 90 deg dipoles and 4 drift spaces. The cell in the simulation exercises is taken as a $\pi/4$ quadrant: l-drift/90°-dipole/l-drift

2848 (c) Additional verifications regarding the model.

Table 9.2 Parameters of Saturne 1 weak focusing synchrotron [14]. ρ_0 denotes the reference bending radius in the dipole; the reference orbit, field index, wave numbers, etc., are taken along that radius

Orbit length, C	cm	6890
Equivalent radius, R	cm	1096.58
Straight section length, $2l$	cm	400
Magnetic radius, ρ_0	cm	841.93
R/ρ_0		1.30246
Field index n , nominal value		0.6
Wave numbers, $\nu_x; \nu_y$		0.724; 0.889
Stability limit		$0.5 < n < 0.757$
Injection energy	MeV	3.6
Field at injection	kG	0.0326
Top energy	GeV	2.94
Field at top energy	kG	14.9
Field ramp at injection	kG/s	20
Synchronous energy gain	keV/turn	1.160
RF harmonic		2

2849 Produce a graph of the field $B(s)$
 2850 - along the on-momentum closed orbit, and along off-momentum chromatic closed
 2851 orbits, across a cell;
 2852 - along orbits at large horizontal excursion;
 2853 - along orbits at large vertical excursion.
 2854 For all these cases, verify qualitatively, from the graphs, that $B(s)$ appears as
 2855 expected.

(d) Justify considering the betatron oscillation as sinusoidal, namely,

$$y(\theta) = A \cos(\nu_y \theta + \phi)$$

2856 wherein $\theta = s/R$, $R = \oint ds/2\pi$.

2857 Find the value of the horizontal and vertical betatron functions, resulting from
 2858 that approximation. Compare with the betatron functions obtained in (b).

2859 (e) Produce an acceleration cycle from 3.6 MeV to 3 GeV, for a few particles
 2860 launched on the a common $10^{-4} \pi m$ vertical initial invariant, with small horizontal
 2861 invariant. Ignore synchrotron motion (CAVITE[IOPT=3] can be used in that case).
 2862 Take a peak voltage $\hat{V} = 200$ kV (unrealistic though, as it would result in a nonphysi-
 2863 cal \hat{B} (Eq. 9.37)) and synchronous phase $\phi_s = 150$ deg (justify $\phi_s > \pi/2$). Add spin,
 2864 using SPNTRK, in view of the next question, (f).

2865 Check the accuracy of the betatron damping over the acceleration range, compared
 2866 to theory.

2867 How close to symplectic the numerical integration is (it is by definition *not*
 2868 symplectic, being a truncated Taylor series method [25, Eq. 1.2.4]), depends on the
 2869 integration step size, and on the size of the flying mesh in the DIPOLE method [25,
 2870 Fig. 20]; check a possible departure of the betatron damping from theory as a function
 2871 of these parameters.

2872 Produce a graph of the the evolution of the horizontal and vertical wave numbers
2873 during the acceleration cycle.

2874 (f) Using the raytracing material developed in (e), but for a peak voltage $\hat{V} =$
2875 20 kV, produce a graph of the value of the vertical spin component of the particles
2876 as a function of $G\gamma$, over the acceleration range from 3.6 MeV to 3 GeV.

2877 Produce a graph of the average value of S_Z over that 200 particle set, as a function
2878 of $G\gamma$. Indicate on that graph the location of the resonant $G\gamma_R$ values.

2879 (g) Based on the simulation file used in (f), simulate the acceleration of a single
2880 particle, through the intrinsic resonance $G\gamma_R = 4 - \nu_Z$, from a few thousand turns
2881 upstream to a few thousand turns downstream.

2882 Perform this resonance crossing for five different values of the particle invariant,
2883 namely: $\varepsilon_Z/\pi = 2, 10, 20, 40, 200 \mu\text{m}$.

2884 Compute P_f/P_i in each case, check the dependence on ε_Z against theory. Compute
2885 the resonance strength in each case, check the dependence on ε_Z against theory.

2886 Re-do this crossing simulation for a different crossing speed (take for instance
2887 $\hat{V} = 10 \text{ kV}$) and a couple of vertical invariant values, compute P_f/P_i so obtained.
2888 Check the crossing speed dependence of P_f/P_i against theory.

2889 (h) Plot the turn-by-turn vertical spin component motion $S_Z(\text{turn})$ across the
2890 resonance $G\gamma_R = 4 - \nu_Z$, in a weakly depolarizing case, $P_f \approx P_i$. Show that it
2891 satisfies Eq. 9.51. Match the data to the latter to get the vertical betatron tune ν_y , and
2892 the location of the resonance $G\gamma_R$.

2893 (i) Track a few particles at fixed energy, at distances from the resonance $G\gamma_R =$
2894 $4 - \nu_y$ of up to a $7 \times \varepsilon_R$ (this distance corresponds to 1% depolarization).

2895 Produce on a common graph the spin motion $S_Z(\text{turn})$ for all these particles, as
2896 observed at some azimuth along the ring.

2897 Produce a graph of $\langle S_y \rangle|_{\text{turn}}(\Delta)$ (as in Fig. 9.21).

Produce the vertical betatron tune ν_y , and the location of the resonance $G\gamma_R$,
obtained from a match of these tracking trials to the theoretical (Eq. 9.49)

$$\langle S_y \rangle (\Delta) = \frac{\Delta}{\sqrt{|\varepsilon_R|^2 + \Delta^2}}$$

2898 9.2 Construct the ZGS synchrotron. Spin Resonances

2899 Solution: page 375

2900 In this exrecie, ZGS synchrotron is modeled in `zgoubi`, and spin resonances in
2901 this weak focusing zero-gradient synchrotron are studied.

2902 (a) Construct an approximate model of the ZGS synchrotron, using `DIPOLE`.
2903 Use Figs. 9.24, 9.25 as a guidance, and parameters given in Tab. 9.3. Assume that
2904 the reference orbit is the same at all energies, on nominal radius, 2076 cm. It is
2905 judicious (although in no way an obligation) to take `RM=2076` in `DIPOLE`. (Note

2906 that in reality, unlike the present assumption for this exercise, the reference orbit in
 2907 ZGS would be moved outward during acceleration [26].)

2908 Check the correctness of the model by producing the lattice parameters of the
 2909 ring. TWISS can be used for that. Compare with the lattice parameters given in
 2910 Tab. 9.3.

2911 (b) Produce a graph of the betatron functions along the ZGS cell. Provide checks
 2912 of the correctness of the computation.

2913 Check the theoretical periodic dispersion (Eq. 9.14) against the radial distance
 2914 between on- and off-momentum closed orbits obtained from tracking. Provide a plot
 2915 of the dispersion function.

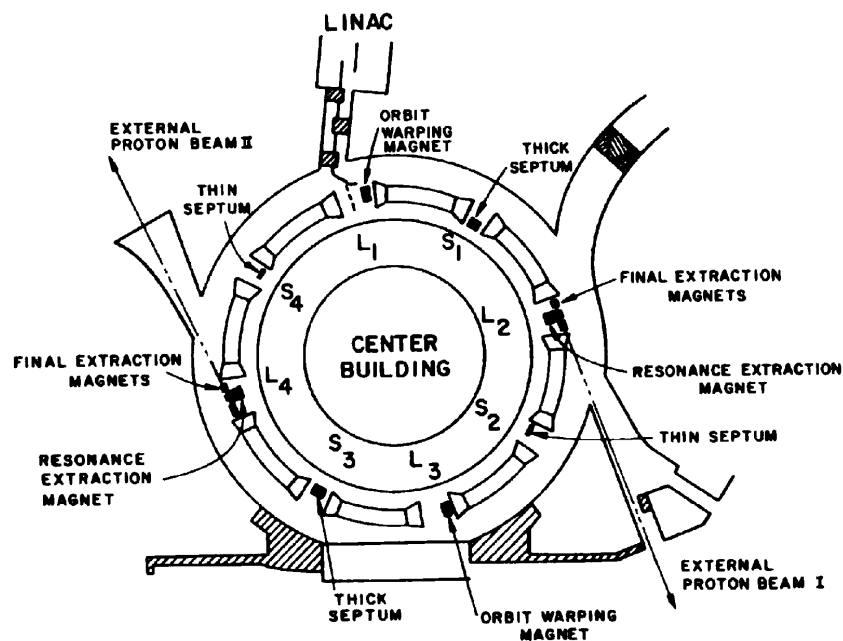


Fig. 9.24 A schematic layout of the ZGS [23], a $\pi/2$ -periodic structure, comprised of 8 zero-index dipoles, 4 long and 4 short straight sections

2916 (c) Additional verifications regarding the model.

2917 Produce a graph of the field $B(s)$

2918 - along the on-momentum closed orbit, and along off-momentum chromatic closed
 2919 orbits, across a cell;

2920 - along orbits at large horizontal excursion;

2921 - along orbits at large vertical excursion.

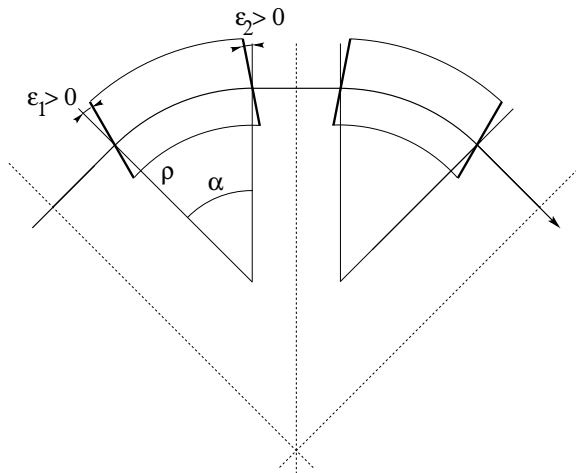


Fig. 9.25 A sketch of Saturne I cell layout. In defining the entrance and exit faces (EFBs) of the magnet, beam goes from left to right. Wedge angles at the long straight sections (ϵ_1) and at the short straight sections (ϵ_2) are different

2922 For all these cases, verify qualitatively, from the graphs, that $B(s)$ appears as
2923 expected.

(d) Justify considering the betatron oscillation as sinusoidal, namely,

$$y(\theta) = A \cos(\nu_y \theta + \phi)$$

2924 wherein $\theta = s/R$, $R = \oint ds/2\pi$.

2925 Find the value of the horizontal and vertical betatron functions, resulting from
2926 that approximation. Compare with the betatron functions obtained in (b).

2927 (e) Produce an acceleration cycle from 50 MeV to 17 GeV about, for a few particles
2928 launched on the a common $10^{-5} \pi \text{m}$ vertical initial invariant, with small horizontal
2929 invariant. Ignore synchrotron motion (CAVITE[IOPT=3] can be used in that case).
2930 Take a peak voltage $\hat{V} = 200 \text{ kV}$ (this is unrealistic but yields 10 times faster
2931 computing than the actual $\hat{V} = 20 \text{ kV}$, Tab. 9.3) and synchronous phase $\phi_s = 150 \text{ deg}$
2932 (justify $\phi_s > \pi/2$). Add spin, using SPNTRK, in view of the next question, (f).

2933 Check the accuracy of the betatron damping over the acceleration range, compared
2934 to theory. How close to symplectic the numerical integration is (it is by definition
2935 *not* symplectic), depends on the integration step size, and on the size of the flying
2936 mesh in the DIPOLE method [25, Fig. 20]; check a possible departure of the betatron
2937 damping from theory as a function of these parameters.

2938 Produce a graph of the the evolution of the horizontal and vertical wave numbers
2939 during the acceleration cycle.

Table 9.3 Parameters of the ZGS weak focusing synchrotron after Refs. [26, 27][23, pp.288-294,p.716] (2nd column, when they are known) and in the present simplified model and numerical simulations (3rd column). Note that the actual orbit is skewed (moves) during ZGS acceleration cycle, tunes change as well - this is not the case in the present modeling

		From Refs. [26, 27]	Simplified model
Injection energy	MeV		50
Top energy	GeV		12.5
$G\gamma$ span		1.888387 - 25.67781	
Length of central orbit	m	171.8	170.90457
Length of straight sections, total	m	41.45	40.44
<i>Lattice</i>			
Wave numbers $\nu_x; \nu_y$		0.82; 0.79	0.849; 0.771
Max. $\beta_x; \beta_y$	m		32.5; 37.1
<i>Magnet</i>			
Length	m	16.3	16.30486 (magnetic)
Magnetic radius	m	21.716	20.76
Field min.; max.	kG	0.482; 21.5	0.4986; 21.54
Field index			0
Yoke angular extent	deg	43.02590	45
Wedge angle	deg	≈ 10	13 and 8
<i>RF</i>			
Rev. frequency	MHz	0.55 - 1.75	0.551 - 1.751
RF harmonic $h=\omega_{rf}/\omega_{rev}$			8
Peak voltage	kV	20	200
B-dot, nominal/max.	T/s	2.15/2.6	
Energy gain, nominal/max.	keV/turn	8.3/10	100
Synchronous phase, nominal	deg		150
<i>Beam</i>			
$\varepsilon_x; \varepsilon_y$ (at injection)	$\pi\mu\text{m}$		25; 150
Momentum spread, rms			3×10^{-4}
Polarization at injection	%	>75	100
Radial width of beam (90%), at inj.	inch	2.5	$\sqrt{\beta_x \varepsilon_x / \pi} = 1.1$

2940 (f) Using the raytracing material developed in (e): produce a graph of the vertical
 2941 spin component of the particles, and the average value over that 200 particle set, as
 2942 a function of $G\gamma$. Indicate on that graph the location of the resonant $G\gamma_R$ values.

2943 (g) Based on the simulation file used in (f), simulate the acceleration of a single
 2944 particle, through one particular intrinsic resonance, from a few thousand turns
 2945 upstream to a few thousand turns downstream.

2946 Perform this resonance crossing for different values of the particle invariant.
 2947 Determine the dependence of final/initial vertical spin component value, on the
 2948 invariant value; check against theory.

2949 Re-do this crossing simulation for a different crossing speed. Check the crossing
 2950 speed dependence of final/initial vertical spin component so obtained, against theory.

2951 (h) Introduce a vertical orbit defect in the ZGS ring.

2952 Find the closed orbit.

2953 Accelerate a particle launched on that closed orbit, from 50 MeV to 17 GeV about,
 2954 produce a graph of the vertical spin component.

2955 Select one particular resonance, reproduce the two methods of (g) to check the
 2956 location of the resonance at $G\gamma_R = \text{integer}$, and to find its strength.

2957 References

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