Homework 6

Problem 1. 10 points. Sylvester formula – dipole/quadrupole
For an uncoupled transverse motion with constant energy and Hamiltonian of a bending magnet with quadrupole term (e.g. field gradient):

$$\tilde{h}_n = \frac{p_x^2 + p_y^2}{2} + f\frac{x^2}{2} + g\frac{y^2}{2};$$

$$f = \left[K_o^2 + K_1\right]; g = -K_1;\ K_o = -\frac{e}{p_o c}B_y; K_1 = \frac{e}{p_o c}\frac{\partial B_y}{\partial x}$$

(a) Define all cases for eigen values of D.
(b) Use Sylvester formula for one-dimensional motions (x and y) when $f \neq 0; g \neq 0;$ (non-degenerated cases) and write explicit form of the 2x2 transport matrices.
(c) Consider a case of pure quadrupole: $K_o = 0$, no bending
(d) Do the same as above using 4x4 matrix formulation (2D case) and show that results are identical

Solutions: since we had considered already quadrupoles without bending, let assume that it is a real bending and the curvature is not zero $K_o = \frac{1}{\rho} = -\frac{e}{p_o c}B_y \neq 0$

(a) We have

$$\mathbf{H} = \begin{bmatrix} \mathbf{H}_x & 0 \\ 0 & \mathbf{H}_y \end{bmatrix}; \mathbf{H}_x = \begin{bmatrix} f & 0 \\ 0 & 1 \end{bmatrix}; \mathbf{H}_y = \begin{bmatrix} g & 0 \\ 0 & 1 \end{bmatrix};$$

$$\mathbf{D} = \mathbf{SH} = \begin{bmatrix} \sigma\mathbf{H}_x & 0 \\ 0 & \sigma\mathbf{H}_y \end{bmatrix} = \begin{bmatrix} \mathbf{D}_x & 0 \\ 0 & \mathbf{D}_y \end{bmatrix}; \mathbf{D}_x = \begin{bmatrix} 0 & 1 \\ -f & 0 \end{bmatrix}; \mathbf{D}_y = \begin{bmatrix} 0 & 1 \\ -g & 0 \end{bmatrix};$$

Since matrix D is block-diagonal,

$$\exp[\mathbf{D}] = \begin{bmatrix} \exp[\mathbf{D}_x] & 0 \\ 0 & \exp[\mathbf{D}_y] \end{bmatrix};$$

and we have an easy job to evaluate 2 1D cases.

$$\det\left[\mathbf{D}_{x,y} - \lambda\mathbf{I}\right] = 0 \rightarrow \lambda_{x,y}^2 = -\det\left[\mathbf{D}_{x,y}\right] = \left\{ \begin{matrix} -\dfrac{eG}{p_o c} - K_o^2 \\ \dfrac{eG}{p_o c} \end{matrix} \right\};\ G = \frac{\partial B_y}{\partial x}; K_o = -\frac{e}{p_o c}B_y$$

It has 5 distinct cases:

(0) drift: $G=0; K_o=0;\ \lambda_x=\lambda_y=0$

(1) bend with gradient: $\frac{eG}{p_oc}+K_o^2>0;\quad \frac{eG}{p_oc}>0; \lambda_x=\pm i\sqrt{K_o^2+\frac{eG}{p_oc}}; \lambda_y=\pm\sqrt{\frac{eG}{p_oc}};$

(2) pure (sector) bend : $G=0;\ \frac{eG}{p_oc}+K_o^2>0;\quad \lambda_x=\pm iK_o;\ \lambda_y=0;$

(3) weak focusing bend: $\frac{eG}{p_oc}+K_o^2>0; \frac{eG}{p_oc}<0;\ \lambda_x=\pm i\sqrt{K_o^2+\frac{eG}{p_oc}}; \lambda_y=\pm i\sqrt{-\frac{eG}{p_oc}};$

(4) bend with gradient: $\frac{eG}{p_oc}+K_o^2=0;\ \lambda_x=0; \lambda_y=\pm i\sqrt{-\frac{eG}{p_oc}}$

(5) bend with gradient: $\frac{eG}{p_oc}+K_o^2<0;\ \lambda_x=\pm\sqrt{-\frac{eG}{p_oc}-K_o^2}; \lambda_y=\pm i\sqrt{-\frac{eG}{p_oc}};$

(b) Use Sylvester formula for one-dimensional motions (x and y) when $f\neq 0; g\neq 0;$ (non-degenerated cases) and write explicit form of the 2x2 transport matrices. There are three cases belong to this (1), (3) and (5). For imaginary values of the eigen values we have:

$$\lambda=\pm i\omega;$$

$$\mathbf{M}=e^{i\omega s}\frac{\mathbf{D}+i\omega\mathbf{I}}{2i\omega}-e^{-i\omega s}\frac{\mathbf{D}-i\omega\mathbf{I}}{2i\omega}=I\cos(\omega s)+\frac{\mathbf{D}}{\omega}\sin(\omega s) \tag{1}$$

$$\mathbf{M}=\begin{bmatrix}\cos(\omega s) & \frac{\sin(\omega s)}{\omega}\\ -\omega\sin(\omega s) & \cos(\omega s)\end{bmatrix}$$

For real values of the eigen values we have:

$$\lambda=\pm\omega;$$

$$\mathbf{M}=e^{\omega s}\frac{\mathbf{D}+\omega\mathbf{I}}{2\omega}-e^{-\omega s}\frac{\mathbf{D}-\omega\mathbf{I}}{2\omega}=I\cosh(\omega s)+\frac{\mathbf{D}}{\omega}\sinh(\omega s) \tag{2}$$

$$\mathbf{M}=\begin{bmatrix}\cosh(\omega s) & \frac{\sinh(\omega s)}{\omega}\\ \omega\sinh(\omega s) & \cosh(\omega s)\end{bmatrix}$$

And in the cases (0), (2) and (4),

$$\mathbf{D}_z=\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}; \mathbf{D}_z^{\ 2}=0\Rightarrow \mathbf{M}_z=e^{\mathbf{D}_z s}=\sum_{n=1}^{\infty}\frac{\mathbf{D}_z^{\ n}s^n}{n!}=\mathbf{I}+\mathbf{D}_z s=\begin{bmatrix}1 & s\\ 0 & 1\end{bmatrix};$$

$$\mathbf{M}_{u\neq z}=\begin{bmatrix}\cos(\omega s) & \frac{\sin(\omega s)}{\omega}\\ -\omega\sin(\omega s) & \cos(\omega s)\end{bmatrix};\ z=\begin{bmatrix}x\,and\,y,\,case(0)\\ y,\ case(2)\\ z,\ case(2)\end{bmatrix}.$$

Note that drift section matrix can be obtained from both focusing (1) and defocusing (2) matrices as limit with $\omega \to 0$:

$$\lim_{\omega\to 0}\begin{bmatrix} \cos(\omega s) & \dfrac{\sin(\omega s)}{\omega} \\ -\omega\sin(\omega s) & \cos(\omega s) \end{bmatrix} = \lim_{\omega\to 0}\begin{bmatrix} \cosh(\omega s) & \dfrac{\sinh(\omega s)}{\omega} \\ \omega\sinh(\omega s) & \cosh(\omega s) \end{bmatrix} = \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix}$$

(c)

(1) $\dfrac{eG}{p_o c} > 0; \lambda_x = \pm i\omega; \lambda_y = \pm\omega; \omega = \sqrt{\left|\dfrac{eG}{p_o c}\right|};$

(2) $\dfrac{eG}{p_o c} = 0;\ \lambda_x = \lambda_y = 0;$

(3) $\dfrac{eG}{p_o c} < 0; \lambda_x = \pm\omega; \lambda_y = \pm i\omega;$

Cases

$$\mathbf{M}_x = \begin{bmatrix} \cos(\omega s) & \dfrac{\sin(\omega s)}{\omega} \\ -\omega\sin(\omega s) & \cos(\omega s) \end{bmatrix}; \mathbf{M}_y = \begin{bmatrix} \cosh(\omega s) & \dfrac{\sinh(\omega s)}{\omega} \\ \omega\sinh(\omega s) & \cosh(\omega s) \end{bmatrix}; \mathbf{M} = \begin{bmatrix} \mathbf{M}_x & 0 \\ 0 & \mathbf{M}_x \end{bmatrix} \quad (1)$$

$$\mathbf{D}_{x,y}{}^2 = 0 \Rightarrow \mathbf{M}_x = \mathbf{M}_y = \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix}; \mathbf{M} = \begin{bmatrix} \mathbf{M}_x & 0 \\ 0 & \mathbf{M}_x \end{bmatrix} \quad (2)$$

$$\mathbf{M}_x = \begin{bmatrix} \cosh(\omega s) & \dfrac{\sinh(\omega s)}{\omega} \\ \omega\sinh(\omega s) & \cosh(\omega s) \end{bmatrix}; \mathbf{M}_y = \begin{bmatrix} \cos(\omega s) & \dfrac{\sin(\omega s)}{\omega} \\ -\omega\sin(\omega s) & \cos(\omega s) \end{bmatrix}; \mathbf{M} = \begin{bmatrix} \mathbf{M}_x & 0 \\ 0 & \mathbf{M}_x \end{bmatrix} \quad (3)$$

(d) Let's do it for non-generated case. Since $\det[\mathbf{D} - \lambda\mathbf{I}] = \det\begin{bmatrix} \mathbf{D}_x - \lambda\mathbf{I} & 0 \\ 0 & \mathbf{D}_y - \lambda\mathbf{I} \end{bmatrix} = \det[\mathbf{D}_x - \lambda\mathbf{I}]\det[\mathbf{D}_y - \lambda\mathbf{I}] = 0$, the eigen values are already known. using 4x4 matrix approach without "thinking".

$$\mathbf{M} = e^{\lambda_x s}\frac{\mathbf{D}-\lambda_x\mathbf{I}}{2\lambda_x}\frac{\mathbf{D}-\lambda_y\mathbf{I}}{\lambda_x-\lambda_y}\frac{\mathbf{D}+\lambda_y\mathbf{I}}{\lambda_x+\lambda_y} + e^{-\lambda_x s}\frac{\mathbf{D}+\lambda_x\mathbf{I}}{2\lambda_x}\frac{\mathbf{D}-\lambda_y\mathbf{I}}{\lambda_x+\lambda_y}\frac{\mathbf{D}+\lambda_y\mathbf{I}}{\lambda_y-\lambda_x} +$$
$$e^{\lambda_y s}\frac{\mathbf{D}-\lambda_y\mathbf{I}}{2\lambda_y}\frac{\mathbf{D}-\lambda_x\mathbf{I}}{\lambda_y-\lambda_x}\frac{\mathbf{D}+\lambda_x\mathbf{I}}{\lambda_y+\lambda_x} + e^{-\lambda_y s}\frac{\mathbf{D}+\lambda_y\mathbf{I}}{2\lambda_y}\frac{\mathbf{D}-\lambda_x\mathbf{I}}{\lambda_y+\lambda_x}\frac{\mathbf{D}+\lambda_x\mathbf{I}}{\lambda_x-\lambda_y}$$

One can multiply matrices and show element by element that it is true. I would suggest to look onto

$$(\mathbf{D}-\lambda_y\mathbf{I})(\mathbf{D}+\lambda_y\mathbf{I}) = \mathbf{D}^2 - \lambda_y^2\mathbf{I} = \begin{bmatrix} 0 & 1 \\ g & 0 \end{bmatrix}^2 - \begin{bmatrix} \lambda_y^2 & 0 \\ 0 & \lambda_y^2 \end{bmatrix} = \begin{bmatrix} g-\lambda_y^2 & 0 \\ 0 & g-\lambda_y^2 \end{bmatrix} = 0$$

One can use the fact that matrix is a root of its characteristic polynomial

$$p(\lambda)=(\lambda-\lambda_y)(\lambda+\lambda_y);\, p(\mathbf{D}_y)=(\mathbf{D}_y-\lambda_y\mathbf{I})(\mathbf{D}_y+\lambda_y\mathbf{I})=0$$

or just multiply directly

$$(\mathbf{D}_y-\lambda_y\mathbf{I})(\mathbf{D}_y+\lambda_y\mathbf{I})=\mathbf{D}_y^2-\lambda_y^2\mathbf{I}=\begin{bmatrix} g-\lambda_y^2 & 0 \\ 0 & g-\lambda_y^2 \end{bmatrix}=0$$

similarly:

$$(\mathbf{D}_x-\lambda_y\mathbf{I})(\mathbf{D}_x+\lambda_y\mathbf{I})=\mathbf{D}_x^2-\lambda_y^2\mathbf{I}=\begin{bmatrix} f-\lambda_y^2 & 0 \\ 0 & f-\lambda_y^2 \end{bmatrix}=\begin{bmatrix} \lambda_x^2-\lambda_y^2 & 0 \\ 0 & \lambda_x^2-\lambda_y^2 \end{bmatrix}$$

Thus, additional term generate

$$\mathbf{M}=\left(e^{\lambda_x s}\frac{\mathbf{D}-\lambda_x\mathbf{I}}{2\lambda_x}-e^{-\lambda_x s}\frac{\mathbf{D}+\lambda_x\mathbf{I}}{2\lambda_x}\right)\begin{bmatrix} \mathbf{I} & 0 \\ 0 & 0 \end{bmatrix}+\left(e^{\lambda_y s}\frac{\mathbf{D}-\lambda_y\mathbf{I}}{2\lambda_y}+e^{-\lambda_y s}\frac{\mathbf{D}+\lambda_y\mathbf{I}}{2\lambda_y}\right)\begin{bmatrix} 0 & 0 \\ 0 & \mathbf{I} \end{bmatrix}$$

The rest of the comparison is trivial.

Problem 2. 10 points. Sylvester formula, SQ-quadrupole

For a coupled transverse motion with constant energy and Hamiltonian of a SQ-quadrupole:

$$\tilde{h}_n=\frac{p_x^2+p_y^2}{2}+Nxy; \qquad N=\frac{e}{p_o c}\frac{\partial B_x}{\partial x}$$

(a) Use Sylvester formula and find matrix of SQ-quadrupole.

(b) Consider a "standard approach" – turn coordinates 45-degrees (use rotation matrix), to turn SQ-quad into a "normal". Then make the product of 45-degree turn, quad matrix, -45 degrees turn. Show that the matrix is the same as in case (a).

Note: for point (b), consider a rotation is around z-axis: it will rotate x,y and p_x,p_y:

$$\begin{pmatrix} x^1 \\ p_x^1 \\ y^1 \\ p_y^1 \end{pmatrix}=R\begin{pmatrix} x \\ p_x \\ y \\ p_y \end{pmatrix}=\begin{bmatrix} I\cdot\cos\varphi & I\cdot\sin\varphi \\ -I\cdot\sin\varphi & I\cdot\cos\varphi \end{bmatrix}\begin{pmatrix} x \\ p_x \\ y \\ p_y \end{pmatrix};\ I=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Thus

$$x^1=x\cos\varphi+y\sin\varphi;\ \ y^1=-x\sin\varphi+y\cos\varphi$$
$$p_x^1=p_x\cos\varphi+p_y\sin\varphi; p_y^1=-p_x\sin\varphi+p_y\cos\varphi$$

Do not forget that you need inverse matrix of R as well.

In rotated coordinates with $\varphi=\pi/2$ the Hamiltonian will have a decoupled form of one in quadrupole and you easily can calculate the matrix M_Q.. Finally, you need to rotate back to initial coordinates. $M_{SQ}=R^{-1}\cdot M_Q\cdot R$

(a) First let define the eigen values. Using lecture notes (or just a straight-forward det)

$$f=g=L=0;\ldots.\quad \mathbf{D}=\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -N & 0 \\ 0 & 0 & 0 & 1 \\ -N & 0 & 0 & 0 \end{bmatrix};$$

$$\lambda_{1,2}{}^2=\pm N \Rightarrow \lambda_1=\sqrt{|N|};\lambda_2=i\sqrt{|N|};$$

$$\mathbf{M}=\left(e^{\lambda_1 s}\frac{\mathbf{D}-\lambda_1\mathbf{I}}{2\lambda_1}-e^{-\lambda_1 s}\frac{\mathbf{D}+\lambda_1\mathbf{I}}{2\lambda_1}\right)\frac{\mathbf{D}^2-\lambda_2^2\mathbf{I}}{\lambda_1^2-\lambda_2^2}+\left(e^{\lambda_2 s}\frac{\mathbf{D}-\lambda_2\mathbf{I}}{2\lambda_2}-e^{-\lambda_2 s}\frac{\mathbf{D}+\lambda_2\mathbf{I}}{2\lambda_2}\right)\frac{\mathbf{D}^2-\lambda_1^2\mathbf{I}}{\lambda_2^2-\lambda_1^2};$$

$$\lambda_1^2=|N|;\lambda_2^2=-|N|;\lambda_1^2-\lambda_2^2=2|N|;$$

$$\mathbf{M}=\left(\mathbf{I}\cosh\sqrt{|N|}s+\frac{\sinh\sqrt{|N|}s}{\sqrt{|N|}}\mathbf{D}\right)\frac{\mathbf{D}^2+|N|\mathbf{I}}{2|N|}-\left(\mathbf{I}\cos\sqrt{|N|}s+\frac{\sin\sqrt{|N|}s}{\sqrt{|N|}}\mathbf{D}\right)\frac{\mathbf{D}^2-|N|\mathbf{I}}{2|N|};$$

the rest us a simple multiplication to use specific expressions for

$$\frac{\mathbf{D}^2+|N|\mathbf{I}}{2|N|}=\frac{signN}{2}\begin{bmatrix} I & -I \\ -I & I \end{bmatrix};\frac{\mathbf{D}^2-|N|\mathbf{I}}{2|N|}=-\frac{signN}{2}\begin{bmatrix} I & I \\ I & I \end{bmatrix}$$

$$\mathbf{M}=\frac{signN}{2}\left\{\left(\mathbf{I}\cosh\sqrt{|N|}s+\frac{\sinh\sqrt{|N|}s}{\sqrt{|N|}}\mathbf{D}\right)\begin{bmatrix} I & -I \\ -I & I \end{bmatrix}+\left(\mathbf{I}\cos\sqrt{|N|}s+\frac{\sin\sqrt{|N|}s}{\sqrt{|N|}}\mathbf{D}\right)\begin{bmatrix} I & I \\ I & I \end{bmatrix}\right\};$$

$$\begin{pmatrix} \mathrm{Cos}[\varphi] & \frac{\mathrm{Sin}[\varphi]}{\sqrt{\mathrm{Abs}[N]}} & 0 & 0 \\ 0 & \mathrm{Cos}[\varphi] & -\frac{N\,\mathrm{Sin}[\varphi]}{\sqrt{\mathrm{Abs}[N]}} & 0 \\ 0 & 0 & \mathrm{Cos}[\varphi] & \frac{\mathrm{Sin}[\varphi]}{\sqrt{\mathrm{Abs}[N]}} \\ -\frac{N\,\mathrm{Sin}[\varphi]}{\sqrt{\mathrm{Abs}[N]}} & 0 & 0 & \mathrm{Cos}[\varphi] \end{pmatrix}\qquad\begin{pmatrix} \mathrm{Cosh}[\varphi] & \frac{\mathrm{Sinh}[\varphi]}{\sqrt{\mathrm{Abs}[N]}} & 0 & 0 \\ 0 & \mathrm{Cosh}[\varphi] & -\frac{N\,\mathrm{Sinh}[\varphi]}{\sqrt{\mathrm{Abs}[N]}} & 0 \\ 0 & 0 & \mathrm{Cosh}[\varphi] & \frac{\mathrm{Sinh}[\varphi]}{\sqrt{\mathrm{Abs}[N]}} \\ -\frac{N\,\mathrm{Sinh}[\varphi]}{\sqrt{\mathrm{Abs}[N]}} & 0 & 0 & \mathrm{Cosh}[\varphi] \end{pmatrix}\quad (1)$$

to reduce it to

$$F=\begin{bmatrix} \cos\sqrt{|N|}s & \frac{\sin\sqrt{|N|}s}{\sqrt{|N|}} \\ -\sqrt{|N|}\sin\sqrt{|N|}s & \cos\sqrt{|N|}s \end{bmatrix};D=\begin{bmatrix} \cosh\sqrt{|N|}s & \frac{\sinh\sqrt{|N|}s}{\sqrt{|N|}} \\ \sqrt{|N|}\sinh\sqrt{|N|}s & \cosh\sqrt{|N|}s \end{bmatrix}$$

$$\mathbf{M}=\frac{1}{2}\begin{bmatrix} F+D & signN(F-D) \\ signN\cdot(F-D) & F+D \end{bmatrix}$$

e.g. a surprising adding and subtracting focusing and defocusing 2x2 matrices, which magically combined out of pieces of (1). Reason for this form becomes clear from exercise below.

(b) Solution is to rotate the coordinate system 45 degrees ($sn=\pm 1$)

$$\begin{Bmatrix} x \\ y \end{Bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -sn \\ sn & 1 \end{bmatrix}\begin{Bmatrix} \tilde{x} \\ \tilde{y} \end{Bmatrix}; X = \frac{1}{\sqrt{2}}\begin{bmatrix} I & -sn\cdot I \\ sn\cdot I & I \end{bmatrix}\tilde{X}$$

$$\begin{Bmatrix} \tilde{x} \\ \tilde{y} \end{Bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & sn \\ -sn & 1 \end{bmatrix}\begin{Bmatrix} x \\ y \end{Bmatrix}; \tilde{X} = \frac{1}{\sqrt{2}}\begin{bmatrix} I & sn\cdot I \\ -sn\cdot I & I \end{bmatrix}X$$

$$\tilde{h}_n = \frac{p_x^2 + p_y^2}{2} + Nxy = \frac{\tilde{p}_x^2 + \tilde{p}_y^2}{2} + \frac{N}{2}(x - sn\cdot y)(y + snx);\ sn = \pm 1$$

$$\frac{N}{2}(x - sn\cdot y)(y + snx) = sn\cdot\frac{N}{2}\left(x^2 - y^2\right); N = \frac{e}{p_o c}\frac{\partial B_x}{\partial x}$$

To define the direction, let's select sign that $sn\cdot N > 0$, e.g. it is focusing in $\tilde{x}$ direction. Than we know that rotated coordinate system the matrix is:

$$\tilde{\mathbf{M}} = \begin{bmatrix} F & 0 \\ O & D \end{bmatrix}$$

Now just need to restore it to original coordinate system: $sn = sign(N)$

$$X_1 = \mathbf{R}\tilde{X}_1; \mathbf{R} = \frac{1}{\sqrt{2}}\begin{bmatrix} I & -sn\cdot I \\ sn\cdot I & I \end{bmatrix}; \tilde{X}_0 = \mathbf{R}^{-1}X_0;$$

$$\tilde{X}_1 = \tilde{\mathbf{M}}\tilde{X}_0;\ X_1 = \mathbf{R}\tilde{\mathbf{M}}\tilde{X}_0 = \mathbf{R}\tilde{\mathbf{M}}\mathbf{R}^{-1}X_0 \to \mathbf{M} = \mathbf{R}\tilde{\mathbf{M}}\mathbf{R}^{-1}$$

$$\mathbf{M}_{SQ} = \frac{1}{2}\begin{bmatrix} I & -sn\cdot I \\ sn\cdot I & I \end{bmatrix}\begin{bmatrix} F & 0 \\ 0 & D \end{bmatrix}\begin{bmatrix} I & sn\cdot I \\ -sn\cdot I & I \end{bmatrix} = \frac{1}{2}\begin{bmatrix} F + D & sn\cdot(F - D) \\ sn\cdot(F - D) & F + D \end{bmatrix}$$

e.g. the diagonal 2x2 blocks are identical as well as off-diagonal one. Sign of the off-diagonal blocks depends on the on direction of rotation of normal focusing quadrupole.

Compare this with matrix of solenoid, whose off-diagonal block have opposite sign.

$$\mathbf{M}_{sol} = \begin{bmatrix} I\cos\varphi & I\sin\varphi \\ -I\sin\varphi & \cos\varphi \end{bmatrix}\begin{bmatrix} F & 0 \\ 0 & F \end{bmatrix} = \begin{bmatrix} F\cos\varphi & F\sin\varphi \\ -F\sin\varphi & F\cos\varphi \end{bmatrix}$$