PHY 564
Advanced Accelerator Physics
Lecture 22

Collective Effects II: Examples of Collective Instabilities

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Outline

- Transverse beam breakup instability (BBU) in linear accelerator
  - Two particle model
  - BNS damping
- Longitudinal microwave instability
  - Dispersion relation
  - Cold beam
  - Warm beam (Keil-Schnell criteria for stability)

Single pass BBU (Two particle model)

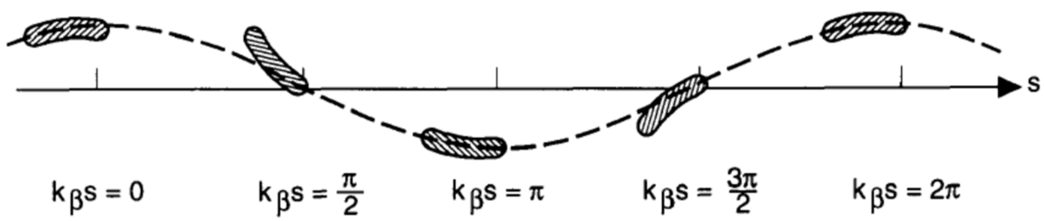
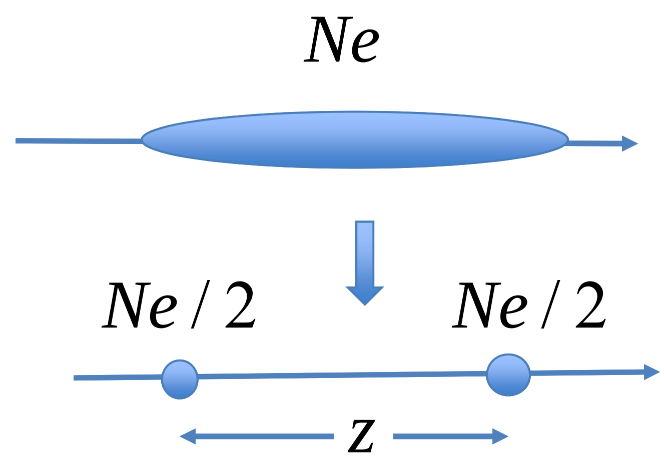


Figure 3.3. Sequence of snapshots of a beam undergoing dipole beam breakup instability in a linac. Values of $k_\beta s$ indicated are modulo 2π. The dashed curves indicate the trajectory of the bunch head.

Leading particles $y_1(s) = \hat{y}\cos(k_\beta s)$

Trailing particles
$$y_2(s)'' + k_\beta^2 y_2(s) = -\frac{Ne^2 W_1(z)}{2EL} y_1(s)$$
$$= -4\pi\varepsilon_0 \frac{N r_0 W_1(z)}{2\gamma L} \hat{y}\cos(k_\beta s)$$



Single pass BBU (Two particle model)

For a linear inhomogenous 2nd order differential equation

$$\frac{d^2x}{dt^2}+a(t)\frac{dx}{dt}+b(t)x=f(t)$$

its solution is given by

$$W(t)=\begin{vmatrix}\phi_1(t) & \phi_2(t)\\ \phi_1'(t) & \phi_2'(t)\end{vmatrix}$$

$$x(t)=c_1\phi_1(t)+c_2\phi_2(t)+\int_{t_0}^{t}\frac{\phi_1(\xi)\phi_2(t)-\phi_2(\xi)\phi_1(t)}{W(\xi)}f(\xi)\,d\xi$$

$$\phi_2=\sin(k_\beta s)\qquad \phi_1=\cos(k_\beta s)$$

$$W(t)=\begin{vmatrix}\cos(k_\beta s) & \sin(k_\beta s)\\ -k_\beta\sin(k_\beta s) & k_\beta\cos(k_\beta s)\end{vmatrix}=k_\beta$$

$$\begin{aligned}y_{2,inh}(s)&=-4\pi\varepsilon_0\frac{Nr_0W_1(z)}{2\gamma Lk_\beta}\hat{y}\int_0^s\sin(k_\beta s-k_\beta\xi)\cos(k_\beta\xi)\,d\xi\\&=-4\pi\varepsilon_0\frac{Nr_0W_1(z)}{2\gamma Lk_\beta}\hat{y}\frac{1}{2}\left[s\sin(k_\beta s)-\int_{-s/2}^{s/2}\sin(2k_\beta\tilde{\xi})\,d\tilde{\xi}\right]\\&=-4\pi\varepsilon_0\frac{Nr_0W_1(z)}{4\gamma Lk_\beta}\hat{y}s\sin(k_\beta s)\end{aligned}$$

Single pass BBU (Two particle model)

$$y_2(s) = c_1\cos(k_\beta s) + c_2\sin(k_\beta s) - 4\pi\varepsilon_0 \frac{N r_0 W_1(z)}{4\gamma L k_\beta}\hat{y} s\sin(k_\beta s)$$

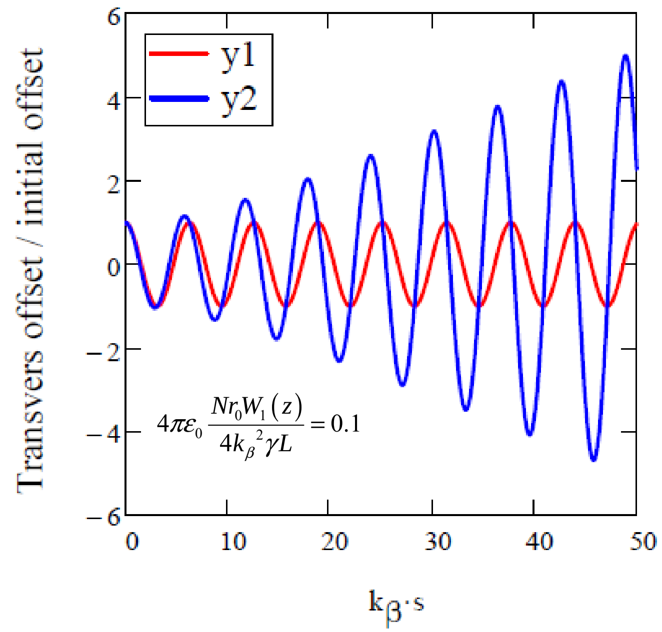
Noticing that before going through the structure, particle 2 has the same trajectory as that of Particle 1, i.e.

$$y_2(0) = y_1(0) = \hat{y}\cos(0) = \hat{y}$$

$$y'_2(0) = y'_1(0) = -\hat{y}k_\beta\sin(0) = 0$$

We obtain $c_1 = \hat{y}$ and $c_2 = 0$. Thus the solution for particle 2 is

$$y_2(s) = \hat{y}\left[\cos(k_\beta s) - 4\pi\varepsilon_0 \frac{N r_0 W_1(z)}{4k_\beta \gamma L} s\sin(k_\beta s)\right]$$



Single pass BBU II

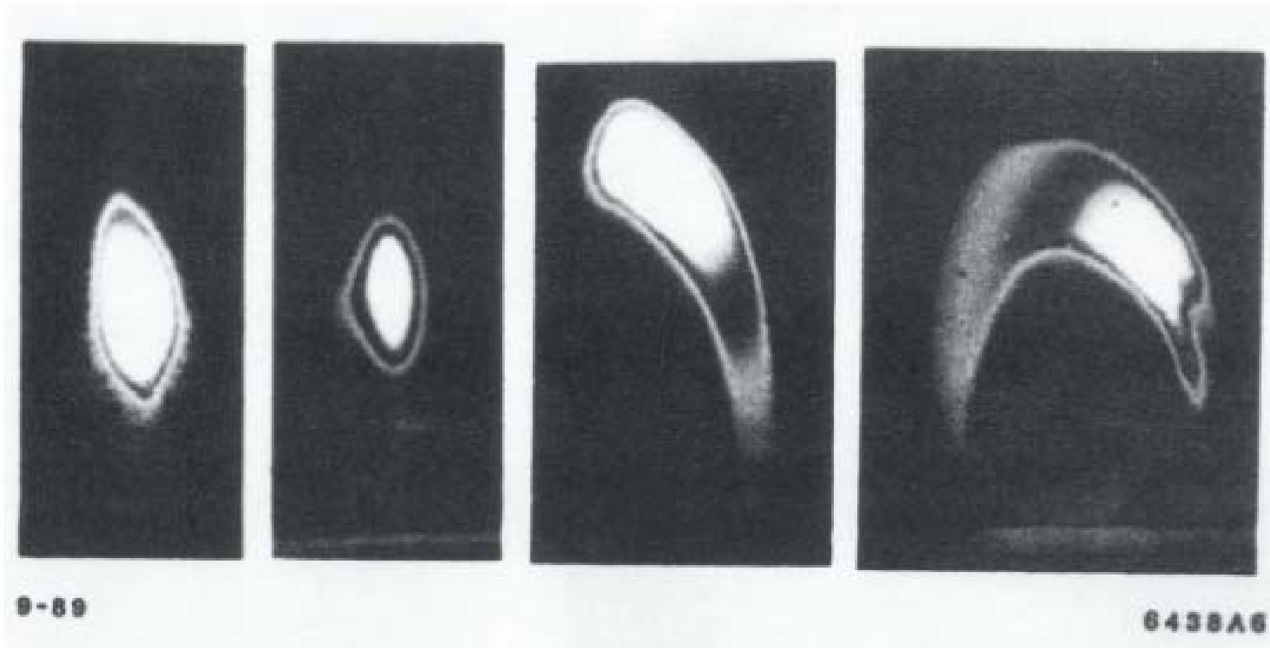


Figure 4.4: Four transverse beam profiles observed at the end of the SLAC linac are shown when the beam was carefully injected, and injected with 0.2, 0.5, and 1 mm offsets. The beam sizes σ_x and σ_y are about 120 μm. (Courtesy John Seeman, 1991)

One possible cure: BNS damping

Introduce focusing variation along the bunch, i.e. head and tail have different focusing strength

$$y_2'' + \left(k_\beta + \Delta k_\beta\right)^2 y_2 = -4\pi\varepsilon_0 \frac{N r_0 W_1(z)}{2\gamma L} \hat{y} \cos\left(k_\beta s\right)$$

$$\tilde{k}_\beta \equiv k_\beta + \Delta k_\beta$$

$$\phi_2 = \sin\left(\tilde{k}_\beta s\right) \quad \phi_1 = \cos\left(\tilde{k}_\beta s\right)$$

$$W(t) = \tilde{k}_\beta$$

$$y_{2,inh}(s) = -4\pi\varepsilon_0 \frac{N r_0 W_1(z)}{2\gamma L \tilde{k}_\beta} \hat{y} \int_0^s \sin\left(\tilde{k}_\beta s - \tilde{k}_\beta \xi\right) \cos\left(k_\beta \xi\right) d\xi$$

$$= 4\pi\varepsilon_0 \frac{N r_0 W_1(z)}{2\gamma L \tilde{k}_\beta} \hat{y} \frac{1}{2} \left[\int_0^s \sin\left(\Delta k_\beta \left(\xi - \frac{\tilde{k}_\beta}{\Delta k_\beta} s \right)\right) d\xi + \int_0^s \sin\left(\left(\tilde{k}_\beta + k_\beta\right) \left(\xi - \frac{\tilde{k}_\beta}{\tilde{k}_\beta + k_\beta} s \right)\right) d\xi \right]$$

$$= -4\pi\varepsilon_0 \frac{N r_0 W_1(z)}{2\gamma L \tilde{k}_\beta} \hat{y} \frac{1}{2} \left[\frac{1}{\Delta k_\beta} + \frac{1}{\tilde{k}_\beta + k_\beta} \right] \left[\cos\left(k_\beta s\right) - \cos\left(\tilde{k}_\beta s\right) \right]$$

$$\approx 4\pi\varepsilon_0 \frac{N r_0 W_1(z)}{4\gamma L k_\beta \Delta k_\beta} \hat{y} \left[\cos\left(\tilde{k}_\beta s\right) - \cos\left(k_\beta s\right) \right]$$

← assume $\Delta k_\beta / k_\beta << 1$

$$y_2(s) = \hat{y} \cos\left(\left(k_\beta + \Delta k_\beta\right) s\right) + 4\pi\varepsilon_0 \frac{N r_0 W_1(z)}{4\gamma L k_\beta \Delta k_\beta} \hat{y} \left[\cos\left(\tilde{k}_\beta s\right) - \cos\left(k_\beta s\right) \right]$$

Condition for complete compensation:

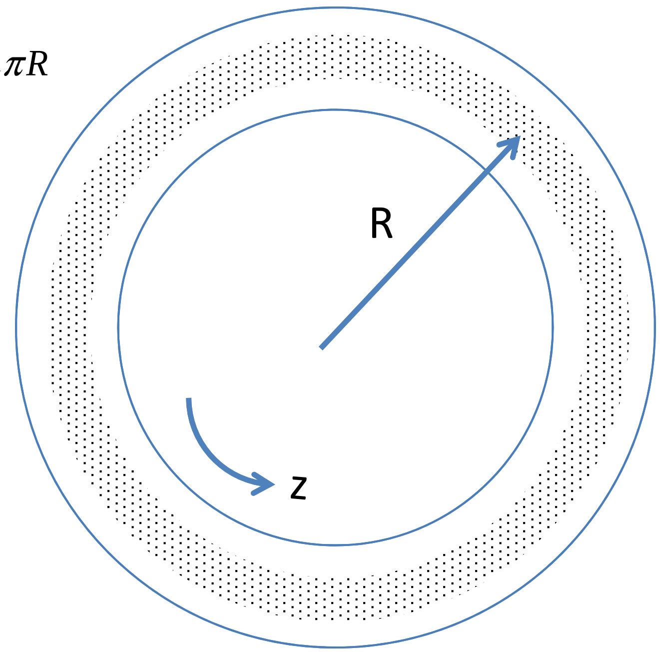
$$4\pi\varepsilon_0 \frac{N r_0 W_1(z)}{4\gamma L k_\beta \Delta k_\beta} = -1 \Rightarrow \Delta k_\beta = -4\pi\varepsilon_0 \frac{N r_0 W_1(z)}{4\gamma L k_\beta} \Rightarrow \boxed{y_2(s) = \hat{y} \cos\left(k_\beta s\right)}$$

Longitudinal Microwave Instability

Unperturbed phase space density:

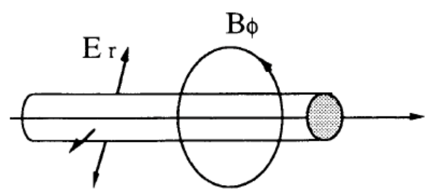
$$\psi_0(z,\Delta E) = \psi_0(\Delta E) = \frac{N}{C_0} f_0(\Delta E) \qquad \rho_0(z) = \rho_0 = \frac{N}{C_0}$$

$C_0 = 2\pi R$



DC current does not excite wake

$$V_{//}(z_0) = \int_{z_0}^{\infty} \lambda(z_1) w_{//}(z_1 - z_0)\, dz_1$$

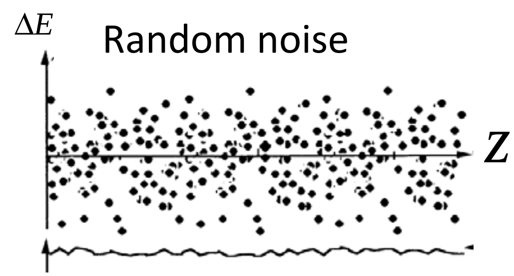


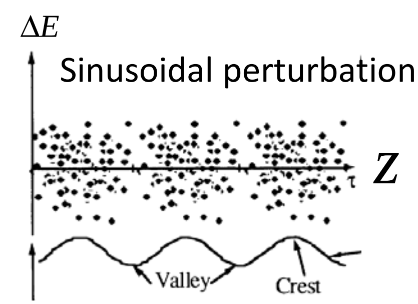
$$= \rho_0 \int_{z_0}^{\infty} W_0'(z_1 - z_0)\, dz_1 = -\rho_0 W_0(0) = 0$$

Consider perturbation in phase space density: n-th azimuthal mode

$$\psi_1(z,\Delta E,0) = \hat{\psi}_1(\Delta E) e^{inz/R}$$

Ansatz: $\psi_1(z,\Delta E,t) = \hat{\psi}_1(\Delta E) e^{inz/R - i\Omega t}$





*Note that if a perturbation is static,

$$\psi_1{*}(z,\Delta E,t) = \hat{\psi}_1{*}(\Delta E) e^{in(z - v_0 t)/R} = \hat{\psi}_1{*}(\Delta E) e^{inz/R - i\Omega^* t} \Rightarrow \Omega^* = nv_0 / R = n2\pi v_0 / C = n\omega_0$$

Longitudinal Microwave Instability

But the system is not likely to be static and we need to solve Vlasov equation self-consistently to know the answer for Ω and hence $\psi_1(s,\Delta E,t)$

$$\frac{\partial}{\partial t}\psi_1(z,\Delta E,t)+\frac{dz}{dt}\cdot\frac{\partial}{\partial z}\psi_1(z,\Delta E,t)+\frac{d\Delta E}{dt}\cdot\frac{\partial}{\partial \Delta E}\psi_0(\Delta E)=0$$

where $$\frac{dz}{dt}=v(\Delta E) \qquad (1)$$

And $\frac{d\Delta E}{dt}$ is obtained by calculating the longitudinal wake potential $\frac{d\Delta E(z,t)}{dt}=-\frac{c\Delta p_z(z,t)}{T_0}$

$$c\Delta p_z(z,t)=-eQ_eV_{//}(z,t)=-e^2v_0\int_{-\infty}^{t}\rho_1(z,t_1)w_{//}(t-t_1)dt_1=-e^2v_0\int_{0}^{\infty}\rho_1(z,t-\tau)w_{//}(\tau)d\tau$$

$\rho_1 v_0 dt$ gives particle number in the slice (t,t+dt).

Longitudinal Microwave Instability

Hence, we obtain
$$\frac{d\Delta E(z,t)}{dt} = -\frac{c\Delta p_z(z,t)}{T_0} = -\frac{e^2 v}{T_0}\int_0^\infty \rho_1(z,t-\tau)w_{//}(\tau)d\tau$$

where $T_0 = \frac{C_0}{v_0}$ is the revolution period. Using the test solution

$$\psi_1(z,\Delta E,t) = \hat{\psi}_1(\Delta E)e^{inz/R-i\Omega t}$$

and the following relations

$$w_{//}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} Z_{//}(\omega)e^{-i\omega\tau}d\omega$$

$$\rho_1(z,t) = \int_{-\infty}^{\infty} \psi_1(z,\Delta E,t)\,d\Delta E = \hat{\rho}_1 e^{inz/R-i\Omega t} \qquad \hat{\rho}_1 \equiv \int_{-\infty}^{\infty} \hat{\psi}_1(\Delta E)\,d\Delta E$$

we can write the energy kick in term of longitudinal impedance

$$\frac{d\Delta E(z,t)}{dt} = -\hat{\rho}_1 \frac{e^2 v_0}{2\pi T_0} e^{inz/R-i\Omega t}\int_{-\infty}^{\infty} d\omega Z_{//}(\omega)\int_{-\infty}^{\infty} e^{i(\Omega-\omega)\tau}\,d\tau = -\hat{\rho}_1 \frac{e^2 v_0}{T_0} e^{inz/R-i\Omega t} Z_{//}(\Omega) \qquad (2)$$

Longitudinal Microwave Instability

Inserting eq. (1) and (2) into Vlasov equation, we obtain

$$-i\Omega\psi_1(z,\Delta E,t)+v(\Delta E)\cdot\frac{in}{R}\psi_1(z,\Delta E,t)-\hat{\rho}_1\frac{e^2v_0}{T_0}e^{inz/R-i\Omega t}Z_{//}(\Omega)\cdot\frac{\partial}{\partial\Delta E}\psi_0(\Delta E)=0$$

, which can be rewritten as

$$\psi_1(z,\Delta E,t)=\frac{ie^2v_0Z_{//}(\Omega)}{T_0}\frac{\hat{\rho}_1e^{inz/R-i\Omega t}}{\Omega-\omega(\Delta E)n}\frac{d\psi_0(\Delta E)}{d\Delta E}\qquad\omega(\Delta E)=\frac{v(\Delta E)}{R}$$

Integrating above equation over energy, i.e. $\int_{-\infty}^{\infty}d\Delta E\rightarrow$, yields

Dispersion relation:

$$1=\frac{ieI_0Z_{//}(\Omega)}{T_0}\int_{-\infty}^{\infty}\frac{f_0'(\Delta E)}{\Omega-\omega(\Delta E)n}d\Delta E$$

$$\psi_0(\Delta E)=\frac{N}{C_0}f_0(\Delta E)$$

$$I_0=eN/T_0$$

Longitudinal Microwave Instability

$$1 = \frac{ieI_0 Z_{//}(\Omega)}{T_0} \int_{-\infty}^{\infty} \frac{f_0'(\Delta E)}{\Omega - \omega(\Delta E) n} d\Delta E$$

*Phase slip factor: $\eta = \frac{1}{\gamma_t^2} - \frac{1}{\gamma^2}$

$$\omega(\Delta E) = \omega_0 + \Delta\omega(\Delta E) = \omega_0 - \eta\omega_0 \frac{\Delta p_z}{p_{0,z}} = \omega_0 - \frac{\eta\omega_0}{\beta^2}\frac{\Delta E}{E_0}$$

*Imaginary part of Ω tell us whether the system is stable

$$\psi_1(z, \Delta E, t) = \hat{\psi}_1(\Delta E) e^{inz/R - i\Omega t}$$

Cold Beam: $f_0(\Delta E) = \delta(\Delta E)$

$$1 = \frac{ieI_0 Z_{//}(\Omega)}{T_0} \frac{\eta n \omega_0}{E_0 \beta^2} \int_{-\infty}^{\infty} \frac{f_0(\Delta E)}{\left(\Omega - n\omega_0 + \frac{\eta n \omega_0}{E_0\beta^2}\Delta E\right)^2} d\Delta E$$

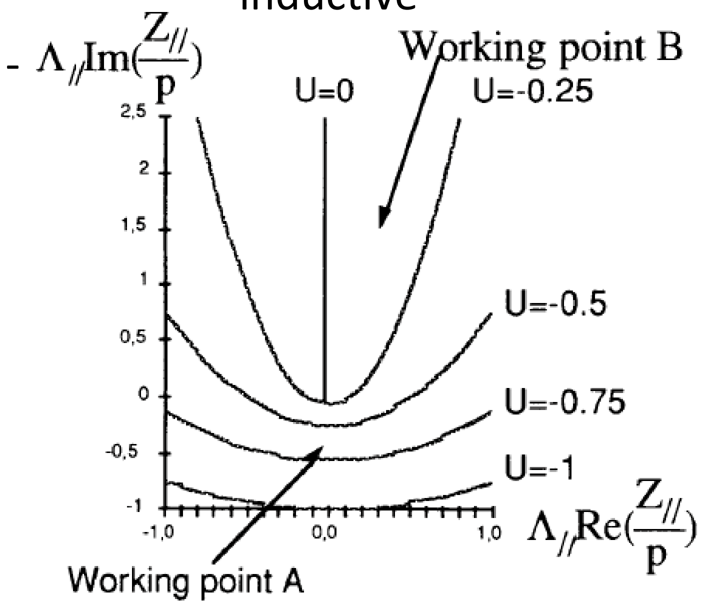
$$\Rightarrow \quad \Omega = n\omega_0 \pm \omega_0 \sqrt{\frac{ieI_0 \eta n Z_{//}(\Omega)}{2\pi E_0 \beta^2}} \approx n\omega_0 \pm \omega_0 \sqrt{\frac{ieI_0 \eta n Z_{//}(n\omega_0)}{2\pi E_0 \beta^2}}$$

Perturbative approach assuming $\frac{|\Omega - n\omega_0|}{n\omega_0} << 1$

Longitudinal Microwave Instabilities

Cold beam continued:
(assuming $\eta > 0$)

inductive



capacitive

$$\Omega \approx n\omega_0 \pm \omega_0 \sqrt{\frac{ieI_0\eta n Z_{//}(n\omega_0)}{2\pi E_0 \beta^2}}$$

For cold beam, the only case for stable beam is the machine impedance is pure inductive, i.e. $\mathrm{Im}(Z_{//}) < 0$, for $\eta > 0$ and capacitive, i.e. $\mathrm{Im}(Z_{//}) > 0$ for $\eta < 0$.

Taken from 'Accelerator Physics' by S.Y. Lee

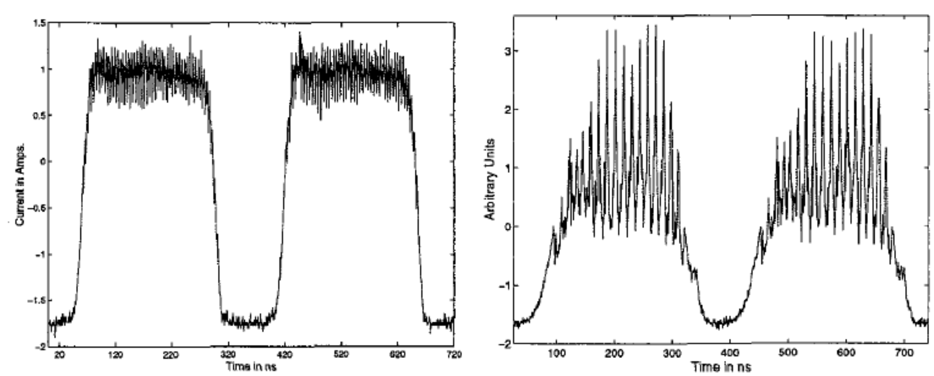


Figure 3.36: The longitudinal beam profiles observed at PSR the bunched coasting beam in the presence of inductive inserts, where three 1-m long ferrite ring cavities were installed in the PSR ring. [Courtesy of R. Macek, LANL]

Longitudinal Microwave Instabilities

Warm Beam: $$f_0(\Delta E) = \frac{1}{\sqrt{2\pi}\sigma_E} \exp\left(-\frac{\Delta E^2}{2\sigma_E^2}\right)$$

Dispersion relation for warm beam

$$1 = \frac{ieI_0 Z_{//}(n\omega_0)}{T_0} \frac{1}{\sqrt{2\pi}\sigma_E^2} \int_{-\infty}^{\infty} \frac{-\frac{\Delta E}{\sigma_E}\exp\left(-\frac{\Delta E^2}{2\sigma_E^2}\right)}{\Omega - \omega(\Delta E)n} d\Delta E = i\frac{1}{2}\left\{\frac{eI_0\left[Z_{//}(n\omega_0)/n\right]E_0\beta^2}{2\pi\eta\sigma_E^2}\right\} J_G(\tilde{\Omega})$$

$$= i\frac{2\ln(2)}{\pi}\left\{\frac{eI_0\left[Z_{//}(n\omega_0)/n\right]E_0\beta^2}{\eta\sigma_{E,FWHM}^2}\right\} J_G(\tilde{\Omega})$$

$$J_G(\tilde{\Omega}) = \sqrt{\frac{2}{\pi}} \int_{-\infty}^{\infty} \frac{-x\exp\left(-\frac{x^2}{2}\right)}{\left[\tilde{\Omega}\frac{E_0\beta^2}{\eta n\omega_0\sigma_E} + x\right]} dx$$

$$U' - iV' \equiv \frac{eI_0\left[Z_{//}(n\omega_0)/n\right]E_0\beta^2}{\eta\sigma_{E,FWHM}^2}$$

$$U' \sim \mathrm{Re}\left(Z_{//}(n\omega_0)\right)$$

$$V' \sim -\mathrm{Im}\left(Z_{//}(n\omega_0)\right)$$

$$\tilde{\Omega} = \mathrm{Re}(\tilde{\Omega}) + \mathrm{Im}(\tilde{\Omega}) \equiv \Omega - \omega_0 n$$

$$U' - iV' = \frac{-i\pi}{2\ln(2) J_G\left(\mathrm{Re}\,\tilde{\Omega} + i\,\mathrm{Im}\,\tilde{\Omega}\right)}$$

- The dispersion relation is solved by numerical plotting the contours for various $\mathrm{Im}\,\tilde{\Omega}$ in the complex impedance plane.

$$\psi_1(z, \Delta E, t) = \hat{\psi}_1(\Delta E) e^{inz/R - i\Omega t}$$

Longitudinal Microwave instability

Gaussian with various growth rate, $\mathrm{Im}(\tilde{\Omega})$

Contours with $\mathrm{Im}(\tilde{\Omega})=0$ for various energy distribution

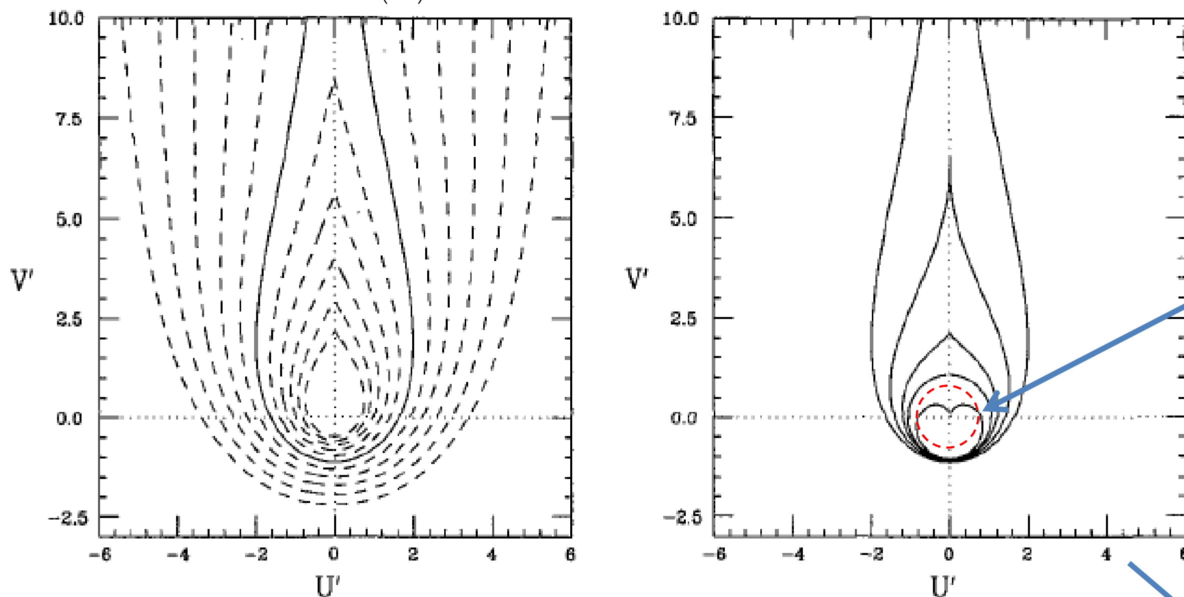


Figure 3.34: Left: The solid line shows the parameters V' vs U' for a Gaussian beam distribution at a zero growth rate. Dashed lines inside the threshold curve are stable. They correspond to $-\mathrm{Im}\,\Omega/(\sqrt{2\ln 2}\,\omega_0\eta\sigma_\delta) = -0.1, -0.2, -0.3, -0.4$, and -0.5. Dashed lines outside the threshold curve have growth rates $-\mathrm{Im}\,\Omega/(\sqrt{2\ln 2}\,\omega_0\eta\sigma_\delta) = 0.1, 0.2, 0.3, 0.4$, and 0.5 respectively. Right: The threshold V' vs U' parameters for various beam distributions.

Simplified estimation for stability condition:

Keil-Schnell criterion

$$\left|Z_{//}(n\omega_0)/n\right| \le \frac{2\pi|\eta|\sigma_E^2}{E_0\beta^2 eI_0}F$$

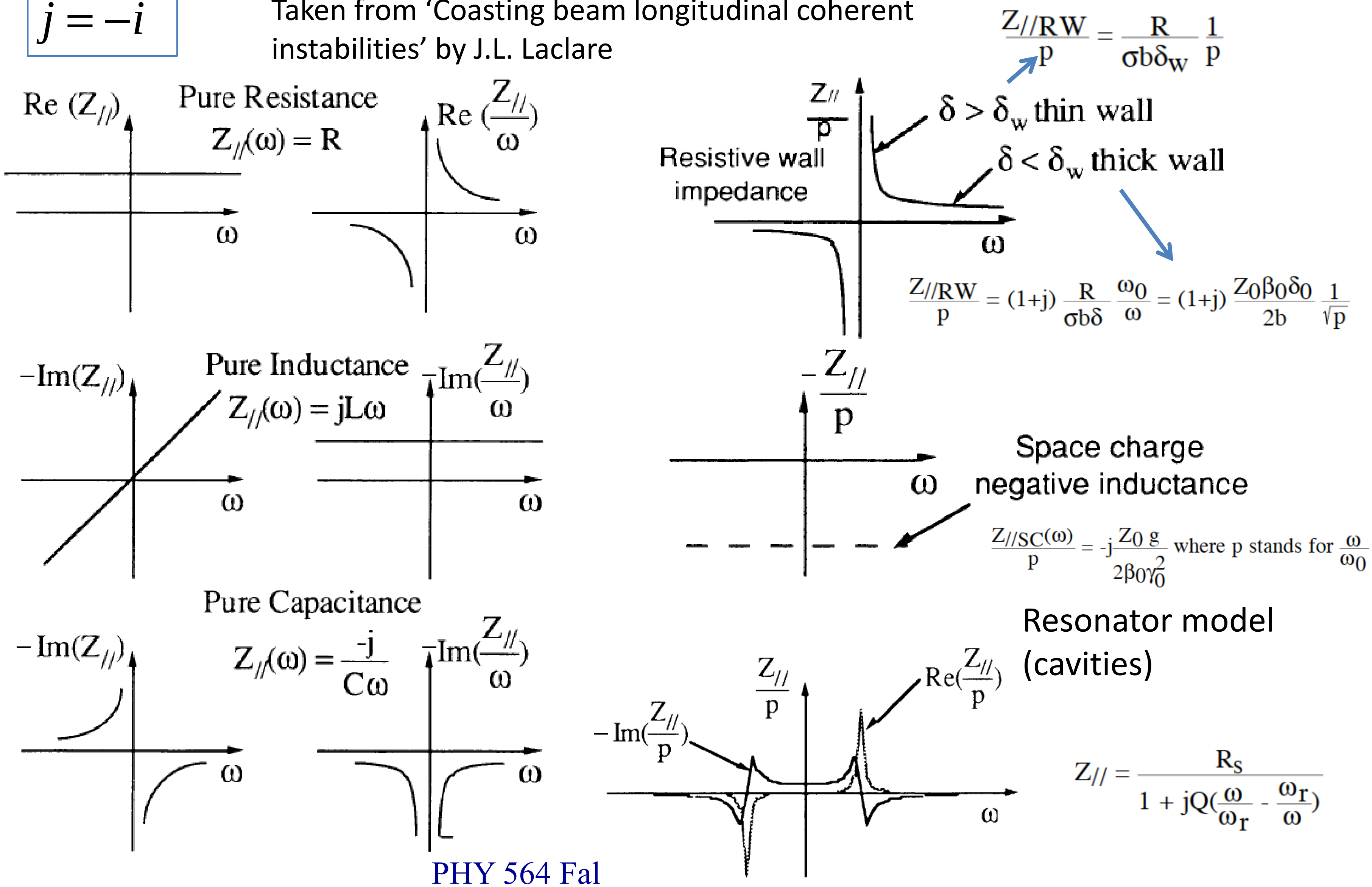
F depends on distributiion and for Gaussian energy distribution, it is 1.

from inside outward, for the normalized distribution functions $\Psi_0(x) = 3(1-x^2)/4$, $8(1-x^2)^{3/2}/3\pi$, $15(1-x^2)^2/16$, $315(1-x^2)^4/32$, and $(1/\sqrt{2\pi})\exp(-x^2/2)$. All dis-

Typical Longitudinal Impedance

$j = -i$

Taken from 'Coasting beam longitudinal coherent instabilities' by J.L. Laclare



Many pictures and derivations used in the slides are taken from the following references:

[1] 'Accelerator Physics' by S.Y. Lee;

[2] 'Physics of Collective Beam Instabilities in High Energy Accelerators' by A. Chao;

[3] 'Coasting beam longitudinal coherent instabilities' by J.L. Laclare

What we learned today

- In linear accelerator, single bunch transverse beam break up instability can develop if the bunches are not carefully injected and machine transverse wake function / impedance is large. Such a instability can be compensated by introducing focusing variation along the bunch, i.e. BNS damping.
- We also derived the dispersion relation for longitudinal microwave instability in a coasting beam. For cold beam, the beam is always unstable unless the impedances is pure inductive above transition or pure capacitive below transition. For warm beam, Landau damping make beam stable if the beam energy spread is sufficiently large. The stability condition can be estimated from Keil-Schnell criteria.