

1. The energy loss per turn is given by

$$U_0 = \frac{e^2 \beta^3 \gamma^4}{3\epsilon_0 \rho} . \quad (1)$$

With $\rho = 8m$ and $\gamma = 4.5GeV / 0.511MeV = 8806$, eq. (1) yields

$$U_0 = \frac{e^2 \beta^3 \gamma^4}{3\epsilon_0 \rho} = 4.535MeV = 7.266 \times 10^{-13} J . \quad (2)$$

The critical photon energy is given by

$$E_c = \hbar \omega_c , \quad (3)$$

where \hbar is the denoted Planck constant and

$$\omega_c = \frac{3}{2} \gamma^3 \frac{c}{\rho} \approx 3.839 \times 10^{19} \text{ rad / s} \quad (4)$$

is the critical angular frequency of the synchrotron radiation. Inserting eq. (4) into eq. (3) yields

$$E_c \approx 25.27 \text{ KeV} = 4.049 \times 10^{-15} J . \quad (5)$$

The total synchrotron radiation power for a beam is given by the 1-turn energy loss of all particles in the ring divided by the time it takes for one circulation (i.e. the revolution period)

$$P_{beam} = (U_0 \cdot N_{ring}) \frac{1}{T_{rev}} = \left(U_0 \cdot \frac{I_b T_{rev}}{e} \right) \frac{1}{T_{rev}} = U_0 \frac{I_b}{e} . \quad (6)$$

where $N_{ring} = I_b T_{rev} / e$ is the total number of electrons in the ring. Inserting eq. (2) and $I_b = 500mA$ into eq. (6) give

$$P_{beam} \approx 2.267 \text{ MW} . \quad (7)$$

2. The angular distribution of radiation power is given by

$$\frac{dP(t_r)}{d\Omega} = \frac{1}{4\pi\epsilon_0} \frac{e^2}{4\pi c} \frac{\dot{\beta}^2}{(1-\beta\cos\theta)^3} \left[1 - \frac{\sin^2\theta\cos^2\phi}{\gamma^2(1-\beta\cos\theta)^2} \right]. \quad (8)$$

For $\frac{1}{\gamma^4} \ll \theta \ll 1$ and $\gamma \gg 1$, we can use the following approximation

$$\begin{aligned} 1 - \beta\cos\theta &\approx 1 - \beta \left(1 - \frac{1}{2}\theta^2 \right) \\ &= 1 - \beta + \frac{1}{2}\beta\theta^2 \\ &= \frac{1}{\gamma^2(1+\beta)} + \frac{1}{2}\theta^2 \\ &= \frac{1}{\gamma^2} \left[\frac{1}{2-(1-\beta)} \right] + \frac{1}{2}\theta^2, \quad (9) \\ &\approx \frac{1}{2\gamma^2} \left[1 + \frac{1-\beta}{2} \right] + \frac{1}{2}\theta^2 \\ &\approx \frac{1}{2\gamma^2} \left[1 + \frac{1}{4\gamma^2} + \dots \right] + \frac{1}{2}\theta^2 \\ &\approx \frac{1}{2\gamma^2} + \frac{1}{2}\theta^2 \end{aligned}$$

and eq. (8) becomes

$$\frac{dP(t_r)}{d\Omega} \approx \frac{1}{4\pi\epsilon_0} \frac{2e^2}{\pi c} \frac{\gamma^6\dot{\beta}^2}{(1+\gamma^2\theta^2)^3} \left[1 - \frac{4\gamma^2\theta^2\cos^2\phi}{(1+\gamma^2\theta^2)^2} \right]. \quad (10)$$

Since the factor inside the square bracket is between 0 and 1, the angular width of eq. (10) is determined by the factor $(1+\gamma^2\theta^2)^{-3}$, i.e. the radiation power drops substantially when $\theta \geq \frac{1}{\gamma}$.

3. (a) The undulator period can be derived from the undulator equation with $\theta = 0$:

$$\lambda = \frac{\lambda_u}{2\gamma^2} \left(1 + \frac{K^2}{2} \right)$$

$$\Rightarrow \lambda_u = \lambda \frac{2\gamma^2}{1 + \frac{K^2}{2}} = 3.8 \text{ cm} .$$

(b) The power radiated into the central cone is (slide #29, Lecture 18)

$$P_{cen} = \frac{\pi e \gamma^2 I_e}{\epsilon_0 \lambda_u} \frac{K^2}{\left(1 + \frac{K^2}{2} \right)^2} f(K) = 6.038 \text{ W} ,$$

where $f(K) = \left[J_0 \left(\frac{K^2}{4 \left(1 + \frac{K^2}{2} \right)} \right) - J_1 \left(\frac{K^2}{4 \left(1 + \frac{K^2}{2} \right)} \right) \right]^2$. The photon flux is then

$$F_{cen} = \frac{P_{cen}}{\hbar \omega_0} = 1.52 \times 10^{16} \text{ s}^{-1} ,$$

with $\omega_0 = 2\pi c / \lambda = 3.767 \times 10^{18} \text{ rad} / \text{s}$. The spectral brightness is given by (slide #33 in Lecture 10)

$$B_{cen} = \frac{F_{cen}}{\Delta A \cdot \Delta \Omega \cdot N^{-1}} = \frac{F_{cen}}{2\pi \sigma_x \sigma_y \pi \theta_{Tx} \theta_{Ty} N^{-1}} = 2.275 \times 10^{35} \text{ m}^{-2} \text{ s}^{-1} = 2.275 \times 10^{20} \frac{1}{\text{s} \cdot \text{mm}^2 \text{ mrad}^2 (0.1\% \text{ BW})}$$

with $\theta_{Tx} = \sqrt{\theta_{cen}^2 + \sigma_{x'}^2} = 33.02 \mu\text{rad}$, $\theta_{Ty} = \sqrt{\theta_{cen}^2 + \sigma_{y'}^2} = 18.85 \mu\text{rad}$, $\theta_{cen} = \frac{1}{\gamma^* \sqrt{N}} = 18.45 \mu\text{rad}$,

$$\gamma^* = \frac{\gamma}{\sqrt{1 + K^2/2}} , \quad \sigma_{x'} = \sqrt{\epsilon_x / \beta_x} = 27.4 \mu\text{rad} , \quad \sigma_{y'} = \sqrt{\epsilon_y / \beta_y} = 3.873 \mu\text{rad} ,$$

$\sigma_x = \sqrt{\epsilon_x \beta_x} = 54.77 \mu\text{m}$, and $\sigma_y = \sqrt{\epsilon_y \beta_y} = 7.75 \mu\text{m}$. One can also use the practical formula in slide #34 of Lecture 18 to get the answer.