1. The energy loss per turn is given by

$$U_0 = \frac{e^2 \beta^3 \gamma^4}{3\varepsilon_0 \rho} \ . \tag{1}$$

With $\rho = 8m$ and $\gamma = 4.5GeV / 0.511MeV = 8806$, eq. (1) yields

$$U_0 = \frac{e^2 \beta^3 \gamma^4}{3\varepsilon_0 \rho} = 4.535MeV = 7.266 \times 10^{-13} J \ . \tag{2}$$

The critical photon energy is given by

$$E_c = \hbar \omega_c \, , \tag{3}$$

where $\hbar$ is the denoted Planck constant and

$$\omega_c = \frac{3}{2} \gamma^3 \frac{c}{\rho} \approx 3.839 \times 10^{19} rad / s \tag{4}$$

is the critical angular frequency of the synchrotron radiation. Inserting eq. (4) into eq. (3) yields

$$E_c \approx 25.27KeV = 4.049 \times 10^{-15} J \ . \tag{5}$$

The total synchrotron radiation power for a beam is given by the 1-turn energy loss of all particles in the ring divided by the time it takes for one circulation (i.e. the revolution period)

$$P_{beam} = \left(U_0 \cdot N_{ring} \right) \frac{1}{T_{rev}} = \left(U_0 \cdot \frac{I_b}{e} T_{rev} \right) \frac{1}{T_{rev}} = U_0 \frac{I_b}{e} \ . \tag{6}$$

where $N_{ring} = I_b T_{rev} / e$ is the total number of electrons in the ring. Inserting eq. (2) and $I_b = 500mA$ into eq. (6) give

$$P_{beam} \approx 2.267MW \ . \tag{7}$$

2. The angular distribution of radiation power is given by

$$\frac{dP(t_r)}{d\Omega} = \frac{1}{4\pi\varepsilon_0}\frac{e^2}{4\pi c}\frac{\dot{\beta}^2}{(1-\beta\cos\theta)^3}\left[1-\frac{\sin^2\theta\cos^2\phi}{\gamma^2(1-\beta\cos\theta)^2}\right]. \tag{8}$$

For $\frac{1}{\gamma^4} << \theta << 1$ and $\gamma >> 1$, we can use the following approximation

$$\begin{aligned} 1-\beta\cos\theta &\approx 1-\beta\left(1-\frac{1}{2}\theta^2\right) \\ &= 1-\beta+\frac{1}{2}\beta\theta^2 \\ &= \frac{1}{\gamma^2(1+\beta)}+\frac{1}{2}\theta^2 \\ &= \frac{1}{\gamma^2}\left[\frac{1}{2-(1-\beta)}\right]+\frac{1}{2}\theta^2 \quad , \\ &\approx \frac{1}{2\gamma^2}\left[1+\frac{1-\beta}{2}\right]+\frac{1}{2}\theta^2 \\ &\approx \frac{1}{2\gamma^2}\left[1+\frac{1}{4\gamma^2}+\ldots\right]+\frac{1}{2}\theta^2 \\ &\approx \frac{1}{2\gamma^2}+\frac{1}{2}\theta^2 \end{aligned} \tag{9}$$

and eq. (8) becomes

$$\frac{dP(t_r)}{d\Omega} \approx \frac{1}{4\pi\varepsilon_0}\frac{2e^2}{\pi c}\frac{\gamma^6\dot{\beta}^2}{\left(1+\gamma^2\theta^2\right)^3}\left[1-\frac{4\gamma^2\theta^2\cos^2\phi}{\left(1+\gamma^2\theta^2\right)^2}\right]. \tag{10}$$

Since the factor inside the square bracket is between 0 and 1, the angular width of eq. (10) is determined by the factor $\left(1+\gamma^2\theta^2\right)^{-3}$, i.e. the radiation power drops substantially when $\theta \geq \frac{1}{\gamma}$.

2. (a) The undulator period can be derived from the undulator equation with $\theta = 0$:

$$\lambda = \frac{\lambda_u}{2\gamma^2}\left(1+\frac{K^2}{2}\right)$$
$$\Rightarrow \lambda_u = \lambda\frac{2\gamma^2}{1+\frac{K^2}{2}} = 3.8cm \quad .$$

(b) The power radiated into the central cone is (slide #29, Lecture 18)

$$P_{cen} = \frac{\pi e\gamma^2 I_e}{\varepsilon_0\lambda_u}\frac{K^2}{\left(1+\frac{K^2}{2}\right)^2} f\left(K\right) = 6.038W \quad ,$$

where $f\left(K\right) = \left[J_0\left(\frac{K^2}{4\left(1+\frac{K^2}{2}\right)}\right) - J_1\left(\frac{K^2}{4\left(1+\frac{K^2}{2}\right)}\right)\right]^2$. The photon flux is then

$$F_{cen} = \frac{P_{cen}}{\hbar\omega_0} = 1.52\times10^{16}s^{-1} \quad ,$$

with $\omega_0 = 2\pi c/\lambda = 3.767\times10^{18}rad/s$. The spectral brightness is given by (slide #33 in Lecture 10)

$$B_{cen} = \frac{F_{cen}}{\Delta A\cdot\Delta\Omega\cdot N^{-1}} = \frac{F_{cen}}{2\pi\sigma_x\sigma_y\pi\theta_{Tx}\theta_{Ty}N^{-1}} = 2.275\times10^{35}m^{-2}s^{-1} = 2.275\times10^{20}\frac{1}{s\cdot mm^2 mrad^2\left(0.1\%BW\right)}$$

with $\theta_{Tx} = \sqrt{\theta_{cen}^2+\sigma_{x'}^2} = 33.02\mu rad$, $\theta_{Ty} = \sqrt{\theta_{cen}^2+\sigma_{x'}^2} = 18.85\mu rad$, $\theta_{cen} = \frac{1}{\gamma^*\sqrt{N}} = 18.45\mu rad$, $\gamma^* = \frac{\gamma}{\sqrt{1+K^2/2}}$, $\sigma_{x'} = \sqrt{\varepsilon_x/\beta_x} = 27.4\mu rad$, $\sigma_{y'} = \sqrt{\varepsilon_y/\beta_y} = 3.873\mu rad$, $\sigma_x = \sqrt{\varepsilon_x\beta_x} = 54.77\mu m$, and $\sigma_y = \sqrt{\varepsilon_y\beta_y} = 7.75\mu m$. One can also use the practical formula in slide #34 of Lecture 18 to get the answer.