PHY 564
Advanced Accelerator Physics
Lecture 4
Accelerator Hamiltonian

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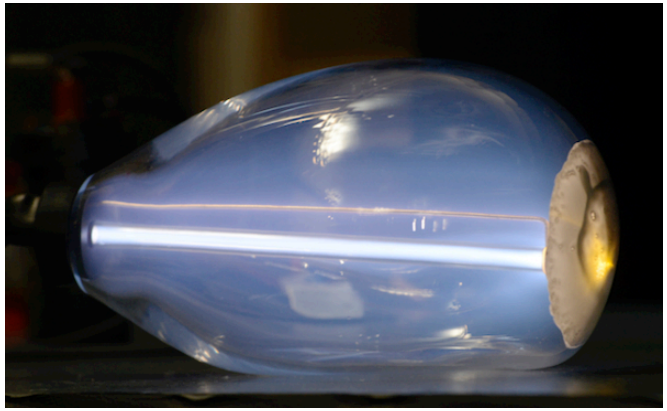
Accelerator Hamiltonian

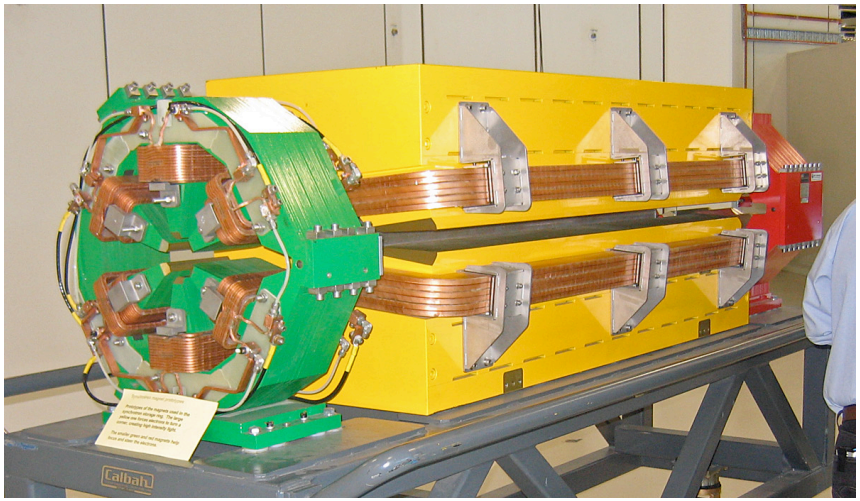
Intro: Before we are indulging ourselves on deriving accelerator Hamiltonian, let's discuss what accelerators are about. In practice they are about accelerating and circulating beams of relativistic charged particles. Beam – by a definition – is a group of particles which propagates along and around a common trajectory, which frequently called "orbit" or "reference trajectory". What is important that their motion is continuous, e.g. particles do not separate from the beam and go backwards. The later is very important, since the distance along the reference trajectory, s, will be used as an independent variable instead of the time. In addition, typical beams are confined transversely and usually propagate inside a vacuum chamber to avoid scattering. Exceptions are exceptions, and one can imagine an "accelerator" in which particles are completely disorganized and go everywhere in space in time – needless to say it most likely will be a useless device. Thus, let's focus on practical accelerators operating confined beams of charged particles.

A beam of particles in a cathode-ray tube



There is a number of very good reason for using *s* as independent variable: most of the accelerator elements are either DC (constant) or slowly varying in time, but always have a specific geometry – in other words all accelerators are bolted to the floor. Thus, arrival time of a particle into an accelerator element can vary, while element position, structure and duration along the reference trajectory is well defined. In circular accelerators (such as synchrotrons or storage rings), particles circulate for billions and billions of turns traversing the same magnetic structure (frequently called magnetic lattice!). This motion is nearly periodic in space along the trajectory.

LHC tunnel: a long periodic lattice of superconducting 80 kGs (8T) magnets

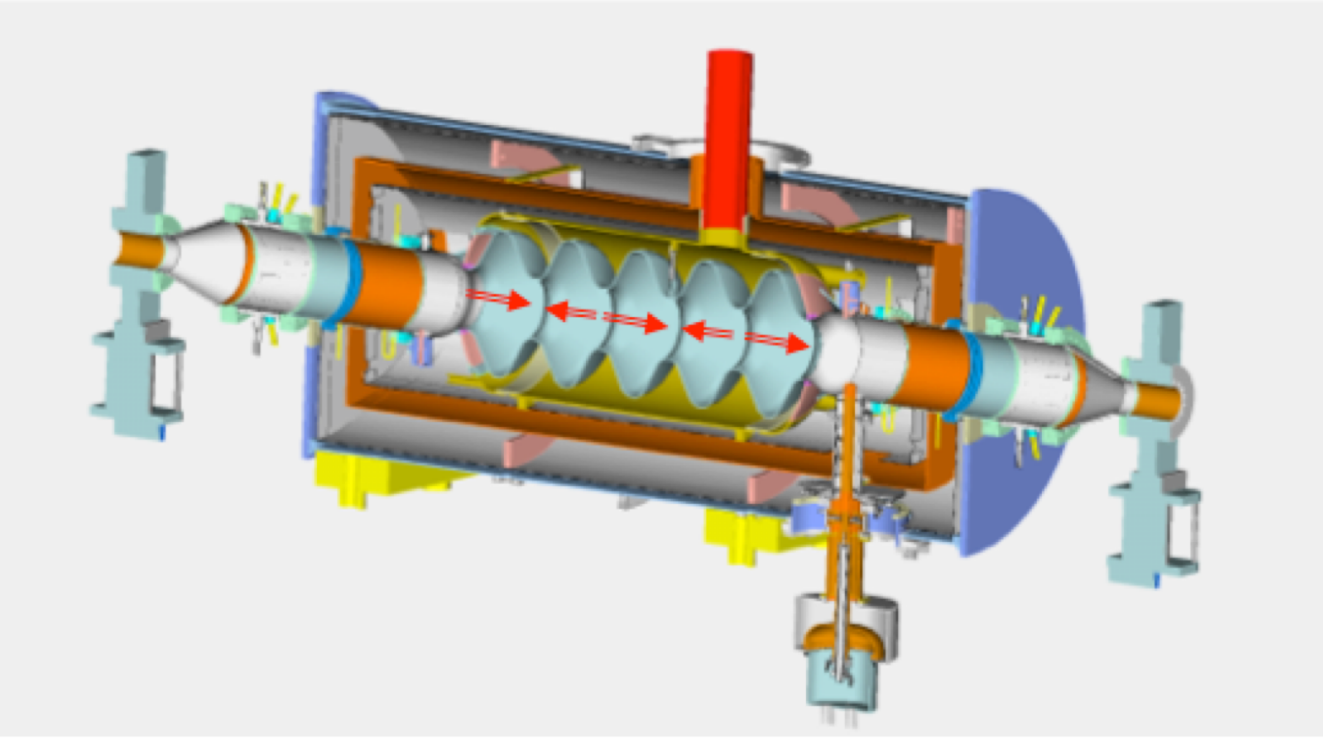


A magnet cell of an Australian light source: Green is a sextupole, yellow is a combined function dipole and red is a quadrupole

At the same time, only electric field can change energy of charged particles:

$$\frac{dE}{dt} \equiv mc^2 \frac{d\gamma}{dt} = e\vec{\mathbf{E}}(\vec{r},t)\cdot\vec{\mathrm{v}};$$

and has to be used for their acceleration (or deceleration).



A 5-cell SRF accelerator – an oscillating electric field is synchronized with the particle passing through that $\left\langle \vec{\mathbf{E}}(\vec{r},t)\cdot\vec{\mathrm{v}} \right\rangle$ is maximized.

While it is possible to accelerate particles in the DC electric field, the maximum energy is proportional to the difference of electric potential (voltage) and in practice is limited by few 10s of MeV. Meanwhile, using oscillating (RF) fields and special RF structures (linacs) it is possible to accelerate charged particles to any desirable energy –for example 7 TeV (7,000,000 MeV) at LHC.

Simple things useful in accelerator physics
sgs <-> SI <-> eV/TeV

- 1 meter = 100 cm; 1kg=10^3 g; 1J = 10^7 erg; seconds are universal
- Speed of the light 2.9979 x 10^{10} cm/sec ~ 3 x 10^{10} cm/sec
- Electron charge, e 4.803 x 10^{-10} ESU 1.602 x 10^{-19} C
- EM field, Gs 1 Gs = 299.79 (~300) V/cm 1 T =10^4 Gs
- Energy 1 eV =1.602 x 10^{-12} erg =1.602 x 10^{-19} J
- Energy/rigidity (pc) e x 1 Gs cm = 299.79 eV ~ 0.3 keV
- e x 1 T m = 299.79 MeV ~ 0.3 GeV

$$E = \sqrt{\vec{p}^2 c^2 + \left(mc^2\right)^2}$$

We will introduce more "handy" formulae/relations in the future

I found one useful unit in old British – modern USA system:

1' = One foot ~ 30 cm ~ c10^{-9} sec*

This how I remember: one foot is 1 nsec x speed of light

2.1 Accelerator coordinate system.

In accelerator physics we usually study beams of particles, i.e. particles moving in approximately the same direction (a huge difference from detectors) with approximately the same momenta. It is traditional, and very useful to choose one particle in the beam as the reference particle and study its trajectory $\vec{r}_o(t)$ as natural reference. Furthermore, most accelerator equipment is bolted to the floor and, hence, can be better described by its position in space that its existence in time. This is the reason why accelerator physicists decided to use length along the reference trajectory, ***s***, as independent coordinate instead of time:

$$s(t) = \int_{t_i}^{t} \left| d\vec{r}_o(t) \right| = \int_{t_i}^{t} \left| \vec{\mathrm{v}}_o(t) \right| dt; \tag{96}$$

$$\vec{\mathrm{v}}_o(t) = \frac{d\vec{r}_o(t)}{dt}; \quad \gamma(t) = 1/\sqrt{1 - \vec{\mathrm{v}}_o^{\,2}(t)/c^2}; \quad \vec{p}_o(t) = \gamma(t) m \vec{\mathrm{v}}_o(t); \quad E_o(t) = \gamma(t) mc^2$$

It is important for independent variable to be a monotonous function (as is time), which requires that the reference particle never stops moving (except possibly at the beginning and the end of the reference trajectory).

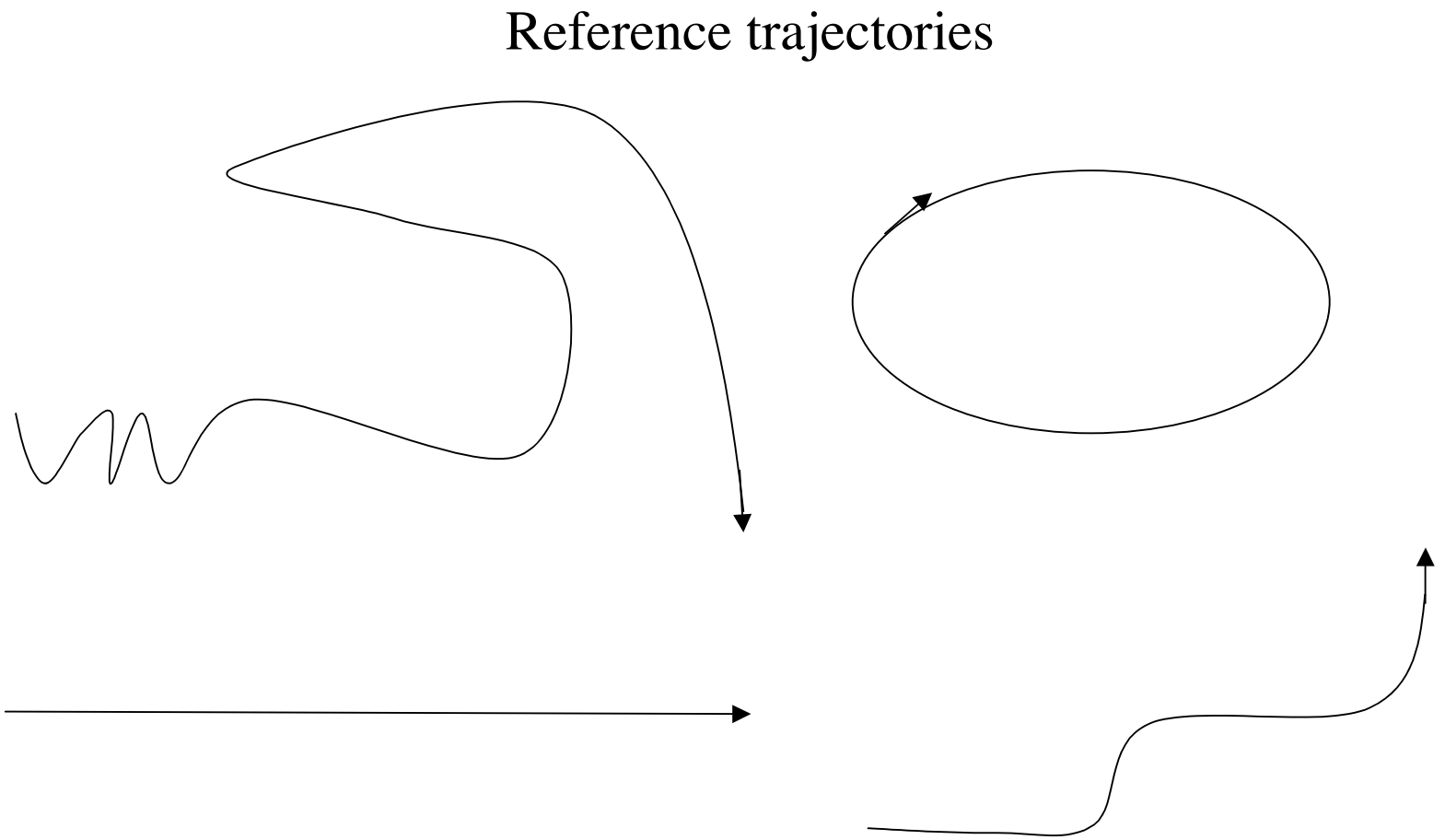


Fig. 1. Various possible reference trajectories, from a simple straight pass to a circular one, though all other possibilities.

The reference trajectory is determined by initial 4-momentum of the reference particle and the EM field along its trajectory. We should consider that trajectory is given (and from $\vec{r}_o(t)$ we also know the particle's 4-momentum in each point of trajectory) and so satisfy the equation of motion.

Usually EM fields are designed for the existence of such a trajectory (within constrains of Maxwell equation). Herein, the words reference trajectory and orbit are used interchangeably.

Inverting (96 we can write the 4D trajectory at the function of s:

$$\vec{r} = \vec{r}_o(s);\ t = t_o(s);\ \vec{p} = \vec{p}_o(s), E = E_o(s). \qquad (97$$

with the charge to the designer of accelerator to make it real trajectory:

$$\frac{d\vec{p}_o(s)}{ds} = \frac{dt_o(s)}{ds}\left(e\vec{E}\left(\vec{r}_o(s), t_o(s)\right) + \frac{e}{c}\left[\vec{\mathrm{v}}_o(s) \times \vec{B}\left(\vec{r}_o(s), t_o(s)\right)\right]\right) \qquad (98)$$

Starting from this point, we use following conventions: Derivatives of any function with respect to the time will be shown by appropriate number of dots, while appropriate number of symbol ′ will be used to indicate derivatives with respect to ***s***:

$$f' = \frac{df}{ds};\ f'' = \frac{d^2 f}{ds^2} \ldots\ldots\ \dot{f} = \frac{df}{dt};\ \ddot{f} = \frac{d^2 f}{dt^2}. \tag{99}$$

There is infinite variety of possible reference trajectories. The most popular ones are flat, i.e. they lie in a plane. A typical example is the circular orbit of a storage ring with a horizontal trajectory. Many of reference orbits are piece-wise combinations of trajectories lying in various planes. Still, there are 3D reference orbits by design. As the matter of fact, all real reference orbits are 3D because of the field errors in magnets, and errors in aligning these magnets.

Hence, there is no good reason not to start this discussion from general 3D reference trajectory. Fortunately two French mathematicians, Jean Frédéric Frenet and Joseph Alfred Serret, in the mid-nineteenth century developed such a coordinate system, which is described by the Frenet-Serret formulas in classical differential geometry (O.Struik, Dirk J., Lectures on Classical Differential Geometry, Addison-Wesley, Reading, Mass, 1961). The Frenet-Serret coordinate system often is called the natural coordinate system. One important feature is that it has non-diagonal metrics. Hence, we have a bit of differential geometry to spice the mix.

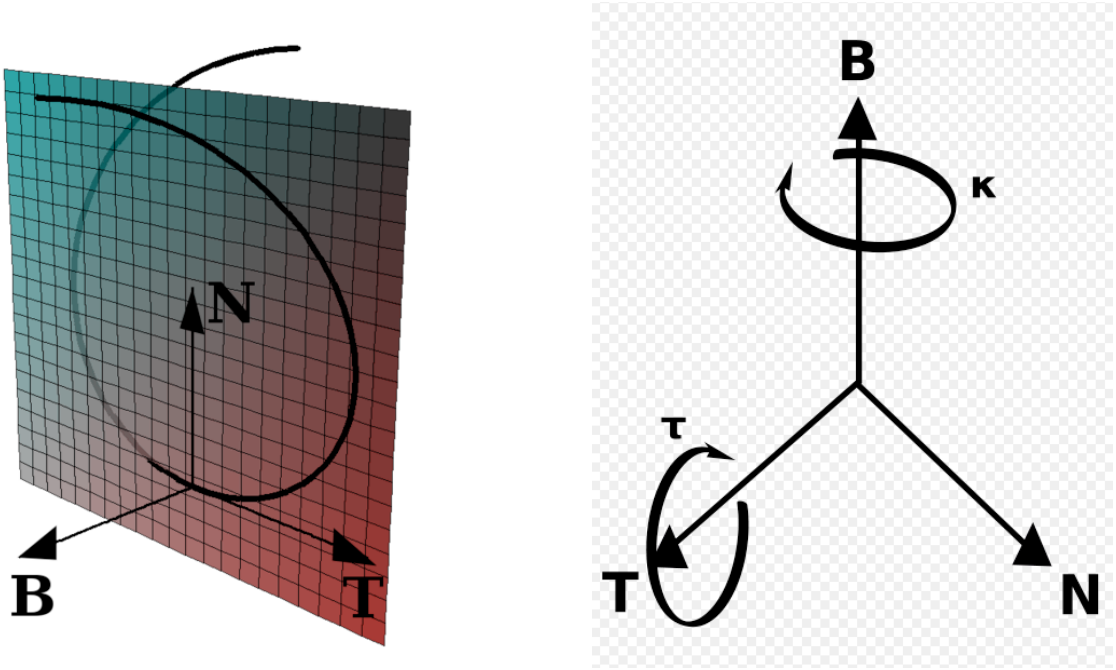
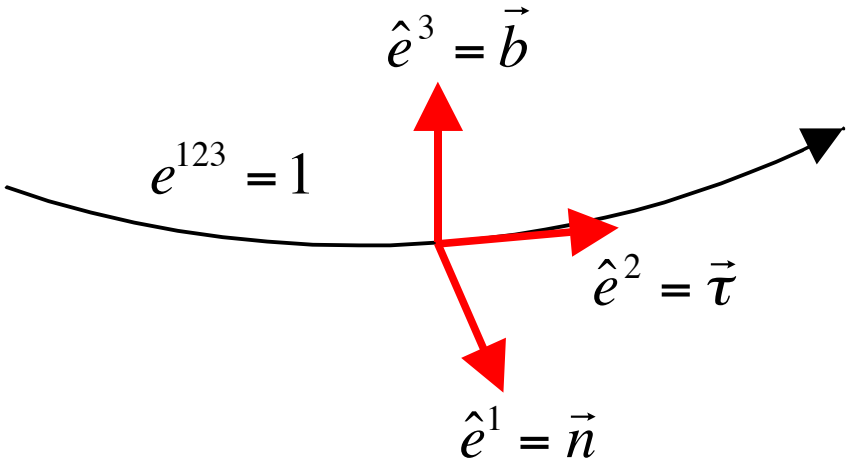


Fig. 2. Illustration of Frenet-Serret formulas and system from http://en.wikipedia.org/wiki/Frenet-Serret

Figures 2 and 3 illustrate the Frenet-Serret coordinate system and define 3 orthogonal unit vectors: Normal $\hat{e}^1 = \vec{n}(s)$, tangent $\hat{e}^2 = \vec{\tau}(s)$, and normal and bi-normal $\hat{e}^3 = \vec{b}(s) = \left[\vec{n} \times \vec{\tau}\right]$:

$$\left(\vec{n} \cdot \vec{\tau}\right) = \left(\vec{b} \cdot \vec{n}\right) = \left(\vec{b} \cdot \vec{\tau}\right) = 0.$$



$$\vec{\tau} = \frac{d\vec{r}_o(s)}{ds} = \vec{r}_o\,'$$

$$\vec{n} = -\frac{\vec{r}_o\,''}{\left|\vec{r}_o\,''\right|}$$

$$\vec{b} = \left[\vec{n} \times \vec{\tau}\right]$$

Fig. 3. Unit vectors in the Frenet-Serret coordinate system and their definitions

The reference trajectory must be smooth, with finite second derivatives, etc….etc… The position of any particle located in close proximity to the reference trajectory can uniquely expressed as

$$\vec{r} = \vec{r}_o(s) + x \cdot \vec{n}(s) + y \cdot \vec{b}(s). \tag{100}$$

i.e., it is fully described by 3 contra-variant coordinates:

$$q^1 = x;\ q^2 = s,\ q^3 = y. \tag{100-1}$$

The vectors $\{\vec{n},\vec{\tau},\vec{b}\}$ satisfy Frenet-Serret formulae:

$$\frac{d\vec{\tau}}{ds} = -K(s)\cdot\vec{n};\ \frac{d\vec{n}}{ds} = K(s)\cdot\vec{\tau} - \kappa(s)\cdot\vec{b};\ \frac{d\vec{b}}{ds} = \kappa(s)\cdot\vec{n};. \tag{101}$$

where

$$K(s) = 1/\rho(s) \tag{101-1}$$

is the curvature of the trajectory, and $\kappa(s)$ is its torsion. If the torsion is equal to zero, the trajectory remains in one plane, as designed for majority of accelerators. Curvature of trajectory is more common – each dipole magnet makes trajectory to curve.

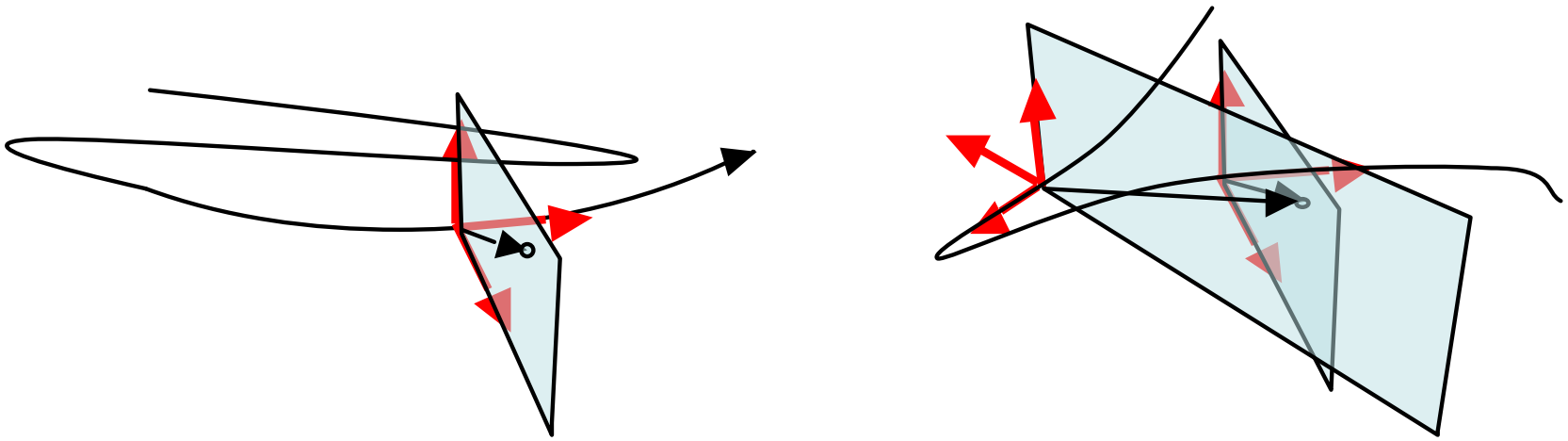
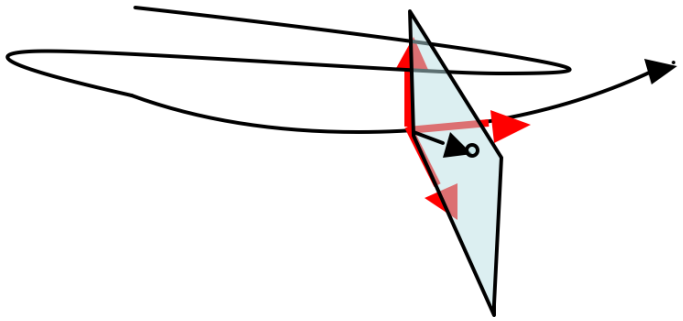


Fig. 4. Expansion of particle's position in Frenet-Serret frame.

Proximity to the reference orbit is important for the uniqueness of the extension (100): As shown on the figure above, equation (101-2) may have multiple solutions if the requirement of proximity is not applied, i.e, the expansion (100) may have multiple branches and mathematically become too involved.



As shown in Fig.4, the transverse part of the position vector $\vec{r}_\perp = x \cdot \vec{n}(s) + y \cdot \vec{b}(s)$ lies in the plane defined by the normal and by-normal unit vectors $(\vec{n}(s),\ \vec{b}(s))$, while s is defined from equation:

$$\left(\vec{r} - \vec{r}_o(s)\right) \cdot \vec{\tau}(s) = 0. \qquad (101\text{-}2)$$

Now we expand the differential geometry:

$$d\vec{r} = \sum_{i=1}^{3} \vec{a}_i dq^i = \vec{n}dx + \vec{b}dy + \left\{(1+Kx)\vec{\tau} + \kappa\left(\vec{n}y - \vec{b}x\right)\right\}ds \qquad (102)$$

with the co-variant basis of

$$\vec{a}_i = \frac{\partial \vec{r}}{\partial q^i};\quad \vec{a}_1 = \vec{n};\ \vec{a}_2 = (1+Kx)\vec{\tau} + \kappa\left(\vec{n}y - \vec{b}x\right);\quad \vec{a}_3 = \vec{b}; \qquad (103)$$

A co-variant basis vector is readily derived from the orthogonal conditions:

$$\vec{a}_i\vec{a}^{\,j}=\delta_i^{\,j};\ \vec{a}^{\,1}=\vec{n}-\frac{\kappa y}{1+Kx}\vec{\tau};\vec{a}^{\,2}=\frac{\vec{\tau}}{1+Kx};\ \vec{a}^{\,3}=\vec{b}+\frac{\kappa x}{1+Kx}\vec{\tau};\tag{104}$$

The components of the co- and contra-variant metric tensors are defined as follows:

$$g_{ik}=\vec{a}_i\cdot\vec{a}_k=\begin{bmatrix}1 & \kappa y & 0\\ \kappa y & (1+Kx)^2+\kappa^2\left(x^2+y^2\right) & -\kappa x\\ 0 & -\kappa x & 1\end{bmatrix}$$

$$g^{ik}=\vec{a}^{\,i}\cdot\vec{a}^{\,k}=\frac{1}{(1+Kx)^2}\cdot\begin{bmatrix}(1+Kx)^2+\kappa^2y^2 & -\kappa y & -\kappa^2xy\\ -\kappa y & 1 & \kappa x\\ -\kappa^2xy & \kappa x & (1+Kx)^2+\kappa^2x^2\end{bmatrix}\tag{105}$$

$$g_o=\det\left[g_{ik}\right]=(1+Kx)^2$$

Any vector can be expanded about both co- and contra-variant bases, as well can $\left\{\vec{n},\vec{\tau},\vec{b}\right\}$:

$$\vec{R} \equiv R_x\vec{n} + R_s\vec{\tau} + R_y\vec{b} \equiv \sum_k R^k\vec{a}_k \equiv \sum_k R_k\vec{a}^k$$

$$R_k = \vec{R}\cdot\vec{a}_k;\ R_1 = R_x; R_2 = (1+Kx)R_s + \kappa\left(R_x y - R_y x\right);\ R_3 = R_y;$$

$$R^k = \vec{R}\cdot\vec{a}^k;\ R^1 = R_x - \frac{\kappa y}{1+Kx}R_s;\ R^2 = \frac{R_s}{1+Kx} + \kappa\left(R_x y - R_y x\right);\ R^3 = R_y + \frac{\kappa x}{1+Kx}R_s;$$

(106)

All this is trivial, and finally differential operators will look like:

$$\vec{\nabla}\varphi = \vec{a}^k\frac{\partial\varphi}{\partial q^k};\quad div\vec{A} = \left(\vec{\nabla}\cdot\vec{A}\right) = \frac{1}{\sqrt{g_o}}\frac{\partial}{\partial q^k}\left(\sqrt{g_o}A^k\right);$$

$$curl\vec{A} = \left[\vec{\nabla}\times\vec{A}\right] = \frac{e^{ikl}}{\sqrt{g_o}}\frac{\partial A_l}{\partial q^k}\vec{a}_i;\quad \Delta\varphi = \vec{\nabla}^2\varphi = \frac{1}{\sqrt{g_o}}\frac{\partial}{\partial q^i}\left(\sqrt{g_o}g^{ik}\frac{\partial\varphi}{\partial q^k}\right). \qquad (107)$$

As discussed before, the Hamiltonian of a charged particle in EM field in Cartesian coordinate system is

$$H(\vec{r},\vec{P},t) = c\sqrt{m^2c^2 + \left(\vec{P} - \frac{e}{c}\vec{A}\right)^2} + e\varphi, \qquad \text{(from Lecture 3/4)}$$

where the canonical momentum is $\vec{P} = \vec{p} + \frac{e}{c}\vec{A}$. Let us explore how we can make the transformation to our "curved and twisted" coordinate system. The easiest way is to apply canonical transformation with generation function

$$F(\vec{P},q^i) = -\vec{P}\cdot\left(\vec{r}_o(s) + x\cdot\vec{n}(s) + y\cdot\vec{b}(s)\right). \qquad (108)$$

to our new coordinates (101):

$$q^1 = x;\ q^2 = s,\ q^3 = y\,. \qquad (109)$$

with new momenta obtained by simple differentiation

$$P_1 = P_x; P_2 = \left(1+Kx\right)P_s + \kappa\left(P_x y - P_y x\right);\ P_3 = P_y; \qquad (110)$$

that alter the appearance of the Hamiltonian (L1.38)

$$H = c\sqrt{\begin{array}{l}(1+Kx)^{-2}\left(\left(P_2 - \frac{e}{c}A_2\right) + \kappa x\left(P_3 - \frac{e}{c}A_3\right) - \kappa y\left(P_1 - \frac{e}{c}A_1\right)\right)^2 \\ + \left(P_1 - \frac{e}{c}A_1\right)^2 + \left(P_3 - \frac{e}{c}A_3\right)^2 + m^2c^2\end{array}} + e\varphi \qquad (111)$$

This is still the Hamiltonian with ***t*** as independent variable and three sets of canonical pairs $\{q^1,P_1\},\{q^2,P_2\},\{q^3,P_3\}$. Now, we change the independent variable to ***s*** by the easiest method, that, as always, is using the least-action principle: we consider the conjugate momentum to s, P_2, as a function of the remaining canonical variables: $\{q^1,P_1\}$, $\{q^3,P_3\}$, $\{-t,H\}$

$$S=\int_A^B P_1dq^1+P_2(....)ds+P_3dq^3-Hdt;\quad \delta S=0; \tag{112}$$

Notably, the coordinates and time, the canonical momenta and the Hamiltonian appear in the 4-D scalar product form in the action integral.

$$P_idx^i;\ x^i=\{ct,x,s,y\};P_i=\{H/c,-P_1,-P_2,-P_3\},i=0,1,2,3.$$

This equivalency of the time and space is fundamental to the relativistic theory.

Let's use s a independent variable and t as one of the coordinates:

$$\delta S_{AB} = \delta\left(\int_A^B P_i dq^i - Hdt\right) = \int_A^B \left(\begin{array}{l} \sum_{i=1,3}\left(\delta P_i dq^i + P_i d\delta q^i + \frac{\partial P_2}{\partial q^i}\delta q^i ds + \frac{\partial P_2}{\partial P_i}\delta P_i ds\right) + \\ \left(-\delta H dt - H d\delta t + \frac{\partial P_2}{\partial t}\delta t ds + \frac{\partial P_2}{\partial H}\delta H ds\right) \end{array}\right) = 0$$

and integrating by parts $\sum_{i=1,3} P_i \delta q^i - H\delta t\Big|_A^B \equiv 0$, equations of motions as functions of s:

$$\delta S_{AB} = \int_A^B \left(\begin{array}{l} \sum_{i=1,3}\left(\delta P_i\left(\frac{\partial P_2}{\partial P_i}ds + dq^i\right) + \delta q^i\left(\frac{\partial P_2}{\partial q^i}ds - dP_i\right)\right) \\ +\left(\delta t\left(dH + \frac{\partial P_2}{\partial t}ds\right) + \delta H\left(\frac{\partial P_2}{\partial H}ds - dt\right)\right) \end{array}\right) = 0 \qquad (113)$$

$$\frac{dq^i}{ds} = -\frac{\partial P_2}{\partial P_i}; \frac{dt}{ds} = \frac{\partial P_2}{\partial H}ds; \quad \frac{dP_i}{ds} = +\frac{\partial P_2}{\partial q_i^i}; \frac{dH}{ds} = -\frac{\partial P_2}{\partial t}$$

Or explicitly:

$$x' = \frac{dx}{ds} = \frac{\partial h^*}{\partial P_1}; \quad \frac{dP_1}{ds} = -\frac{\partial h^*}{\partial x}; \qquad y' = \frac{dy}{ds} = \frac{\partial h^*}{\partial P_3}; \quad \frac{dP_3}{ds} = -\frac{\partial h^*}{\partial y}$$

$$t' = \frac{dt}{ds} = \frac{\partial h^*}{\partial P_t} \equiv -\frac{\partial h^*}{\partial H}; \quad \frac{dP_t}{ds} = -\frac{\partial h^*}{\partial t} \rightarrow \frac{dH}{ds} = \frac{\partial h^*}{\partial t} \tag{114}$$

$$h^* = -(1+Kx)\sqrt{\frac{(H-e\varphi)^2}{c^2} - m^2c^2 - \left(P_1 - \frac{e}{c}A_1\right)^2 - \left(P_3 - \frac{e}{c}A_3\right)^2} + \frac{e}{c}A_2 + \kappa x\left(P_3 - \frac{e}{c}A_3\right) - \kappa y\left(P_1 - \frac{e}{c}A_1\right) \tag{115}$$

Thus, by choosing one of coordinates as independent variable, the new Hamiltonian is nothing but its conjugate canonical momentum with a minus sign.

The same result can be obtained indirectly (the way frequently used in text books) by using equivalency of the Canonical pairs:

$$H = H(x^i, P_1, P_2, P_3) \overset{solve}{\rightarrow} P_2 = P_2(x^i, P_1, P_2, H)$$

$$rename \quad P_t = -H; h^* = -P_2(x^i, P_1, P_2, H)$$

$$S = \int P_1\, dx + P_3\, dy + P_z\, dz - H\, dt \equiv$$

$$\int P_1\, dx - P_3\, dy - h^*\, dz + P_t\, dt$$

While this gives the same result, it has an appearance of a trick, not direct derivation. Hence, we did it from the least action principle.

Applying a canonical transformation that exchanges the coordinate with momentum and then employs a new coordinate (old momentum) as the independent variable it would turn the old coordinate into the new Hamiltonian. In all cases, the Hamiltonian is the function of the remaining canonical variables. This capability of the Hamiltonian systems is unique and one we can take advantage of. An important restriction is the monotonous behavior of independent variable. Otherwise, some or all of the derivatives can be infinite in the point where the independent variable stumbles (i.e., where the new time stops).

The equations (114) and (115) are the general form of the single-particle Hamiltonian equation in an accelerator. It undoubtedly is nonlinear (the square root signifies relativistic mechanics), and cannot be solved analytically in general. Only few specific cases allow such solutions.

Most General Form of the Accelerator Hamiltonian

$$h^* = -(1+Kx)\sqrt{\frac{(H-e\varphi)^2}{c^2} - m^2c^2 - \left(P_1 - \frac{e}{c}A_1\right)^2 - \left(P_3 - \frac{e}{c}A_3\right)^2}$$

$$+\frac{e}{c}A_2 + \kappa x\left(P_3 - \frac{e}{c}A_3\right) - \kappa y\left(P_1 - \frac{e}{c}A_1\right)$$

$$x' = \frac{dx}{ds} = \frac{\partial h^*}{\partial P_1}; \quad \frac{dP_1}{ds} = -\frac{\partial h^*}{\partial x}; \qquad y' = \frac{dy}{ds} = \frac{\partial h^*}{\partial P_3}; \quad \frac{dP_3}{ds} = -\frac{\partial h^*}{\partial y}$$

$$t' = \frac{dt}{ds} = \frac{\partial h^*}{\partial P_t} \equiv -\frac{\partial h^*}{\partial H}; \quad \frac{dP_t}{ds} = -\frac{\partial h^*}{\partial t} \rightarrow \frac{dH}{ds} = \frac{\partial h^*}{\partial t}$$

We always have a choice of the reference orbit (e.g. K and $\varkappa$) as well as of the gauge of 4-potential. We can use this flexibility for our benefit!
We will use a specific gauge to express components of 4-potentail as explicit functions of electric and magnetic fields

Choosing a specific gauge

This allows us to express 4-potential as explicit function of electric and magnetic fields – it is useful when you explain what you need to build to engineers…

The equations (114) and (115) are the general form of the single-particle Hamiltonian equation in an accelerator. It undoubtedly is nonlinear (the square root signifies relativistic mechanics), and cannot be solved analytically in general. Only few specific cases allow such solutions.

The only additional option we have is to choose a gauge for the 4-potential. One good choice (my preference) is to make the vector potential equal to zero at the reference trajectory. Two other auxiliary conditions will allow us to express the components of the 4-vector potential in a form of the Taylor series:

$$a)\ \vec{A}(s,0,0,t)=0;\ b)\ \partial_x^n A_1\Big|_{s,0,0,t}=\partial_y^n A_3\Big|_{s,0,0,t}=0;\ c)\ \frac{\partial A_1}{\partial x}\Big|_{s,0,0,t}+\frac{\partial A_3}{\partial y}\Big|_{s,0,0,t}=0 \quad (116)$$

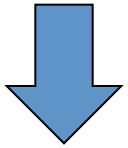
that can be achieved by gauge transformation

$$\vec{A} = \tilde{\vec{A}} - \vec{\nabla} f ; \varphi = \varphi + \frac{1}{c}\frac{\partial f}{\partial t}; \; f = f_a = f_b + f_c$$

$$f_a = \int_0^s \tilde{A}_2(s_1,0,0,t) ds_1 + \tilde{A}_1(s_1,0,0,t) \cdot x + \tilde{A}_3(s_1,0,0,t) \cdot y$$

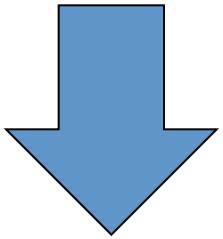
$$f_b = \sum_{n=1} \left(\partial_x^n \tilde{A}_1 \Big|_{s,0,0,t} \frac{x^{n+1}}{(n+1)!} + \partial_y^n \tilde{A}_3 \Big|_{s,0,0,t} \frac{y^{n+1}}{(n+1)!} \right) \qquad (117)$$

$$f_c = \frac{1}{2} \sum_{n,k=0} \partial_x^n \partial_y^k \left(\partial_y \tilde{A}_1 + \partial_x \tilde{A}_3 \right) \frac{x^{n+1}}{(n+1)!} \frac{y^{k+1}}{(n+1)!}$$



$$a) \;\; \vec{A}(s,0,0,t) = 0; \;\; b) \;\; \partial_x^n A_1 \Big|_{s,0,0,t} = \partial_y^n A_3 \Big|_{s,0,0,t} = 0; \;\; c) \;\; \frac{\partial A_1}{\partial x}\Big|_{s,0,0,t} + \frac{\partial A_3}{\partial y}\Big|_{s,0,0,t} = 0$$

$$a)\ \ \vec{A}(s,0,0,t)=0;\ \ b)\ \ \partial_x^n A_1\Big|_{s,0,0,t}=\partial_y^n A_3\Big|_{s,0,0,t}=0;\ \ c)\ \ \frac{\partial A_1}{\partial x}\Big|_{s,0,0,t}+\frac{\partial A_3}{\partial y}\Big|_{s,0,0,t}=0$$



Conditions (116) have following important consequences:

$$a)\,\partial_s^k\partial_t^l\vec{A}(s,0,0,t)\equiv 0;\ b)A_1(s,x,0,t)\equiv 0;$$

$$A_3(s,0,y,t)\equiv 0;\ c)\partial_s^k\partial_t^l\partial_x^m\partial_y^n\left(\frac{\partial A_1}{\partial x}+\frac{\partial A_3}{\partial y}\right)\equiv 0 \qquad (116{+}{+})$$

After a one-page-long exercise, using the first pair of Maxwell equations and conditions (116), one can express the 4-potential in this gauge though the components of the magnetic- and electric- fields, in other words, make an unique vector potential:

$$A_1 = \frac{1}{2}\sum_{n.k=0}^{\infty} \partial_x^k \partial_y^n B_s\big|_{ro} \frac{x^k}{k!}\frac{y^{n+1}}{(n+1)!}; \; A_3 = -\frac{1}{2}\sum_{n.k=0}^{\infty} \partial_x^k \partial_y^n B_s\big|_{ro} \frac{x^{k+1}}{(k+1)!}\frac{y^n}{n!}$$

$$A_2 = \sum_{n=1}^{\infty}\left\{\partial_x^{n-1}\Big(\big(1+Kx\big)B_y + \kappa x B_s\Big)_{ro}\frac{x^n}{n!} - \partial_y^{n-1}\Big(\big(1+Kx\big)B_x - \kappa y B_s\Big)_{ro}\frac{y^n}{n!}\right\} +$$

$$+\frac{1}{2}\sum_{n.k=1}^{\infty}\left\{\partial_x^{n-1}\partial_y^k\Big(\big(1+Kx\big)B_y + \kappa x B_s\Big)_{ro}\frac{x^n}{n!}\frac{y^k}{k!} - \partial_x^{n}\partial_y^{k-1}\Big(\big(1+Kx\big)B_x - \kappa y B_s\Big)_{ro}\frac{x^n}{n!}\frac{y^k}{k!}\right\};$$

$$\varphi = \varphi_o(s,t) - \sum_{n=1}^{\infty}\partial_x^{n-1}E_x\big|_{ro}\frac{x^n}{n!} - \sum_{n=1}^{\infty}\partial_y^{n-1}E_y\big|_{ro}\frac{y^n}{n!} - \frac{1}{2}\sum_{n.k=1}^{\infty}\left(\partial_x^{n-1}\partial_y^k E_x\big|_{ro} + \partial_x^n\partial_y^{k-1}E_y\big|_{ro}\right)\frac{x^n}{n!}\frac{}{}$$

(118)

where $f\big|_{ro}$; $(f)_{ro}$ denotes that the value of the function f is taken at the reference orbit $r_o(s)$: i.e., at $x=0;\;\; y=0$, but in an arbitrary moment of time t.

We reserve the notions $f\big|_{ref}$; $(f)_{ref}$ for values taken at the reference trajectory $\vec{r} = \vec{r}_o(s)$ at the reference time $t = t_o(s)$. It is noteworthy that the value of our new Hamiltonian for the reference particle is the full particle's momentum with the minus sign:

$$h^*\big|_{ref} = -p_o(s) \tag{119}$$

We should note that $\varphi_o(s,t)$ is determined with the accuracy of an arbitrary constant, which can be eliminated by requesting $\varphi_o\left(s_o, t_o\left(s_o\right)\right) = 0$ at some point along the reference trajectory. The coefficients in (118) can be expanded further using a trivial time series

$$f(t) = f(t_o(s)) + \sum_{n=1}^{\infty} \left.\frac{d^n f}{dt^n}\right|_{t=t_o(s)} \frac{(t - t_o(s))^n}{n!}.$$

One important feature of this expansion that no conditions in the EM field are assumed; thus, it can be in free-space field (typical for single-particle dynamics) or a field with sources (for example, charges and currents of beam are examples). Hence, the expansion is applicable to any arbitrary accelerator problem.

An equilibrium particle and a reference trajectory.

A particle that follows the reference trajectory is called an equilibrium (or reference) one:

$$\vec{r} = \vec{r}_o(s); \quad t = t_o(s); \quad H = H_o(s) = E_o(s) + \varphi_o(s, t_o(s)), \tag{120}$$

with $x \equiv 0; \quad y \equiv 0; \; p_x \equiv 0; p_y \equiv 0$. This is where condition (L2.20a) $\vec{A}\Big|_{ref} = 0$ is useful, i.e., for

$$x\big|_{ref} = 0; \; y\big|_{ref} = 0; \; P_1\Big|_{ref} = p_x\big|_{ref} + \frac{e}{c} A_1\big|_{ref} \equiv 0; \quad P_3\big|_{ref} = p_y\big|_{ref} + \frac{e}{c} A_3\big|_{ref} \equiv 0. \tag{121}$$

The differential form of (121)

$$\begin{aligned} \left.\frac{dx}{ds}\right|_{ref} &= \left.\frac{\partial h^*}{d P_1}\right|_{ref} = 0; \quad \left.\frac{dy}{ds}\right|_{ref} = \left.\frac{\partial h^*}{dP_3}\right|_{ref} = 0; \\ \left.\frac{dP_1}{ds}\right|_{ref} &= -\left.\frac{\partial h^*}{dx}\right|_{ref} = 0; \quad \left.\frac{dP_3}{ds}\right|_{ref} = -\left.\frac{\partial h^*}{dy}\right|_{ref} = 0; \end{aligned} \tag{122}$$

should be combined with the expression for the Hamiltonian (L2.19). The two first equations in (122) give us the already known conditions, viz., that of the zero transverse component of momentum. The following two equations are not as trivial; they set the two conditions at the reference orbit.

Completing a trivial differentiation on $\boldsymbol{x}$ (where most of the terms are turned into zero at the reference orbit, except $\partial_x \varphi$ and $\partial_x A_2$) we have

$$-\left.\frac{\partial h^*}{\partial x}\right|_{ref} = K\sqrt{G}\Big|_{ref} - (1+Kx)_{ref} \frac{\left(\left[\frac{eE}{c^2}\frac{\partial \varphi}{\partial x}\right] + p_x \frac{e}{c}\frac{\partial A_1}{\partial x} + p_y \frac{e}{c}\frac{\partial A_3}{\partial x}\right)_{ref}}{\sqrt{G}\Big|_{ref}}$$

$$+\left[\frac{e}{c}\frac{\partial A_2}{\partial x}\right]_{ref} + \kappa\left(p_y\right)_{ref} + \kappa\left(\frac{e}{c}\frac{\partial A_1}{\partial x}y - \frac{e}{c}\frac{\partial A_3}{\partial x}x\right)_{ref} = 0$$

$$E \equiv (H - e\varphi);\;\; G = \frac{E^2}{c^2} - m^2c^2 - p_x^2 - p_y^2;\; \sqrt{G}\Big|_{ref} = p_o \ldots$$

Note: The term(s) that do not vanish at the limit are identified by the square brackets […]

and using the above expansions, we derive the well-know equation for the curvature of the trajectory:

$$K(s) \equiv \frac{1}{\rho(s)} = -\frac{e}{p_o c}\left(B_y\Big|_{ref} - \frac{E_o}{p_o c}E_x\Big|_{ref}\right). \qquad (123)$$

Differentiation on ***y*** is similar

$$-\left.\frac{\partial h^*}{\partial y}\right|_{ref} = -(1+Kx)_{ref}\frac{\left(\left[\frac{eE}{c^2}\frac{\partial\varphi}{\partial y}\right]+p_x\frac{e}{c}\frac{\partial A_1}{\partial y}+p_y\frac{e}{c}\frac{\partial A_3}{\partial y}\right)_{ref}}{\left.\sqrt{G}\right|_{ref}}$$

$$+\left[\frac{e}{c}\frac{\partial A_2}{\partial y}\right]_{ref}-(p_x)_{ref}+\kappa\left(\frac{e}{c}\frac{\partial A_1}{\partial y}y-\frac{e}{c}\frac{\partial A_3}{\partial y}x\right)_{ref}=0$$

and yields

$$\left.B_x\right|_{ref} = \frac{E_o}{p_o c}\left.E_y\right|_{ref}, \qquad (124)$$

That represents only the absence of "vertical curvature". The difference between (123) and (124) arises from the choice of coordinates in Frenet-Serret system: ***x*** (i.e., $\boldsymbol{q_1}$) corresponds to the plane where trajectory bends.

The conditions in (120) for the arrival time of the reference particle and values of its Hamiltonian are also informative, but not surprising. First, the condition on the arrival time

$$\frac{dt_o(s)}{ds} = -\left.\frac{\partial h^*}{dH}\right|_{ref} = \left.\frac{H-e\varphi}{c^2\sqrt{G}}\right|_{ref} = \frac{H_o-e\varphi_o}{p_o c^2} \equiv \frac{E_o}{p_o c^2} = \frac{1}{\mathrm{v}_o(s)} \qquad (125)$$

gives an understandable definition of velocity along trajectory: ***v=ds/dt,*** and the velocity of the reference particle $\mathrm{v}_o = p_o c^2 / E_o$.

The condition on energy (3D Hamiltonian) gives

$$\left.\frac{\partial h^*}{\partial t}\right|_{ref} = (1+Kx)_{ref} \frac{\left(\left[\frac{eE}{c^2}\frac{\partial\varphi}{\partial t}\right] + p_x \frac{e}{c}\frac{\partial A_1}{\partial t} + p_y \frac{e}{c}\frac{\partial A_3}{\partial t}\right)_{ref}}{\left.\sqrt{G}\right|_{ref}}$$

$$+\left(\frac{e}{c}\frac{\partial A_2}{\partial t}\right)_{ref} + \kappa\left(\frac{e}{c}\frac{\partial A_1}{\partial x}y - \frac{e}{c}\frac{\partial A_3}{\partial x}x\right)_{ref} = \frac{eE_o}{p_o c^2}\left.\frac{\partial\varphi}{\partial t}\right|_{ref} \qquad (126)s$$

$$\frac{dH_o(s)}{ds} = \left.\frac{\partial h^*}{dt}\right|_{ref} = \frac{eE_o}{p_o c^2}\left.\frac{\partial\varphi}{\partial t}\right|_{ref};$$

which can be transferred using $H = E + e\varphi$ and $d\varphi_o(s,t_o(s)) = \frac{\partial\varphi_o}{\partial s}ds + \frac{\partial\varphi_o}{\partial t}\frac{ds}{\mathrm{v}_o(s)}$

into the energy gain of the reference particle along is trajectory:

$$\frac{dE_o(s)}{ds} = \frac{d\left(H_o(s) - \varphi_o(s,t_o(s))\right)}{ds} = -e\left.\frac{\partial\varphi}{\partial s}\right|_{ref} \equiv eE_2(s,t_o(s)). \qquad (127)$$

As discussed before, accelerator designers face the problem of ensuring that the reference particle faithfully follows the reference trajectory. Our goal is to use the above conditions to the maximum, and, as we see below, to eliminate zero- order terms from the equations of motion. By selecting the reference trajectory as basis for our coordinate system, we set the transverse coordinates and momenta at zero at the reference orbit. Hence, two canonical pairs have a good and solid origin.

The third pair ***(-t,H)*** is odd; it is not zero for the reference particle. Furthermore, it has different units. Hence, we can move step forward with a more natural Canonical pair $\{q_\tau = -ct, p_\tau = H/c\}$ - whose generating function is obvious: $\Phi\left(q = -t, \tilde{P} = p_\tau\right) = -ct \cdot p_\tau$. In this case, the analogy is complete: $q_\tau = -ct$ has the dimension of distance and is just $-\boldsymbol{x_o}$ in 4D space, while $p_\tau = H/c$ has the dimension of momentum and is just $\boldsymbol{P_o}$ in 4D space.

We also should select variables that are zero at the reference orbit. The following pair is one of better choices:

$$\left\{\tau = -c(t - t_o(s)),\ \delta = \left(H - E_o(s) - e\varphi_o(s,t)\right)/c\right\}, \tag{128}$$

which are zero for the reference particle. Generation function is easily to come with:

$$\Phi(q,\tilde{P},s) = \tilde{P}_1 x + \tilde{P}_3 y - \left(E_o(s) + c\delta\right)\left(t - t_o(s)\right) - e\int^t \varphi_o(s,t_1)dt_1, \tag{129}$$

and it produces what is desired:

$$P_1 = \frac{\partial\Phi}{\partial x} = \tilde{P}_1;\ P_3 = \frac{\partial\Phi}{\partial y} = \tilde{P}_3; H = \frac{\partial\Phi}{\partial(-t)} = E_o + c\delta + e\varphi_o(s,t);$$

$$\tilde{q}_1 = \frac{\partial\Phi}{\partial\tilde{P}_1} = x;\ \ \tilde{q}_3 = \frac{\partial\Phi}{\partial\tilde{P}_3} = y;\ \tilde{q}_\delta = \frac{\partial\Phi}{\partial\delta} = -c\left(t - t_o(s)\right) = \tau$$

$$\tilde{h} = h + \frac{\partial\Phi}{\partial s} = h + \frac{E_o(s) + c\delta}{\mathrm{v}_o(s)} + E_o'(s)\tau/c - e\int^t \varphi_o'(s,t_1)dt_1$$

The change to the Hamiltonian comprised only of meaningful terms as well as just a trivial function of ***s***, ***g(s):***

$$\frac{\partial \Phi}{\partial s} = \frac{c}{\mathrm{v}_o(s)}\delta - e\varphi_{//}(s,\tau) + g(s);$$

$$g(s) = E_o(s)/\mathrm{v}_o(s) - e\int^{t_o(s)} \varphi'_o(s,t_1)dt_1$$

$$\varphi_{//}(s,\tau) =_{def} \frac{\partial}{\partial s}\int_0^{-\tau/c}\left(\varphi_o(s,t_o(s)+\zeta) - \varphi_o(s,t_o(s))\right)d\zeta \equiv \tag{131}$$

$$-\int_0^{-\tau/c}\left(E_2(s,t_o(s)+\zeta) - E_2\big|_{ref}\right)d\zeta$$

where we used eq. (127) as $E'_o(s) = -e\left.\frac{\partial \varphi}{\partial s}\right|_{ref}$. Additive ***g(s)*** simply can be dropped from the Hamiltonian - it does not change equations of motion.

Now the only remaining task is to express the new Hamiltonian function with an updated canonical pair (130) and (115):

$$\tilde{h} = -(1+Kx)\sqrt{p_o{}^2 + \frac{2E_o}{c}\left(\delta - \frac{e}{c}\varphi_\perp\right) + \left(\delta - \frac{e}{c}\varphi_\perp\right)^2 - \left(P_1 - \frac{e}{c}A_1\right)^2 - \left(P_3 - \frac{e}{c}A_3\right)^2} + \tag{132}$$
$$+ \frac{e}{c}A_2 + \kappa x\left(P_3 - \frac{e}{c}A_3\right) - \kappa y\left(P_1 - \frac{e}{c}A_1\right) + \frac{c}{\mathrm{v}_o}\delta - \frac{e}{c}\varphi_{//}(s,\tau)$$

where we used following trivial expansion and definition:

$$\frac{\left(E_o + c\delta + e\varphi_o(s,t) - e\varphi\right)^2}{c^2} - m^2c^2 = p_o{}^2 + \frac{2E_o}{c}\left(\delta - \frac{e}{c}\varphi_\perp\right) + \left(\delta - \frac{e}{c}\varphi_\perp\right)^2;$$

$$\varphi_{\perp def} = \varphi(s,x,y,t) - \varphi_o(s,t) \equiv \varphi(s,x,y,t) - \varphi(s,0,0,t)$$

Scaling variables.

Frequently, it is useful to scale one of canonical variables. Typical scaling in accelerator physics involves dividing the canonical momenta P_1, P_3, δ by the momentum of the reference particle:

$$\pi_1 = \frac{P_1}{p_o};\ \pi_3 = \frac{P_3}{p_o};\quad \pi_o = \frac{\delta}{p_o}. \tag{134}$$

These variables are dimensionless and also are close to $x', y',\ \delta E / p_o c$ for small deviations. Such scaling only is allowed in Hamiltonian mechanics when the scaling parameter is constant, i.e., is not function of s. Thus, scaling by the particle's momentum remains within the framework of Hamiltonian mechanics only if the reference particle's momentum is constant, that is, when the longitudinal electric field is zero along the reference particle's trajectory (i.e. at moment $t=t_o$(s)). One similarly can scale the coordinates by a constant.

$$\xi_1 = \frac{x}{L};\ \xi_3 = \frac{y}{L};\quad \xi_o = \frac{\tau}{L}.$$

Scaling by a constant is easy; divide the Hamiltonian by the constant and rename the variables. Hence, transforming (134) with constant, called p_o, will make Hamiltonian (132) into

$$\tilde{h} = -(1+Kx)\sqrt{1+\frac{2E_o}{p_o c}\left(\delta - \frac{e}{p_o c}\varphi_\perp\right) + \left(\delta - \frac{e}{p_o c}\varphi_\perp\right)^2 - \left(\pi_1 - \frac{e}{p_o c}A_1\right)^2 - \left(\pi_3 - \frac{e}{p_o c}A_3\right)^2} +$$
$$+ \frac{e}{p_o c}A_2 + \kappa x\left(\pi_3 - \frac{e}{p_o c}A_3\right) - \kappa y\left(\pi_1 - \frac{e}{p_o c}A_1\right) + \frac{c}{v_o}\delta - \frac{e}{p_o c}\varphi_{//}(s,\tau)$$

(132 @ **constant energy**)

Usage of this Hamiltonian is very popular for storage rings or transport channels, wherein the energy of the particles remains constant in time. It should not be employed for particles undergoing an acceleration.

Expanding the Hamiltonian.

Expanding the Hamiltonian (132) is a nominal tool in accelerator physics that allows the separation of the effects of various orders and sometimes the use of perturbation-theory approaches. Having completed the process of creating canonical variables, which are zero for the reference particle, the next step is to assume (which is true for operational accelerators) that the relative deviations of momenta are small

$$\left|\frac{P_1}{p_o}\right| << 1; \left|\frac{P_3}{p_o}\right| << 1; \left|\frac{\delta}{p_o}\right| << 1;$$

and that the EM fields are sufficiently smooth around the reference trajectory to allow expansion in terms of x; y; τ. We will consider* all six variables to be of the same order (of infinitesimally, α). We call the order of expansion to be the maximum total power in a product that is any combination of $x, y, \tau, P_1, P_2, \delta$. Unless there is a good reason not to do so, we truncate the series using this rule.

*Sometimes, one can keep explicit the time dependence of fields and expand only the rest of the variables. One such case is an approximate, and useful, description of synchrotron oscillation.

The general expansion of Hamiltonian (132) can be accomplished via the already derived expansion for 4-potential and the well-known expansion of the square root:

$$\tilde{h} = -(1+Kx)\,p_o \sqrt{1+\frac{2E_o}{p_o c}\left(\frac{\delta}{p_o}-\frac{e}{p_o c}\varphi_\perp\right)+\left(\frac{\delta}{p_o}-\frac{e}{p_o c}\varphi_\perp\right)^2 - \left(\frac{P_1}{p_o}-\frac{e}{p_o c}A_1\right)^2 - \left(\frac{P_3}{p_o}-\frac{e}{p_o c}A_3\right)^2} + \tag{135}$$

$$+\frac{e}{c}A_2+\kappa x\left(P_3-\frac{e}{c}A_3\right)-\kappa y\left(P_1-\frac{e}{c}A_1\right)+\frac{c}{v_o}\delta-\frac{e}{c}\varphi_{//}(s,\tau)$$

$$\sqrt{1+g}=1+\sum_{n=1}^{\infty}(-1)^{n-1}\frac{g^n}{2^n}\frac{(2n-3)!!)}{n!}=1+\frac{g}{2}+O\left(g^2\right).$$

Let us discuss a few general issues first. If we expand (134) into an finite series

$$\tilde{h} = -p_o(s) + \sum_{\nu=1}^{N} \sum_{\substack{i+j+k+ \\ l+m+n=\nu}} C_{\text{ijklnm}} x^i y^j \tau^k P_1^l P_3^m \delta^n + O(\alpha^{N+1}) =$$

$$-p_o + \sum_{\nu=1}^{N} \sum_{\sum_{i=1,6} p_i = \nu} C_{\nu\{p_i\}} \prod_{i=1}^{6} x_i^{p_i} + O(\alpha^{N+1}) \quad ; \qquad (136)$$

where we introduce our phase-space vector

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} \equiv \begin{bmatrix} x \\ P_1 \\ y \\ P_2 \\ \tau \\ \delta \end{bmatrix}; \qquad (137)$$

which we will use in its 2n-dimensional phase-space appearance.

$$X^T = \begin{bmatrix} q^1 & P_1 & ..... & .... & q^n & P_n \end{bmatrix} = \begin{bmatrix} x_1 & x_2 & ..... & .... & x_{2n-1} & x_{2n} \end{bmatrix}, \quad (138)$$

where ^{T} stands for "transposed". Using this notion, Hamiltonian equations can be written as one

$$\frac{dX}{ds} = S \cdot \frac{\partial H}{\partial X} \quad \Leftrightarrow \frac{dx_i}{ds} = S_{ij} \cdot \frac{\partial H}{\partial x_j} \equiv \sum_{j=1}^{2n} S_{ij} \cdot \frac{\partial H}{\partial x_j} \quad (139)$$

wherein we introduce matrix S – a generator of the symplectic group (see further). The matrix, S, is asymmetric, with $S_{2m-1,2m} = 1 = -S_{2m,2m-1}$, $m = 1,...,n$, and other elements are zero.

In matrix form **S** has n diagonal blocks with a 2x2 matrix σ, and the rest is the field of zeros:

$$S = \begin{vmatrix} \sigma & 0 & .... & 0 \\ 0 & \sigma & .... & 0 \\ .... & .... & .... & .... \\ .... & .... & .... & \sigma \end{vmatrix}_{2nx2n} \quad ; \quad \sigma = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad (140)$$

This is one of the very important objects in Hamiltonian mechanics.

Using expansion (136), we would dispose of p_o (s) as valueless, and look initially at the first-order terms in the expansion:

$$\tilde{h} = \sum_{i=1.6} C_{1i} x_i + O(\alpha^2); \tag{136-0}$$

That should give us the zero- order terms in the equation of motion:

$$\frac{dx_k}{ds} = \sum_{j=1}^{2n} S_{kj} \cdot C_{1j} + O(\alpha)$$

By design of our variables (section 2.2) $\left.\frac{dx_k}{ds}\right|_{ref} = 0$, i.e. all zero order terms are equal to zero. Hence, the expansion of our Hamiltonian does not have first-order terms:

$$h = \sum_{\nu=2}^{\mathrm{N}} \sum_{\substack{\sum p_i = \nu \\ i=1,6}} C_{\nu\{p_i\}} \prod_{i=1}^{6} x_i^{p_i} + O(\alpha^{N+1}); \tag{141}$$

Next, we look at the second-order term that is of foremost importance in accelerator physics. We can write it as a quadratic form in matrix notations:

$$h = \frac{1}{2} \sum_{i=1}^{6} \sum_{i=1}^{6} h_{ij} x_i x_j + O(\alpha^3) \equiv \frac{1}{2} X^T \cdot H \cdot X + O(\alpha^3); \tag{142}$$

Matrix H can be chosen to be symmetric by a simple $H = (\tilde{H}^T + H)/2$ and noting the obvious $Z^T(\tilde{H}^T - H)Z \equiv 0, \ \forall Z$ equivalence of the asymmetric matrix. Applying (139) with $\partial_{z_i}(\sum h_{jk} x_j x_k) = \left(h_{jk} + h_{kj}\right) x_i = 2 h_{jk} x_i$ the linear part of the equations of motion is obtained:

$$\frac{dX}{ds} = D \cdot X + O(\alpha^2); \ D = S \cdot H \tag{142}$$

We finish this section with the explicit form of the first non-trivial term in the expansion of (135):

$$\tilde{h} = \frac{P_1^2 + P_3^2}{2p_o} + F\frac{x^2}{2} + Nxy + G\frac{y^2}{2} + L\left(xP_3 - yP_1\right) + \frac{\delta^2}{2p_o}\cdot\frac{m^2c^2}{p_o^2} + U\frac{\tau^2}{2} + g_x x\delta + g_y y\delta + F_x x\tau + F_y y\tau \quad ; \tag{143}$$

with

$$\frac{F}{p_o} = \left[-K\cdot\frac{e}{p_o c}B_y - \frac{e}{p_o c}\frac{\partial B_y}{\partial x} + \left(\frac{eB_s}{2p_o c}\right)^2\right] - \frac{e}{p_o \mathrm{v}_o}\frac{\partial E_x}{\partial x} - 2K\frac{eE_x}{p_o \mathrm{v}_o} + \left(\frac{meE_x}{p_o^2}\right)^2;$$

$$\frac{G}{p_o} = \left[\frac{e}{p_o c}\frac{\partial B_x}{\partial y} + \left(\frac{eB_s}{2p_o c}\right)^2\right] - \frac{e}{p_o \mathrm{v}_o}\frac{\partial E_y}{\partial y} + \left(\frac{meE_z}{p_o^2}\right)^2; \quad ;$$

$$\frac{2N}{p_o} = \left[\frac{e}{p_o c}\frac{\partial B_x}{\partial x} - \frac{e}{p_o c}\frac{\partial B_y}{\partial y}\right] - K\cdot\frac{e}{p_o c}B_x - \frac{e}{p_o \mathrm{v}_o}\left(\frac{\partial E_x}{\partial y} + \frac{\partial E_y}{\partial x}\right) - 2K\frac{eE_y}{p_o \mathrm{v}_o} + \left(\frac{meE_z}{p_o^2}\right)\left(\frac{meE_x}{p_o^2}\right) \tag{144}$$

$$L = \kappa + \frac{e}{2p_o c}B_s; \qquad \frac{U}{p_o} = \frac{e}{pc^2}\frac{\partial E_s}{\partial t}; \quad g_x = \frac{(mc)^2\cdot eE_x}{p_o^3} - K\frac{c}{\mathrm{v}_o}; \; g_y = \frac{(mc)^2\cdot eE_y}{p_o^3};$$

$$F_x = \frac{e}{c}\frac{\partial B_y}{\partial ct} + \frac{e}{\mathrm{v}_o}\frac{\partial E_x}{\partial ct}; F_y = -\frac{e}{c}\frac{\partial B_x}{\partial ct} + \frac{e}{\mathrm{v}_o}\frac{\partial E_y}{\partial ct}.$$

If momentum p_o is constant, we can use (134) and rewrite Hamiltonian of the linearized motion (143) as

$$\tilde{h}_n = \frac{{\pi_1}^2 + {\pi_3}^2}{2} + f\frac{x^2}{2} + n\cdot xy + g\frac{y^2}{2} + L\left(x\pi_3 - y\pi_1\right) + \frac{{\pi_o}^2}{2}\cdot\frac{m^2c^2}{{p_o}^2} + u\frac{\tau^2}{2} + g_x x\pi_o + g_y y\pi_o + f_x x\tau + f_y y\tau \quad ; \qquad (144\text{-n})$$

with

$$f = \frac{F}{p_o};\ n = \frac{N}{p_o};\ g = \frac{G}{p_o};\ u = \frac{U}{p_o};\ f_x = \frac{F_x}{p_o};\ f_y = \frac{F_y}{p_o}; \qquad (145\text{-n})$$

Note that

$$x' = \frac{\partial h_n}{\partial \pi_1} = \pi_1 - Ly; \quad y' = \frac{\partial h_n}{\partial \pi_3} = \pi_3 + Lx; \quad ; \qquad (146)$$

i.e. as soon as L=0, we can use traditional x' and y' as reduced momenta.

For a flat reference orbit - $\kappa = 0$, in the absence of transverse coupling (L=0, N=0) and transverse electric fields, the accelerator Hamiltonian has the form which is used in most of the text books and papers:

$$\tilde{h} = \frac{P_1^2 + P_3^2}{2p_o} + F\frac{x^2}{2} + G\frac{y^2}{2} + \frac{\delta^2}{2p_o}\cdot\frac{m^2c^2}{p_o^2} + U\frac{\tau^2}{2} + g_x x\delta$$

or ; (146)

$$\tilde{h}_n = \frac{x'^2 + y'^2}{2p_o} + f\frac{x^2}{2} + g\frac{y^2}{2} + \frac{\pi_o^2}{2}\cdot\frac{m^2c^2}{p_o^2} + u\frac{\tau^2}{2} + g_x x\delta$$

with

$$f = \left[-K\cdot\frac{e}{p_o c}B_y - \frac{e}{p_o c}\frac{\partial B_y}{\partial x}\right];\quad g = \frac{e}{p_o c}\frac{\partial B_x}{\partial y};\quad u = \frac{e}{pc^2}\frac{\partial E_s}{\partial t};\quad g_x = -K\frac{c}{v_o}. \qquad (147)$$

Finally, see an Additional Materials, where Mathematica used for expanding the Accelerator Hamiltonian to an arbitrary order

What we learned today?

- That distance along the reference particle trajectory, *s*, is a natural choice of independent coordinate in accelerator physics
  - Remember, magnets and vacuum chambers are bolted to the floor and are not floating in time
- Accelerator coordinate system than determined by *curvilinear* Frenet-Serret coordinates, e.g. we can not use just simple Cartesian coordinates in most of the cases
- With *s* as independent variable, $-P_2$ becomes the accelerator Hamiltonian with (x,P_1), (y,P_3) and $(t,-H)$ being Canonical pairs
- The Hamiltonian can be expanded to any arbitrary order "about" reference particle's trajectory, momentum/energy and "arriving time" to azimuth *s*
- The condition for reference particle remove first order terms in the Hamiltonian expansion
- Second order is the lowest remaining term in the Hamiltonian.
- It plays fundamental role in accelerator physics since leads to a set of linear s-dependent ordinary differential equation – subject for next class