PHY 564
Advanced Accelerator Physics
Lectures 16

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Lecture 16. Parameterization of linearized particle's motion. Action-angle variables.

Because the parameterization and the action-angle are very useful tools for solving many standard accelerator problems, we will go through steps in a logical manner: from simple to more complicated cases.

Let's start from a simple 1D oscillator with Hamiltonian:

$$H = \frac{p^2}{2m} + k\frac{x^2}{2};\ \mathbf{H} = \begin{bmatrix} \frac{1}{m} & 0 \\ 0 & k \end{bmatrix}; \mathbf{D} = \begin{bmatrix} 0 & \frac{1}{m} \\ -k & 0 \end{bmatrix} \tag{1}$$

in our standard matrix form. The equations of motion are:

$$X = \begin{bmatrix} x \\ p_x \end{bmatrix}; \frac{dX}{dt} = \mathbf{D}\cdot X \Leftrightarrow \frac{d^2x}{dt^2} + \frac{k}{m}x = 0$$
$$\lambda = i\omega \to -\lambda^2 = \omega^2 = \det \mathbf{D} = \frac{k}{m} \tag{2}$$

This is something you did studies many times before. The stability criteria is

$$k > 0$$

and general solution is

$$\mathbf{M}(t) = e^{\mathbf{D}t} = e^{i\omega t}\frac{i\omega I + D}{2i\omega} - e^{-i\omega t}\frac{i\omega I - D}{2i\omega} = I\cos\omega t + \frac{D}{\omega}\sin\omega t = \begin{bmatrix} \cos\omega t & \frac{\sin\omega t}{m\omega} \\ -\frac{k}{\omega}\sin\omega t & \cos\omega t \end{bmatrix}$$

$$X(t) = \begin{bmatrix} x \\ p_x \end{bmatrix} = \begin{bmatrix} \cos\omega t & \frac{\sin\omega t}{m\omega} \\ -\frac{k}{\omega}\sin\omega t & \cos\omega t \end{bmatrix} \begin{bmatrix} x_o \\ p_{xo} \end{bmatrix}; \frac{k}{\omega} = m\omega = \sqrt{mk} \tag{3}$$
$$x = x_o\cos\omega t + \frac{\sin\omega t}{m\omega}p_{xo};\ \ p_x = p_{xo}\cos\omega t - x_o\frac{k}{\omega}\sin\omega t.$$

We know another form of these equations of motions:

$$x = A\cos(\omega t + \varphi); \quad p_x = -m\omega \cdot A \cdot \sin(\omega t + \varphi) \tag{4}$$

which is a consequence of

$$\mathbf{M}Y = e^{i\omega t}Y; \; Y = \begin{bmatrix} w \\ \dfrac{i}{w} \end{bmatrix}; w = \sqrt{m\omega}; \; X(t) = a\,\mathrm{Re}\,Ye^{i\omega t + \varphi} = \begin{bmatrix} \dfrac{a}{\sqrt{m\omega}}\cos(\omega t + \varphi) \\ -a\sqrt{m\omega}\sin(\omega t + \varphi) \end{bmatrix}; \tag{5}$$

which coincides with (4) using $A = a\sqrt{m\omega}$. Naturally, we know that amplitude and phase are constant of motion.

Let's make a Canonical transformation to $\left\{\tilde{q} = \varphi, \tilde{p} = \dfrac{a^2}{2}\right\}$ using

$$\begin{aligned} &F(q = x, \tilde{q} = \varphi) = -m\omega\frac{x^2}{2}\tan(\omega t + \varphi); \\ &\frac{\partial F}{\partial x} = -m\omega x \tan(\omega t + \varphi) = -m\omega\frac{a}{\sqrt{m\omega}}\sin(\omega t + \varphi) = p_x; \\ &\tilde{p} = -\frac{\partial F}{\partial \varphi} = m\omega\frac{x^2}{2\cos^2(\omega t + \varphi)} = \frac{a^2}{2} = I \end{aligned} \tag{6}$$

Finally, the reduced Hamiltonian is

$$\tilde{H} = H + \frac{\partial F}{\partial t} = \frac{p^2}{2m} + k\frac{x^2}{2} - m\omega^2 \frac{x^2}{2\cos^2(\omega t + \varphi)} =$$
$$\frac{a^2}{2}\left(\frac{k}{m\omega}\cos^2(\omega t + \varphi) + \omega \sin(\omega t + \varphi)\right) - \omega\frac{a^2}{2} = 0 \tag{7}$$

Thus, if

$$H = \frac{p^2}{2m} + k\frac{x^2}{2} + H_1(x, p) \rightarrow \tilde{H}(\varphi, I) = H - \frac{p^2}{2m} + k\frac{x^2}{2} = H_1(x(I,\varphi), p(I,\varphi))$$
$$\frac{d\varphi}{dt} = \frac{\partial \tilde{H}}{\partial I}; \frac{dI}{dt} = -\frac{\partial \tilde{H}}{\partial \varphi}. \tag{8}$$

General parameterization of motion of an arbitrary linear Hamiltonian system:

$$H = \frac{1}{2}\sum_{i=1}^{2n}\sum_{i=1}^{2n} h_{ij}(s)x_i x_j \equiv \frac{1}{2}X^T \cdot \mathbf{H}(s) \cdot X,$$

$$X^T = \left[q^1 \quad P_1 \quad ..... \quad .... \quad q^n \quad P_n\right] = \left[x_1 \quad x_2 \quad ..... \quad .... \quad x_{2n-1} \quad x_{2n}\right].$$

$$\frac{dX}{ds} = \mathbf{D}(s)\cdot X;\ \mathbf{D} = \mathbf{S}\cdot\mathbf{H}(s),\ X(s) = \mathbf{M}(s_o|s)\cdot X_o. \qquad \backslash \qquad \text{(I)}$$

$$\mathbf{M}' \equiv \frac{d\mathbf{M}}{ds} = \mathbf{D}(s)\cdot\mathbf{M};\ \ \mathbf{M}(s_o) = \mathbf{I},\ \mathbf{M}^{\mathbf{T}}\cdot\mathbf{S}\cdot\mathbf{M} = \mathbf{S},\ \ \mathbf{M}^{-1} = -\mathbf{S}\cdot\mathbf{M}^{\mathbf{T}}\cdot\mathbf{S}. \qquad \text{(II)}$$

Parameterization of motion: a periodic system with period C $\mathbf{H}(s+C) = \mathbf{H}(s)$

$$\mathbf{T}(s) = \mathbf{M}(s|s+C),\ \det[\mathbf{T} - \lambda_i\cdot\mathbf{I}] = 0 \qquad \text{(III)}$$

Stable system $|\lambda_i| = 1.\ \lambda_k \equiv 1/\lambda_{k+n} \equiv \lambda^*_{k+n} \equiv e^{i\mu_k};\ \mu_k \equiv 2\pi\nu_k,\ \{k = 1,...n\}.$ (IV)

We proved that

$$\tilde{Y}_k(s) = Y_k(s) e^{\psi_k(s)}; \quad Y_k(s+C) = Y_k(s); \qquad \psi_k(s+C) = \psi_k(s) + \mu_k$$

$$\tilde{\mathbf{U}} = \left[\ldots \tilde{Y}_k, \tilde{Y}^*_k .. \right]; \ \tilde{\mathbf{U}}(s) = \mathbf{U}(s) \cdot \Psi(s), \ \Psi(s) = \begin{pmatrix} e^{i\psi_1(s)} & 0 & & 0 \\ 0 & e^{-i\psi_1(s)} & & 0 \\ & & \ldots & 0 \\ 0 & 0 & 0 & e^{-i\psi_n(s)} \end{pmatrix}$$

$$X = \frac{1}{2} \sum_{k=1}^{n} \left(a_k \tilde{Y}_k + a^*_k \tilde{Y}^*_k \right) = \mathrm{Re} \sum_{k=1}^{n} a_k \tilde{Y}_k = \frac{1}{2} \tilde{\mathbf{U}} A = \frac{1}{2} \mathbf{U} \Psi A = \frac{1}{2} \mathbf{U} \tilde{A}$$

$$a_i = \frac{1}{i} Y_i^{*T} S X; \ \tilde{a}_i \equiv a_i e_i^{i\psi} = \frac{1}{i} Y_i^{*T} S X;$$

$$A = 2 \tilde{\mathbf{U}}^{-1} \cdot X = -i \Psi^{-1} \cdot \mathbf{S} \cdot \mathbf{U}^{T*} \cdot \mathbf{S} \cdot X; \ \tilde{A} = \Psi A = -i \cdot \mathbf{S} \cdot \mathbf{U}^{T*} \cdot \mathbf{S} \cdot X.$$

$$\mathbf{T}(s)=\mathbf{U}(s)\Lambda\mathbf{U}^{-1}(s);\ \Lambda=\begin{pmatrix}\lambda_1 & 0 & & 0\\ 0 & \lambda_1^* & & 0\\ & & ... & 0\\ 0 & 0 & 0 & \lambda_n^*\end{pmatrix};\ \ \mathbf{T}\cdot\mathbf{U}=\mathbf{U}\cdot\Lambda;\ \Lambda=\mathbf{U}^{-1}\cdot\mathbf{T}\cdot\mathbf{U}.... \qquad \text{(V)}$$

$$\mathbf{U}(s)=\left[Y_1,Y_1^*....Y_n,Y_n^*\right];\qquad \mathbf{T}(s)Y_k(s)=\lambda_k Y_k(s)\quad\Leftrightarrow\quad \mathbf{T}(s)Y_k^*(s)=\lambda_k^* Y_k^*(s) \qquad \text{(VI)}$$

$$\tilde{Y}_k(s_1)=\mathbf{M}\left(s|s_1\right)\tilde{Y}_k(s)\ \Leftrightarrow\ \frac{d}{ds}\tilde{Y}_k=\mathbf{D}\left(s\right)\cdot\tilde{Y}_k;\qquad \tilde{\mathbf{U}}(s_1)=\mathbf{M}\left(s|s_1\right)\tilde{\mathbf{U}}(s)\ \Leftrightarrow\ \frac{d}{ds}\tilde{\mathbf{U}}=\mathbf{D}\left(s\right)\cdot\tilde{\mathbf{U}} \qquad \text{(VII)}$$

$$\tilde{\mathbf{U}}(s+C)=\tilde{\mathbf{U}}(s)\cdot\Lambda,\ \tilde{Y}_k(s+C)=\lambda_k\tilde{Y}_k(s)=e^{i\mu_k}\tilde{Y}_k(s) \qquad \text{(VIII)}$$

$$\tilde{Y}_k(s)=Y_k(s)e^{\psi_k(s)};\quad Y_k(s+C)=Y_k(s);\qquad \psi_k\left(s+C\right)=\psi_k\left(s\right)+\mu_k \qquad \text{(IX)}$$

$$\tilde{\mathbf{U}}(s)=\mathbf{U}(s)\cdot\Psi(s),\ \Psi(s)=\begin{pmatrix}e^{i\psi_1(s)} & 0 & & 0\\ 0 & e^{-i\psi_1(s)} & & 0\\ & & ... & 0\\ 0 & 0 & 0 & e^{-i\psi_n(s)}\end{pmatrix} \qquad \text{(X)}$$

$$Y_k^{T*}\cdot\mathbf{S}\cdot Y_{j\neq k}=0;\quad Y_k^{T}\cdot\mathbf{S}\cdot Y_j=0;\quad Y_k^{T*}\cdot\mathbf{S}\cdot Y_k=2i \qquad \text{(XI)}$$

$$\mathbf{U}^T\cdot\mathbf{S}\cdot\mathbf{U}\equiv\tilde{\mathbf{U}}^T\cdot\mathbf{S}\cdot\tilde{\mathbf{U}}=-2i\mathbf{S},\ \ \mathbf{U}^{-1}=\frac{1}{2i}\mathbf{S}\cdot\mathbf{U}^T\cdot\mathbf{S} \qquad \text{(XII)}$$

$$X(s)=\frac{1}{2}\sum_{k=1}^{n}\left(a_k\tilde{Y}_k+a_k{}^*\tilde{Y}_k{}^*\right)\equiv\operatorname{Re}\sum_{k=1}^{n}a_kY_ke^{i\psi_k}\equiv\frac{1}{2}\tilde{\mathbf{U}}\cdot A=\frac{1}{2}\mathbf{U}\cdot\boldsymbol{\Psi}\cdot A=\frac{1}{2}\mathbf{U}\cdot\tilde{A} \qquad \text{(XIII)}$$

$$a_i=\frac{1}{2i}Y_i^{*T}SX;\ \tilde{a}_i\equiv a_ie^{i\psi}{}_i=\frac{1}{2i}Y_i^{*T}SX;$$

$$A=2\tilde{\mathbf{U}}^{-1}\cdot X=-i\boldsymbol{\Psi}^{-1}\cdot\mathbf{S}\cdot\mathbf{U}^{T*}\cdot\mathbf{S}\cdot X;\ \tilde{A}=\Psi A=-i\cdot\mathbf{S}\cdot\mathbf{U}^{T*}\cdot\mathbf{S}\cdot X. \qquad \text{(XIV)}$$

and finished with fact that this parameterization is equivalent to Canonical transformation to action-angle variables.

$$\left\{\varphi_k, I_k = \frac{a_k^2}{2}\right\}; k = 1,2..,n$$

Than if $\mathcal{H} = \frac{1}{2} X^T \mathbf{H}(s) X + \mathcal{H}_1(X,s)$

$$\tilde{H}(\varphi_k, I_k, s) = \mathcal{H}_1\left(X(\varphi_k, I_k, s), s\right);$$

$$\frac{d\varphi_k}{ds} = \frac{\partial \tilde{H}}{\partial I_k}; \frac{dI_k}{ds} = -\frac{\partial \tilde{H}}{\partial \varphi_k}.$$

1D – ACCELERATOR

$$\tilde{h} = \frac{p^2}{2} + K_1(s)\frac{y^2}{2};\ \mathbf{H} = \begin{bmatrix} K_1 & 0 \\ 0 & 1 \end{bmatrix};\ \mathbf{D} = \mathbf{SH} = \begin{bmatrix} 0 & 1 \\ -K_1 & 0 \end{bmatrix};$$

$$\frac{d}{ds}\begin{bmatrix} x \\ p \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -K_1 & 0 \end{bmatrix} \cdot \begin{bmatrix} x \\ p \end{bmatrix} = \begin{bmatrix} p \\ -K_1 x \end{bmatrix}\ (i.e.\ x' \equiv p).$$

$$Y = \begin{bmatrix} \mathrm{w} \\ \mathrm{w}' + i/\mathrm{w} \end{bmatrix};\ \psi' = \frac{1}{\mathrm{w}^2};\tilde{Y} = Ye^{i\psi}$$

$$\mathbf{T}(s) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \mathbf{U}(s)\Lambda\mathbf{U}^{-1}(s);\ \Lambda = \begin{pmatrix} \lambda & 0 \\ 0 & 1/\lambda \end{pmatrix} = \begin{pmatrix} e^{i\mu} & 0 \\ 0 & e^{-i\mu} \end{pmatrix};$$

$$\mathbf{T} = \mathbf{I}\cos\mu + \mathbf{J}\sin\mu;$$

$$\mathbf{J} = \begin{bmatrix} -ww' & w^2 \\ -w'^2 - \frac{1}{w^2} & ww' \end{bmatrix} = \begin{bmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{bmatrix};\quad \mathbf{J}^2 = -\mathbf{I}$$

Note: for not normalized Hamiltonian $H = \frac{p_y^2}{2p_o} + p_o K_1(s)\frac{y^2}{2}$ and

$$Y = \begin{bmatrix} \dfrac{w}{\sqrt{p_o}} \\ \sqrt{p_o}\left(w' + i/w\right) \end{bmatrix}; \psi' = \frac{1}{w^2}; \tilde{Y} = Ye^{i\psi}$$

$$\mathbf{T} = \mathbf{I}\cos\mu + \mathbf{J}\sin\mu;$$

$$\mathbf{J} = \begin{bmatrix} -ww' & w^2 \\ -w'^2 - \dfrac{1}{w^2} & ww' \end{bmatrix} = \begin{bmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{bmatrix}; \quad \mathbf{J}^2 = -\mathbf{I}; \quad \gamma = \left(1+\alpha^2\right)/\beta$$

$$\cos\mu = Trace(\mathbf{T})/2 = \frac{T_{11}+T_{22}}{2}$$

$$Stability\ if: -1 < Trace(\mathbf{T})/2 < 1$$

$$w^2 \equiv \beta = \frac{T_{12}}{\sin\mu} = \frac{\left|T_{12}\right|}{\sqrt{1-\left(Trace(\mathbf{T})/2\right)^2}}; w = \sqrt{\frac{\left|T_{12}\right|}{\sqrt{1-\left(Trace(\mathbf{T})/2\right)^2}}};$$

$$ww' \equiv -\alpha = \frac{T_{22}-T_{11}}{2\cos\mu} = -\frac{T_{11}-T_{22}}{T_{11}+T_{22}}$$

1D - ACCELERATOR

$$Y=\begin{bmatrix} \mathrm{w} \\ \mathrm{w}'+i/\mathrm{w} \end{bmatrix};\ \tilde{Y}=\begin{bmatrix} \mathrm{w} \\ \mathrm{w}'+i/\mathrm{w} \end{bmatrix}e^{i\psi};\ \mathbf{U}=\begin{bmatrix} \mathrm{w} & \mathrm{w} \\ \mathrm{w}'+i/\mathrm{w} & \mathrm{w}'-i/\mathrm{w} \end{bmatrix};\tilde{\mathbf{U}}=\mathbf{U}\cdot\begin{pmatrix} e^{i\psi} & 0 \\ 0 & e^{-i\psi} \end{pmatrix}$$

$$\mathrm{w}''+K_1(\mathrm{s})\mathrm{w}=\frac{1}{\mathrm{w}^3},\quad \psi'=1/\mathrm{w}^2;\ \begin{bmatrix} x \\ x' \end{bmatrix}=\mathrm{Re}\left(ae^{i\varphi}\begin{bmatrix} \mathrm{w} \\ \mathrm{w}'+i/\mathrm{w} \end{bmatrix}e^{i\psi}\right);$$

$$x=a\cdot\mathrm{w}(\mathrm{s})\cdot\cos(\psi(s)+\varphi)$$

$$x'=a\cdot\left(\mathrm{w}'(\mathrm{s})\cdot\cos(\psi(s)+\varphi)-sin(\psi(s)+\varphi)/\mathrm{w}(\mathrm{s})\right)$$

$$\beta\equiv\mathrm{w}^2\ \Rightarrow\ \psi'=1/\beta;\alpha\equiv-\beta'\equiv-\mathrm{w}\,\mathrm{w}',\ \gamma\equiv\frac{1+\alpha^2}{\beta}\ \text{- definitions}$$

$$x=a\cdot\sqrt{\beta(\mathrm{s})}\cdot\cos(\psi(s)+\varphi)$$

$$x'=-\frac{a}{\sqrt{\beta(\mathrm{s})}}\cdot\left(\alpha(\mathrm{s})\cdot\cos(\psi(s)+\varphi)+sin(\psi(s)+\varphi)\right)$$

Complex amplitude and real amplitude and phase are easy to calculate. Expression for a^2 is called Currant-Snyder invariant.

Thus, the final form of the eigen vector can be rewritten as

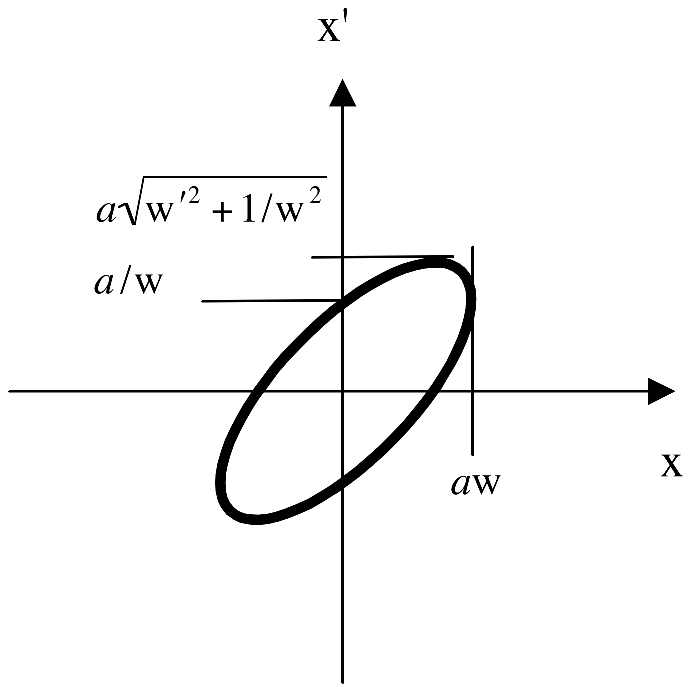
$$Y = \begin{bmatrix} \mathrm{w} \\ \mathrm{w}' + i/\mathrm{w} \end{bmatrix}; \psi' = \frac{1}{\mathrm{w}^2}; \tilde{Y} = Ye^{i\psi} \tag{34}$$

The parameterization of the linear 1D motion is

$$\begin{bmatrix} x \\ x' \end{bmatrix} = \mathrm{Re}\left(ae^{i\varphi} \begin{bmatrix} \mathrm{w} \\ \mathrm{w}' + i/\mathrm{w} \end{bmatrix} e^{i\psi} \right);$$

$$\begin{aligned} x &= a \cdot \mathrm{w(s)} \cdot \cos(\psi(s) + \varphi) \\ x' &= a \cdot (\mathrm{w}'(\mathrm{s}) \cdot \cos(\psi(s) + \varphi) - sin(\psi(s) + \varphi)/\mathrm{w(s)}) \end{aligned} \tag{35}$$

where $\boldsymbol{a}$ and $\boldsymbol{\varphi}$ are the constants of motion.



$$X = \mathrm{Re}\,\tilde{a}Y;$$

$$ae^{i\varphi} = -\mathrm{i}Y^{*T}SX = \begin{bmatrix} \mathrm{w} \\ \mathrm{w}' - \mathrm{i}/\mathrm{w} \end{bmatrix}^T \cdot \begin{bmatrix} \mathrm{x}' \\ \text{-x} \end{bmatrix} = \mathrm{x}/\mathrm{w} + \mathrm{i}(\mathrm{w}'\mathrm{x} - \mathrm{w}\mathrm{x}')$$

$$a^2 = \frac{\mathrm{x}^2}{\mathrm{w}^2} + (\mathrm{w}'\mathrm{x} - \mathrm{w}\mathrm{x}')^2 \equiv \frac{\mathrm{x}^2 + (\alpha\mathrm{x} + \beta\mathrm{x}')^2}{\beta} \equiv \gamma\mathrm{x}^2 + 2\alpha\mathrm{x}\mathrm{x}' + \beta\mathrm{x}'^2$$

$$\varphi = \arg\left(\mathrm{x}/\mathrm{w} + \mathrm{i}(\mathrm{w}'\mathrm{x} - \mathrm{w}\mathrm{x}')\right) = \tan^{-1}\frac{\mathrm{w}\mathrm{w}'\mathrm{x} - \mathrm{w}^2\mathrm{x}'}{\mathrm{x}} = -\tan^{-1}\frac{\alpha\mathrm{x} + \beta\mathrm{x}'}{\mathrm{x}};$$

$$\varphi = \sin^{-1}\frac{\mathrm{w}'\mathrm{x} - \mathrm{w}\mathrm{x}'}{\sqrt{\mathrm{x}^2 + \mathrm{w}^2(\mathrm{w}'\mathrm{x} - \mathrm{w}\mathrm{x}')^2}} = -\sin^{-1}\frac{\alpha\mathrm{x} + \beta\mathrm{x}'}{\sqrt{\gamma\mathrm{x}^2 + 2\alpha\mathrm{x}\mathrm{x}' + \beta\mathrm{x}'^2}}$$

Inverse ratios – matrices through parameterization: reverse of eq (VII), where U is propagated by M.

$$\mathbf{M}\left(s_1|s_2\right)=\tilde{\mathbf{U}}(s_2)\tilde{\mathbf{U}}^{-1}(s_1)=\frac{i}{2}\tilde{\mathbf{U}}(s_2)\cdot\mathbf{S}\cdot\tilde{\mathbf{U}}^{T}(s_1)\cdot\mathbf{S}=\frac{i}{2}\mathbf{U}(s_2)\cdot\Psi(s_2)\cdot\mathbf{S}\cdot\Psi^{-1}(s_1)\cdot\mathbf{U}^{T}(s_1)\cdot\mathbf{S}$$

$$\Psi(s_2)\cdot\mathbf{S}\cdot\Psi^{-1}(s_1)\equiv\Psi(s_2-s_1)\cdot\mathbf{S};\quad \mathbf{M}\left(s_1|s_2\right)=\frac{i}{2}\mathbf{U}(s_2)\cdot\Psi(s_2-s_1)\cdot\mathbf{S}\cdot\mathbf{U}^{T}(s_1)\cdot\mathbf{S} \qquad \text{(XV)}$$

$$\mathbf{T}=\mathbf{U}\Lambda\mathbf{U}^{-1}=\frac{i}{2}\mathbf{U}\Lambda\mathbf{S}\mathbf{U}^{T}\mathbf{S}$$

Specific case of $\boldsymbol{s_1=s+C}$

$$\mathbf{T}=\frac{1}{2i}\mathbf{U}\Lambda\mathbf{S}\mathbf{U}^{T}\mathbf{S}=\frac{1}{2i}\begin{bmatrix} w & w \\ w'+\frac{i}{w} & w'-\frac{i}{w}\end{bmatrix}\cdot\begin{bmatrix} e^{i\mu} & 0 \\ 0 & e^{-i\mu}\end{bmatrix}\cdot$$

$$\begin{bmatrix} 0 & 1 \\ -1 & 0\end{bmatrix}\cdot\begin{bmatrix} w & w'+\frac{i}{w} \\ w & w'-\frac{i}{w}\end{bmatrix}\cdot\begin{bmatrix} 0 & 1 \\ -1 & 0\end{bmatrix}=$$

$$\frac{1}{2i}\begin{bmatrix} we^{i\mu} & we^{-i\mu} \\ \left(w'+\frac{i}{w}\right)e^{i\mu} & \left(w'-\frac{i}{w}\right)e^{-i\mu}\end{bmatrix}\cdot\begin{bmatrix} \frac{i}{w}-w' & w \\ \frac{i}{w}+w' & -w\end{bmatrix}=$$

$$\begin{bmatrix} \frac{e^{i\mu}+e^{-i\mu}}{2}-\frac{e^{i\mu}-e^{-i\mu}}{2i}ww' & w^2\frac{e^{i\mu}-e^{-i\mu}}{2i} \\ -\left(w'^2+\frac{1}{w^2}\right)\frac{e^{i\mu}-e^{-i\mu}}{2} & \frac{e^{i\mu}+e^{-i\mu}}{2}+\frac{e^{i\mu}-e^{-i\mu}}{2i}ww'\end{bmatrix}$$

$$\mathbf{T} = \frac{1}{2i}\mathbf{U\Lambda SU}^T\mathbf{S} = \begin{bmatrix} \cos\mu - ww'\cdot\sin\mu & w^2\cdot\sin\mu \\ -\left(w'^2 + \frac{1}{w^2}\right)\cdot\sin\mu & \cos\mu + ww'\cdot\sin\mu \end{bmatrix} = \mathbf{I}\cos\mu + \mathbf{J}\sin\mu;$$

$$\mathbf{J} = \begin{bmatrix} -ww' & w^2 \\ -w'^2 - \frac{1}{w^2} & ww' \end{bmatrix} = \begin{bmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{bmatrix};$$

$$\mathbf{J}^2 = \begin{bmatrix} -ww' & w^2 \\ -\left(w'^2 + \frac{1}{w^2}\right) & ww' \end{bmatrix}\begin{bmatrix} -ww' & w^2 \\ -\left(w'^2 + \frac{1}{w^2}\right) & ww' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -\mathbf{I}$$

$$M(s_1|s_2) = \frac{1}{2i}\begin{bmatrix} \mathrm{w}_2 & \mathrm{w}_2 \\ \mathrm{w}'_2 + i/\mathrm{w}_2 & \mathrm{w}'_2 - i/\mathrm{w}_2 \end{bmatrix} \cdot \begin{pmatrix} e^{i\Delta\psi} & 0 \\ 0 & e^{-i\Delta\psi} \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \cdot \begin{bmatrix} \mathrm{w}_1 & \mathrm{w}'_1 + i/\mathrm{w}_1 \\ \mathrm{w}_1 & \mathrm{w}'_1 - i/\mathrm{w}_1 \end{bmatrix} \cdot \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$\Delta\psi = \psi(s_2) - \psi(s_1);$$

$$M(s_1|s_2) = \frac{1}{2i}\begin{bmatrix} \mathrm{w}_2 & \mathrm{w}_2 \\ \mathrm{w}'_2 + i/\mathrm{w}_2 & \mathrm{w}'_2 - i/\mathrm{w}_2 \end{bmatrix} \cdot \begin{bmatrix} -(\mathrm{w}'_1 - i/\mathrm{w}_1)e^{i\Delta\psi} & \mathrm{w}_1 e^{i\Delta\psi} \\ (\mathrm{w}'_1 + i/\mathrm{w}_1)e^{-i\Delta\psi} & -\mathrm{w}_1 e^{-i\Delta\psi} \end{bmatrix} =$$

$$\begin{bmatrix} \mathrm{w}_2/\mathrm{w}_1 \cos\Delta\psi - \mathrm{w}_2\mathrm{w}'_1 \sin\Delta\psi & \mathrm{w}_1\mathrm{w}_2 \sin\Delta\psi \\ (\mathrm{w}'_2/\mathrm{w}_1 - \mathrm{w}'_1/\mathrm{w}_2)\cos\Delta\psi - (\mathrm{w}'_1\mathrm{w}'_2 + 1/(\mathrm{w}_2\mathrm{w}_1))\sin\Delta\psi & \mathrm{w}_1/\mathrm{w}_2 \cos\Delta\psi + \mathrm{w}_1\mathrm{w}'_2 \sin\Delta\psi \end{bmatrix}$$

use $\mathrm{w} = \sqrt{\beta}$; $\mathrm{w}' = -\alpha/\sqrt{\beta}$ to get a standard

$$M(s_1|s_2) = \begin{bmatrix} \dfrac{\cos\Delta\psi + \alpha_1 \sin\Delta\psi}{\sqrt{\beta_1/\beta_2}} & \sqrt{\beta_1\beta_2}\sin\Delta\psi \\ -\dfrac{(\alpha_2 - \alpha_1)\cos\Delta\psi + (1 + \alpha_1\alpha_2)\sin\Delta\psi}{\sqrt{\beta_1\beta_2}} & \dfrac{\cos\Delta\psi - \alpha_2 \sin\Delta\psi}{\sqrt{\beta_2/\beta_1}} \end{bmatrix}$$

with obvious simplification for one-turn matrix:

$$\mathbf{T} = M(s|s+C) = \begin{bmatrix} t11 & t12 \\ t21 & t22 \end{bmatrix} = \begin{bmatrix} \cos\mu + \alpha\sin\mu & \beta\sin\mu \\ -\dfrac{(1+\alpha^2)\sin\mu}{\beta} & \cos\mu - \alpha\sin\mu \end{bmatrix} \mathbf{I}\cos\mu + \mathbf{J}\sin\mu;\ \mathbf{J} = \begin{bmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{bmatrix}$$

from where one can get easily all functions and constants:

$$\mu = \cos^{-1}\left(\frac{t11 + t22}{2}\right) \textit{with sign}(\sin\mu) = \textit{sign}(t12);\ \beta = t12/\sin\mu;\ \alpha = \frac{t11 - t22}{2\sin\mu}$$

A little bit more complex is fully coupled 2D case.

$$\mathbf{T}(s) = \begin{bmatrix} N_1 & N_2 \\ N_3 & N_4 \end{bmatrix}; \ \det[\mathbf{T} - \lambda I] = \lambda^4 - \lambda^3 Tr[T] + \lambda^2 (2 + a) - \lambda Tr[T] + 1 =$$

$$a = Tr[N_1] \cdot Tr[N_4] - Tr[N_2 N_3] - 2 \det N_2$$

$$(note \det N_2 = \det N_3 = 1 - \det N_1 = 1 - \det N_4)$$

Finding roots:

$$z_k = \lambda_k + \lambda_k^{-1}; \ z_k = \frac{Tr[N_1 + N_1]}{2} \pm \left\{ \frac{Tr^{\,2}[N_1 - N_1]}{4} + Tr[N_2 N_3] + 2 \det N_2 \right\}$$

$$X = \begin{bmatrix} x \\ P_x \\ y \\ P_y \end{bmatrix} = \operatorname{Re} \tilde{a}_1 Y_1 + \operatorname{Re} \tilde{a}_1 Y_2 = \operatorname{Re} a_1 \tilde{Y}_1 + \operatorname{Re} \tilde{a}_1 \tilde{Y}_2$$

$$Y_k = R_k + iQ_k; \quad \tilde{Y}_k = \begin{bmatrix} w_{kx} e^{i\psi_{kx}} \\ \left(u_{kx} + iv_{kx}\right) e^{i\psi_{kx}} \\ w_{ky} e^{i\psi_{ky}} \\ \left(u_{ky} + iv_{ky}\right) e^{i\psi_{ky}} \end{bmatrix};$$

$$\psi_{kx}(s+C) = \psi_{kx}(s) + \mu_k; \psi_{ky}(s+C) = \psi_{ky}(s) + \mu_k;$$

$$w_{kx} v_{kx} + w_{ky} v_{ky} = 1;$$

Conditions: there are

$$Y_k^{*T} S Y_k = 2i; \quad Y_1^{*T} S Y_2 = 0; \; Y_1^T S Y_2 = 0; \quad \theta_k = \psi_{kx} - \psi_{ky}$$

$$a) \;\; w_{1x} v_{1x} = w_{2y} v_{2y} = 1 - q \qquad \Rightarrow v_{1x} = \frac{1-q}{w_{1x}}; \;\; v_{2y} = \frac{1-q}{w_{2y}}$$

$$b) \;\; w_{1y} v_{1y} = w_{2x} v_{2x} = q \quad \Rightarrow v_{2x} = \frac{q}{w_{2x}}; \;\; w_{1y} = \frac{q}{w_{1y}}$$

$$c) \;\; c = w_{1x} w_{1y} \sin\theta_1 = -w_{2x} w_{2y} \sin\theta_2$$

$$d) \;\; d = w_{1x}\left(u_{1y} \sin\theta_1 - v_{1y} \cos\theta_1\right) = -w_{2x}\left(u_{2y} \sin\theta_2 - v_{2y} \cos\theta_2\right)$$

$$e) \;\; e = w_{1y}\left(u_{1x} \sin\theta_1 + v_{1x} \cos\theta_1\right) = -w_{2y}\left(u_{2x} \sin\theta_2 + v_{2x} \cos\theta_2\right)$$

Conditions are result of symplecticity. Conditions a) and b) are equivalent to Poincaré's invariants conserving sum of projections on (x-px) and (y-py) planes.

$$Y_1 = \begin{bmatrix} w_{1x} e^{i\varphi_{1x}} \\ \left(u_{1x} + i \frac{q}{w_{1x}} \right) e^{i\varphi_{1x}} \\ w_{1y} e^{i\varphi_{1y}} \\ \left(u_{1y} + i \frac{1-q}{w_{1y}} \right) e^{i\varphi_{1y}} \end{bmatrix}; \quad Y_2 = \begin{bmatrix} w_{2x} e^{i\varphi_{2x}} \\ \left(u_{2x} + i \frac{1-q}{w_{2x}} \right) e^{i\varphi_{2x}} \\ w_{2y} e^{i\varphi_{2y}} \\ \left(u_{2y} + i \frac{q}{w_{2y}} \right) e^{i\varphi_{2y}} \end{bmatrix}$$

One should note for completeness that there is another way of parameterization of coupled motion proposed by Edward and Teng parameterization (*D.A.Edwards, L.C.Teng, IEEE Trans. Nucl. Sci.* ***NS-20*** *(1973) 885*), which differs from what we are discussing here.