

**Homework 11.**

**Problem 1. 20 points. A weak transverse coupling.**

**\*\*\* STAR part - 50 points**

Consider a fully uncoupled x and y betatron motion in a storage ring with circumference C:

$$\tilde{h}_o = \frac{\pi_1^2 + \pi_3^2}{2} + f(s) \frac{x^2}{2} + g(s) \frac{y^2}{2}$$

described by eigen vectors:

$$\mu_{x,y} = 2\pi Q_{x,y}; \quad Y_x(s) = \begin{bmatrix} w_x \\ w'_x + \frac{i}{w_x} \\ 0 \\ 0 \end{bmatrix}; \quad Y_y(s) = \begin{bmatrix} 0 \\ 0 \\ w_y \\ w'_y + \frac{i}{w_y} \end{bmatrix}$$

The eigen vectors and tunes are considered to be known. Introduce a weak coupling by SQ-quadrupole and solenoidal fields (for torsion equal zero):

$$\delta\tilde{h} = \delta f \frac{x^2}{2} + \delta n \cdot xy + \delta g \frac{y^2}{2} + \delta L(x\pi_3 - y\pi_1)$$

with

$$\delta n(s) = \frac{e}{2p_o c} \left[ \frac{\partial B_x}{\partial x} - \frac{\partial B_y}{\partial y} \right]; \quad \delta L(s) = \frac{e\delta B_s}{2p_o c}; \quad \delta f(s) = \delta g(s) = \delta L^2(s);$$

- (a) Write explicitly expressions for new betatron tunes using our developed perturbation method. Show that there is linear term on  $\delta n, \delta L$  only in case of coupling resonance when  $\mu_x = \pm\mu_y + 2\pi m$ .
- (b) For the case  $\mu_x \neq \mu_y$  write expressions for new eigen vectors. Use perturbation method developed in class. Normalize eigen vectors symplectically.

Solution - use perturbation theory from lecture 18, **Sample III:**

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Here is a short re-collection:

$$\frac{dX}{ds} = (\mathbf{D}(s) + \varepsilon \mathbf{D}_1(s)) \cdot X = (\mathbf{SH}(s) + \varepsilon \mathbf{SH}_1(s)) \cdot X$$

$$\frac{d\tilde{Y}_k(s)}{ds} = \mathbf{D}(s) \tilde{Y}_k(s); k = 1, \dots, n.$$

$$\begin{aligned}\tilde{Y}_{1k} &= \tilde{Y}_k e^{i\delta\phi_k} + \varepsilon c_k \tilde{Y}_k^* + \varepsilon \sum_{j \neq k} (a_{kj} \tilde{Y}_j + b_{kj} \tilde{Y}_j^*) + O(\varepsilon^2); \quad k = 1, \dots, n \\ \tilde{Y}_{1k}^* &= \tilde{Y}_k^* e^{-i\delta\phi_k} + \varepsilon c_k^* \tilde{Y}_k + \varepsilon \sum_{j \neq k} (a_{kj}^* \tilde{Y}_j^* + b_{kj}^* \tilde{Y}_j) + O(\varepsilon^2); \\ \frac{d\tilde{Y}_{1k}}{ds} &= (\mathbf{D}(s) + \varepsilon \mathbf{D}_1(s)) \cdot \tilde{Y}_{1k} + o(\varepsilon^2);\end{aligned}$$

leads to

$$\delta\phi'_k \tilde{Y}_k e^{i\delta\phi_k} + \varepsilon c'_k \tilde{Y}_k^* + \varepsilon \sum_{j \neq k} (a'_{kj} \tilde{Y}_j + b'_{kj} \tilde{Y}_j^*) = \varepsilon \mathbf{D}_1(s) \tilde{Y}_k e^{i\delta\phi_k}$$

and symplectic orthogonality of the eigen vectors

$$\tilde{Y}_k^* S \tilde{Y}_j = -\tilde{Y}_k S \tilde{Y}_j^* = 2i\delta_{ik}; \quad \tilde{Y}_k S \tilde{Y}_j = \tilde{Y}_k^* S \tilde{Y}_j^* = 0$$

multiplying by  $\tilde{Y}_m^* S$  or  $\tilde{Y}_m S$  from the left yields:

$$\begin{aligned}-2\delta\phi'_k &= \varepsilon \tilde{Y}_k^* S \mathbf{D}_1(s) \tilde{Y}_k \rightarrow \delta\phi'_k = \frac{\varepsilon}{2} Y_k^{*T} \mathbf{H}_1(s) Y_k; \quad S \mathbf{D}_1 = -\mathbf{H}_1; \\ -2ic' &= \tilde{Y}_k^T S \mathbf{D}_1(s) \tilde{Y}_k e^{i\delta\phi_k} \rightarrow c' = \frac{1}{2i} Y_k^T \mathbf{H}_1(s) Y_k e^{i(2\psi_k + \delta\phi_k)} \cong \frac{1}{2i} Y_k^T \mathbf{H}_1 Y_k e^{2i\psi_k} \\ 2ia'_{kj} &= \tilde{Y}_j^* S \mathbf{D}_1(s) \tilde{Y}_k e^{i\delta\phi_k} \rightarrow a'_{kj} = \frac{-1}{2i} Y_j^{*T} \mathbf{H}_1(s) Y_k e^{i(\psi_k - \psi_j + \delta\phi_k)} \cong \frac{-1}{2i} Y_j^{*T} \mathbf{H}_1(s) Y_k e^{i(\psi_k - \psi_j)}; \quad j \neq k \\ -2ib'_{kj} &= \tilde{Y}_j^* S \mathbf{D}_1(s) \tilde{Y}_k e^{i\delta\phi_k} \rightarrow b'_{kj} = \frac{1}{2i} Y_j^T \mathbf{H}_1(s) Y_k e^{i(\psi_k + \psi_j + \delta\phi_k)} \cong \frac{1}{2i} Y_j^T \mathbf{H}_1(s) Y_k e^{i(\psi_k + \psi_j)}; \quad j \neq k.\end{aligned}$$

$$\begin{aligned}\delta\phi(s) &= \phi_o + \frac{\varepsilon}{2} \int_0^s Y_k^{*T} \mathbf{H}_1 Y_k d\xi; \quad c(s) = c_o + \frac{1}{2i} \int_0^s d\xi Y_k^T \mathbf{H}_1 Y_k e^{i(2\psi_k + \delta\phi_k)}; \\ a_{kj} &= a_{kj_o} - \frac{1}{2i} \int_0^s d\xi Y_j^{*T} \mathbf{H}_1 Y_k e^{i(\psi_k - \psi_j + \delta\phi_k)}; \quad b_{kj} = b_{kj_o} + \frac{1}{2i} \int_0^s d\xi Y_j^T \mathbf{H}_1 Y_k e^{i(\psi_k + \psi_j + \delta\phi_k)}; \\ \tilde{Y}_{1k} e^{-i(\psi_k + \delta\phi_k)} &= Y_k + \varepsilon c_k Y_k^* e^{-i(2\psi_k + \delta\phi_k)} \left( c_o + \frac{1}{2i} \int_0^s d\xi Y_k^T \mathbf{H}_1 Y_k e^{i(2\psi_k + \delta\phi_k)} \right) + \\ &\quad \varepsilon \sum_{j \neq k} \left( Y_j e^{-i(\psi_k - \psi_j + \delta\phi_k)} \left( a_{kj_o} - \frac{1}{2i} \int_0^s d\xi Y_j^{*T} \mathbf{H}_1 Y_k e^{i(\psi_k - \psi_j + \delta\phi_k)} \right) + \right. \\ &\quad \left. Y_j^* e^{-i(\psi_k + \psi_j + \delta\phi_k)} \left( b_{kj_o} + \frac{1}{2i} \int_0^s d\xi Y_j^T \mathbf{H}_1 Y_k e^{i(\psi_k + \psi_j + \delta\phi_k)} \right) \right) + O(\varepsilon^2)\end{aligned}$$

Now we want to have periodic eigen vectors, e.g.

$$\tilde{Y}_{1k}(s+C) = \tilde{Y}_{1k}(s) e^{i\mu_{1k} C}; \quad \mu_{1k} = \mu_k + \frac{\varepsilon}{2} \int_0^C Y_k^{*T} \mathbf{H}_1 Y_k d\xi;$$

we need to choose the initial conditions to make a coefficient looking like:

$$d(s) = e^{-i\theta(s)} \left( d_o - \frac{1}{2i} \int_o^s d\xi f(\xi) e^{i\theta(\xi)} \right);$$

$$\rightarrow \left( d_o + \int_o^s d\xi f(\xi) e^{i\theta(\xi)} \right) = \frac{1}{e^{i\Delta\theta(C)} - 1} \int_o^{s+C} d\xi f(\xi) e^{i\theta(\xi)}.$$

giving us:

$$\tilde{Y}_{1k} e^{-i(\psi_k + \delta\phi_k)} = Y_{1k}(s) = Y_k + \varepsilon \frac{Y_k^* e^{-i(2\psi_k + \delta\phi_k)}}{2i(1 - e^{i(2\mu_k + \delta\mu_k)})} \int_s^{s+C} d\xi Y_k^T \mathbf{H}_1 Y_k e^{i(2\psi_k + \delta\phi_k)} +$$

$$\varepsilon \sum_{j \neq k} \left( \begin{aligned} & - \frac{Y_j e^{i(\psi_j - \psi_k - \delta\phi_k)}}{2i(1 - e^{i(\mu_k - \mu_j + \delta\mu_k)})} \int_s^{s+C} d\xi Y_j^* \mathbf{H}_1 Y_k e^{i(\psi_k - \psi_j + \delta\phi_k)} + \\ & \frac{Y_j^* e^{-i(\psi_j + \psi_k + \delta\phi_k)}}{2i(1 - e^{i(\mu_j + \mu_k + \delta\mu_k)})} \int_s^{s+C} d\xi Y_j^T \mathbf{H}_1 Y_k e^{i(\psi_k + \psi_j + \delta\phi_k)} \end{aligned} \right) + O(\varepsilon^2) \quad (18-15)$$

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We shall start from writing deviation of the Hamiltonian (desirably in the matrix form):

$$\delta K = \begin{bmatrix} \delta f & 0 & \delta n & \delta L \\ 0 & 0 & -\delta L & 0 \\ \delta n & -\delta L & \delta g & 0 \\ \delta L & 0 & 0 & 0 \end{bmatrix}; \delta D = \begin{bmatrix} 0 & 0 & -\delta L & 0 \\ -\delta f & 0 & -\delta n & -\delta L \\ \delta L & 0 & 0 & 0 \\ -\delta n & \delta L & -\delta g & 0 \end{bmatrix}; \quad (1)$$

In our case we have only 2D case

$$\mu_{x,y} = 2\pi Q_{x,y}; \quad Y_x(s) = \begin{bmatrix} w_x \\ w'_x + \frac{i}{w_x} \\ 0 \\ 0 \end{bmatrix}; \quad Y_y(s) = \begin{bmatrix} 0 \\ 0 \\ w_y \\ w'_y + \frac{i}{w_y} \end{bmatrix}$$

First, let's find tune changes

$$\tilde{Y}_x^* \delta \mathbf{H}(\zeta) \tilde{Y}_x = w_x^2 \delta f = \beta_x \cdot \delta L^2$$

$$\tilde{Y}_y^* \delta \mathbf{H}(\zeta) \tilde{Y}_y = w_y^2 \delta g = \beta_y \cdot \delta L^2$$

$$\delta Q_{x,y} \cong \frac{1}{4\pi} \int_0^C ds \beta_{x,y} \cdot \delta L^2.$$

to prove that they indeed change only in the second order of  $\delta L$ . If one gets to the next (second order) perturbation – there will be also second order term of  $\delta n$ . Still, no first order change. A special case of the resonance  $\mu_x = \pm \mu_y + 2\pi m$  - we will consider later.

To find perturbed eigen vectors using straight-forward convolutions

$$Y_y^{*T} \delta \mathbf{H} Y_x = \delta n w_y w_x + \delta L \left( w_x w'_y - w_y w'_x - i \frac{w_x}{w_y} - i \frac{w_y}{w_x} \right) = (Y_x^{*T} \delta \mathbf{H} Y_y)^*$$

$$Y_y^T \delta \mathbf{H} Y_x = \delta n w_y w_x + \delta L \left( w_x w'_y - w_y w'_x + i \frac{w_x}{w_y} - i \frac{w_y}{w_x} \right) = Y_x^T \delta \mathbf{H} Y_y$$

$$Y_x^T \delta \mathbf{H} Y_x = (w_x \delta L)^2; Y_y^T \delta \mathbf{H} Y_y = (w_y \delta L)^2$$

and dropping the last two, since they are of the second order we have

$$Y_1 \cong Y_x + a_x Y_y + b_x Y_y^*$$

$$Y_2 \cong Y_y + a_y Y_x + b_y Y_x^*$$

$$a_x(s) = \frac{-e^{-i(\psi_x - \psi_y)}}{2i \left( e^{i(\mu_x - \mu_y)} - 1 \right)} \int_s^{s+C} e^{i(\psi_x - \psi_y)} \left( \delta n \cdot w_y w_x + \delta L \left( w_x w'_y - w_y w'_x - i \frac{w_x}{w_y} - i \frac{w_y}{w_x} \right) \right) d\xi$$

$$b_x(s) = \frac{e^{-i(\psi_x + \psi_y)}}{2i \left( e^{i(\mu_x + \mu_y)} - 1 \right)} \int_o^C e^{i(\psi_x + \psi_y)} \left( \delta n \cdot w_y w_x + \delta L \left( w_x w'_y - w_y w'_x + i \frac{w_x}{w_y} - i \frac{w_y}{w_x} \right) \right) d\xi$$

$$a_x(s) = \frac{-e^{-i(\psi_y - \psi_x)}}{2i \left( e^{i(\mu_y - \mu_x)} - 1 \right)} \int_o^C e^{i(\psi_y - \psi_x)} \left( \delta n \cdot w_y w_x + \delta L \left( -w_x w'_y + w_y w'_x - i \frac{w_x}{w_y} - i \frac{w_y}{w_x} \right) \right) d\xi$$

$$b_x(s) = \frac{e^{-i(\psi_x + \psi_y)}}{2i \left( e^{i(\mu_x + \mu_y)} - 1 \right)} \int_o^C e^{i(\psi_x + \psi_y)} \left( \delta n \cdot w_y w_x + \delta L \left( -w_x w'_y + w_y w'_x - i \frac{w_x}{w_y} + i \frac{w_y}{w_x} \right) \right) d\xi$$

It is easy to see that attained expression is periodic. We can show that new set is symplectic-orthogonal for a general case - it is not more complicated that for specific case. We just need to re-write (8-15) in a compact form.

$$Y_{1k} = Y_k + \varepsilon \left( c_k Y_k^* + \sum_{j \neq k} (a_{kj} Y_j + b_{kj} Y_j^*) \right);$$

$$Y_{1k}^* = Y_k^* + \varepsilon \left( c_k^* Y_k + \varepsilon \sum_{j \neq k} (a_{kj}^* Y_j^* + b_{kj}^* Y_j) \right);$$

With obvious  $Y_{1k}^T \mathbf{S} Y_{1k} \cong 0$  let's check first normal pairs  $Y_{1k}^*, Y_{1k}$ :

$$Y_{1k}^{*T} \mathbf{S} Y_{1k} = Y_k^{*T} \mathbf{S} Y_k + \varepsilon \left( c_k Y_k^{*T} \mathbf{S} Y_k^* + \sum_{j \neq k} (a_{kj} Y_k^{*T} \mathbf{S} Y_j + b_{kj} Y_k^{*T} \mathbf{S} Y_j^*) \right); Y_k^*$$

$$+ \varepsilon \left( c_k^* Y_k^T \mathbf{S} Y_k + \sum_{j \neq k} (a_{kj}^* Y_j^{*T} \mathbf{S} Y_k + b_{kj}^* Y_j^T \mathbf{S} Y_k) \right) + O(\varepsilon^2) = 2i$$

where all term is red are equal zero. One can also evaluate second order term to be

$$O(\varepsilon^2) = 2i\varepsilon^2 \sum_{j \neq k} (|a_{kj}|^2 - |b_{kj}|^2)$$

Let's now look at  $Y_{1k}^T \mathbf{S} Y_{1m}; m \neq k$

$$\begin{aligned}
Y_{1k} &= Y_k + \varepsilon \left( c_k Y_k^* + \sum_{j \neq k} (a_{kj} Y_j + b_{kj} Y_j^*) \right); \\
Y_{1m} &= Y_m + \varepsilon \left( c_m Y_m^* + \sum_{j \neq k} (a_{mj} Y_j + b_{mj} Y_j^*) \right); \\
Y_{1k}^T \mathbf{S} Y_{1m} &= Y_k^T \mathbf{S} Y_m + \varepsilon \left( c_m Y_k^T \mathbf{S} Y_m^* + \sum_{j \neq k} (a_{mj} Y_k^T \mathbf{S} Y_j + b_{mj} Y_k^T \mathbf{S} Y_j^*) \right) + \\
&\varepsilon \left( c_k Y_k^{*T} \mathbf{S} Y_m + \sum_{j \neq k} (a_{kj} Y_j^T \mathbf{S} Y_m + b_{kj} Y_j^{*T} \mathbf{S} Y_m) \right) + O(\varepsilon^2); \\
Y_k^T \mathbf{S} Y_j^* &= -2i \delta_{kj}; Y_j^{*T} \mathbf{S} Y_m = 2i \delta_{jm} \\
Y_{1k}^T \mathbf{S} Y_{1m} &= 2i \varepsilon (b_{km} - b_{mk}); \\
b_{km} = b_{mk} &= \frac{e^{-i(\psi_m + \psi_k)}}{2i(1 - e^{i(\mu_m + \mu_k)})} \int_s^{s+C} d\xi Y_j^T \mathbf{H}_1 Y_k e^{i(\psi_k + \psi_m)}
\end{aligned}$$

Again, red term are equal zero and can be dropped. Similar equation work for  $Y_{1k}^{*T} \mathbf{S} Y_{1m}; m \neq k$

$$\begin{aligned}
Y_{1m}^{*T} \mathbf{S} Y_{1k} &= 2i \varepsilon (a_{mk}^* - a_{kn}); \\
a_{km} = a_{mk}^* &= \frac{-e^{i(\psi_m - \psi_k)}}{2i(1 - e^{i(\mu_k - \mu_m)})} \int_s^{s+C} d\xi Y_j^{*T} \mathbf{H}_1 Y_k e^{i(\psi_k - \psi_m)}
\end{aligned}$$

Note, that this is all approximation with error of  $O(\varepsilon^2)$ . Naturally, not to have any errors, we should get the transport matrices for the motion and then do everything perfectly – but not analytically!

STAR PART – we will discuss it when we are discussing resonances.