PHY 554
Fundamentals of Accelerator Physics

Lecture 18: Synchrotron Radiation Sources

November 2, 2016

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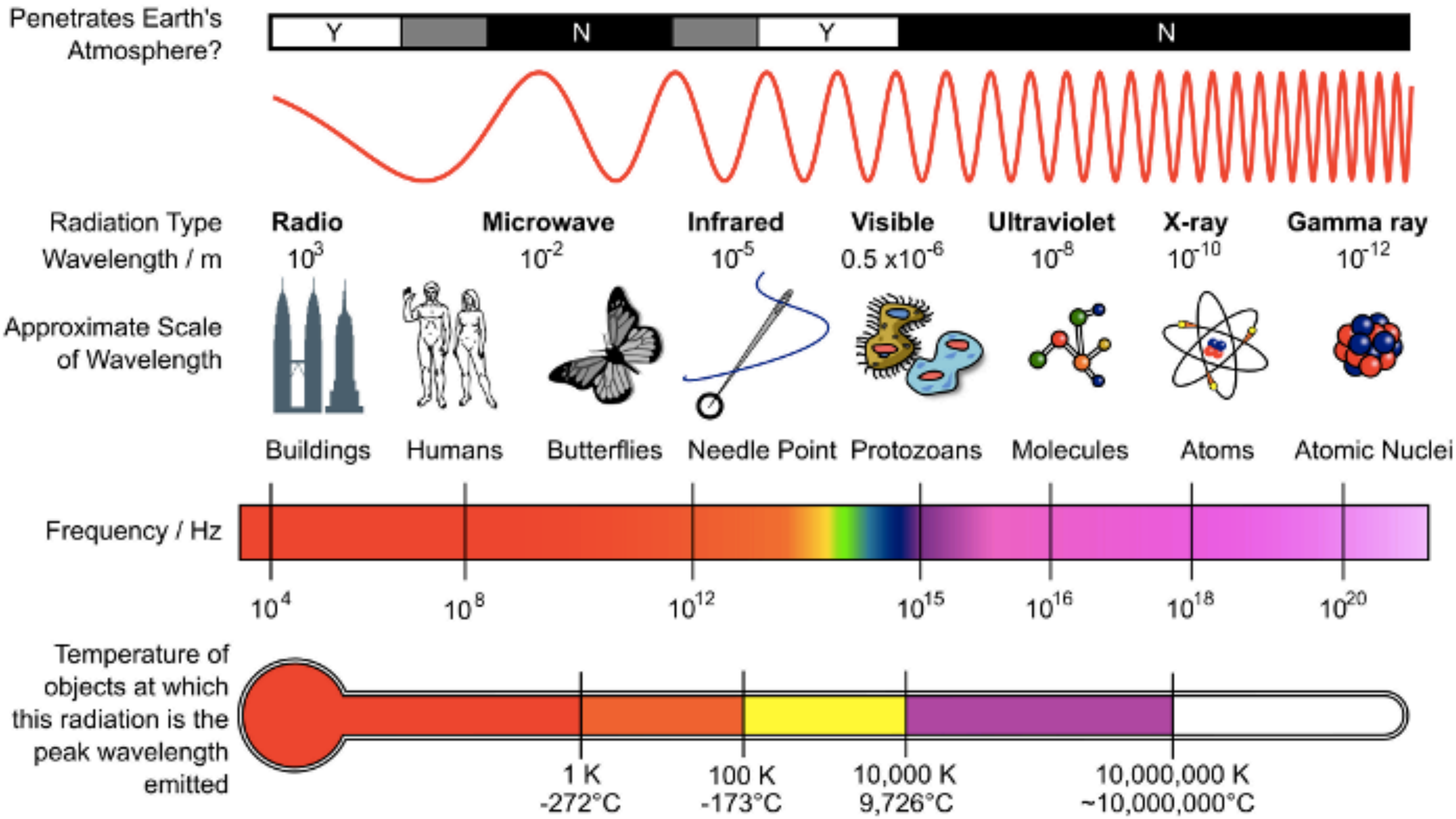
There is a large number of dedicated courses on Synchrotron Radiation Sources and Their Applications. If you are interested in this topic, I would strongly recommend lectures given by Prof. D.T. Attwood at UC Berkeley, https://people.eecs.berkeley.edu/~attwood/srms//

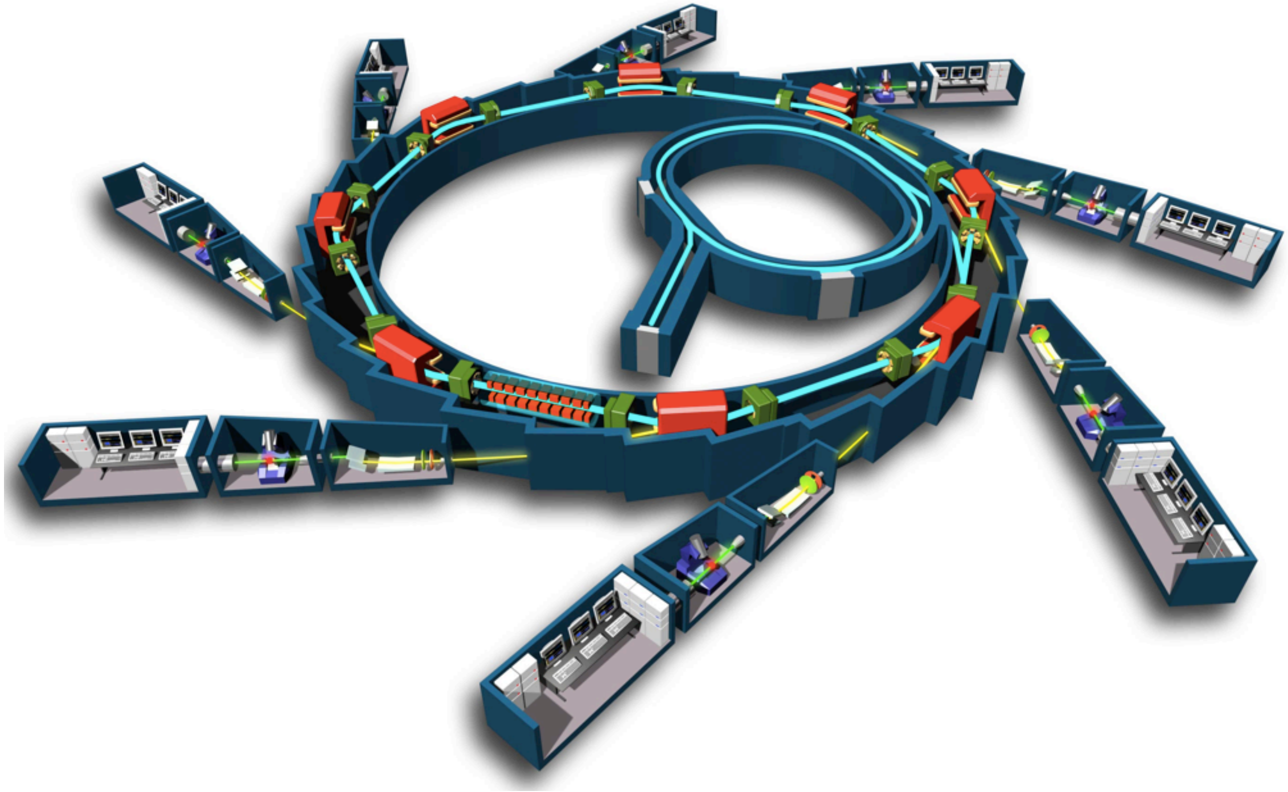
http://case.physics.stonybrook.edu/index.php/PHY554_fall_2016

Radiation Spectrum



SR Light Sources

- To generate IR, UV and X-ray radiation
  - From dipoles, undulators/wigglers



VERY POPULAR SCIENTIFIC TOOL:
With thousands of users

LIGHT INTERACTS with the MATTER

List of operational light sources

http://www.lightsources.org/regions

Name	Country	Website
Center for the Advancement of Natural Discoveries using Light Emission	Armenia	http://www.candle.am/index.html
Australian Synchrotron	Australia	http://www.synchrotron.org.au
Laboratorio Nacional de Luz Sincrotron	Brazil	http://www.lnls.br/
Canadian Light Source	Canada	http://www.lightsource.ca
Beijing Synchrotron Radiation Facility	China	http://bsrf.ihep.cas.cn/
National Synchrotron Radiation Laboratory	China	http://www.nsrl.ustc.edu.cn/
SSRF – Shanghai Synchrotron Radiation Facility	China	http://ssrf.sinap.ac.cn/english/
Institute for Storage Ring Facilities	Denmark	http://www.isa.au.dk/
European Synchrotron Radiation Facility	France	http://www.esrf.eu
SOLEIL	France	http://www.synchrotron-soleil.fr/
Angstromquelle Karlsruhe – ANKA	Germany	http://anka.kit.edu
BESSY II – Helmholtz-Zentrum Berlin	Germany	http://www.helmholtz-berlin.de/
Dortmund Electron Storage Ring Facility	Germany	http://www.delta.tu-dortmund.de/
ELSA – Electron Stretcher Accelerator	Germany	http://www-elsa.physik.uni-bonn.de/elsa-facility_en.html
Metrology Light Source	Germany	http://www.ptb.de/mls/
PETRA III at DESY	Germany	http://photon-science.desy.de
Centre for Advanced Technology	India	http://www.cat.ernet.in/technology/accel/indus/index.html
Iranian Light Source Facility	Iran	http://ilsf.ipm.ac.ir/
DAFNE	Italy	http://web.infn.it/Dafne_Light/
Elettra Synchrotron Light Laboratory	Italy	http://www.elettra.eu
Aichi Synchrotron Radiation Center	Japan	http://www.astf-kha.jp/synchrotron/en/
Hiroshima Synchrotron Radiation Center	Japan	http://www.hsrc.hiroshima-u.ac.jp/index.html
Photon Factory	Japan	http://pfwww.kek.jp/
Ritsumeikan University SR Center	Japan	http://www.ritsumei.ac.jp/acd/re/src/index.htm
Saga Light Source	Japan	http://www.saga-ls.jp/?page=206
SPring-8	Japan	http://www.spring8.or.jp/en/
Ultraviolet Synchrotron Orbital Radiation Facility	Japan	http://www.uvsor.ims.ac.jp/defaultE.html
Synchrotron-light for Experimental Science and Applications in the Middle East	Jordan	http://www.sesame.org.jo/sesame/
Pohang Light Source	Korea	http://paleng.postech.ac.kr
Dubna Electron Synchrotron	Russia	http://wwwinfo.jinr.ru/delsy/
Kurchatov Synchrotron Radiation Source	Russia	http://www.nrcki.ru/e/engl.html
Siberian Synchrotron Research Centre	Russia	http://ssrc.inp.nsk.su/
TNK	Russia	http://www.niifp.ru/page/sinhrotron
Singapore Synchrotron Light Source	Singapore	http://ssls.nus.edu.sg/index.html
ALBA	Spain	http://www.cells.es/
MAX IV Laboratory	Sweden	https://www.maxiv.se
Swiss Light Source	Switzerland	http://www.psi.ch/sls/
National Synchrotron Radiation Research Center	Taiwan	http://www.nsrrc.org.tw/
Synchrotron Light Research Institute	Thailand	http://www.slri.or.th
Diamond Light Source	United Kingdom	http://www.diamond.ac.uk/
Advanced Light Source	USA	https://als.lbl.gov/
Advanced Photon Source	USA	http://www.aps.anl.gov
Center for Advanced Microstructures and Devices	USA	http://www.camd.lsu.edu/
Cornell High Energy Synchrotron Source	USA	http://www.chess.cornell.edu/
National Synchrotron Light Source II	USA	http://www.bnl.gov/ps/
Stanford Synchrotron Radiation Lightsource	USA	http://www-ssrl.slac.stanford.edu
Synchrotron Ultraviolet Radiation Facility	USA	http://physics.nist.gov/MajResFac/SURF/SURF/index.html


How a Synchrotron Works
The First Synchrotron Light from TPS Storage Ring
December 31, 2014
10⁶ × Sun
10⁹-10¹² × X-ray
10¹¹ × Light Bulb
The Electromagnetic Spectrum
The many colours of light
"Brightness"
Synchrotron Light
Millions of times brighter than sunlight
What are X-Rays Good For?
DIAMOND SYNCHROTRON
Why use a synchrotron ?
Lorentz transformation
NSLS II
ANKA: Integrating multidisciplinary R&D with synchrotron technology

SR Light Sources Worldwide

ESRF, 6 GeV

SPring-8, 8 GeV

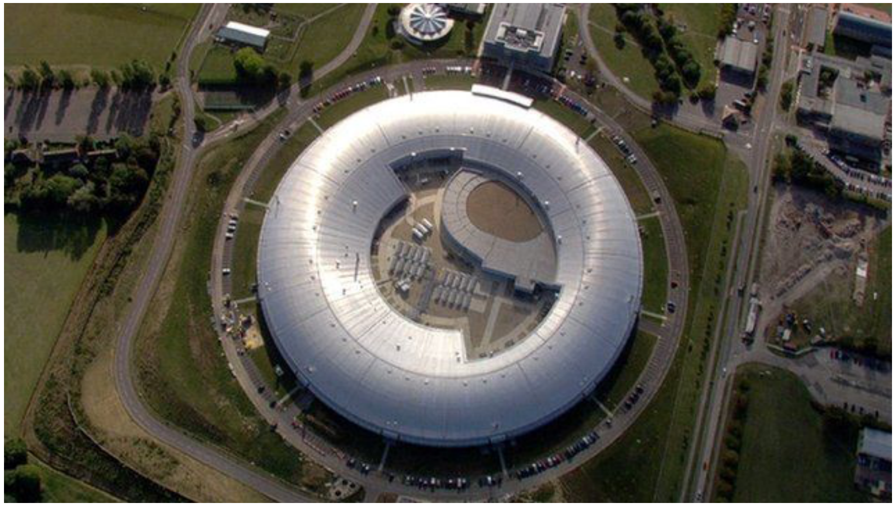
APS, 7 GeV

SSRF, 3.5 GeV

SR Light Sources Worldwide

MAX IV, 3 GeV, Sweden

NSLS II, 3 GeV, BNL, USA



Diamond, 3 GeV, England

ALBA, 3 GeV, Spain

SR Light Sources Worldwide

SSRF, China, 3.5 GeV

Soleil, France, 2.75 GeV

SLS, Switzerland, 2.4 GeV

PLS, Korea, 3 GeV

Australian Synchrotron, 3 GeV

BESSY II, Germany, 1.7 GeV

NSRRC, Taiwan, 3GeV

Indus II, India, 2.5GeV

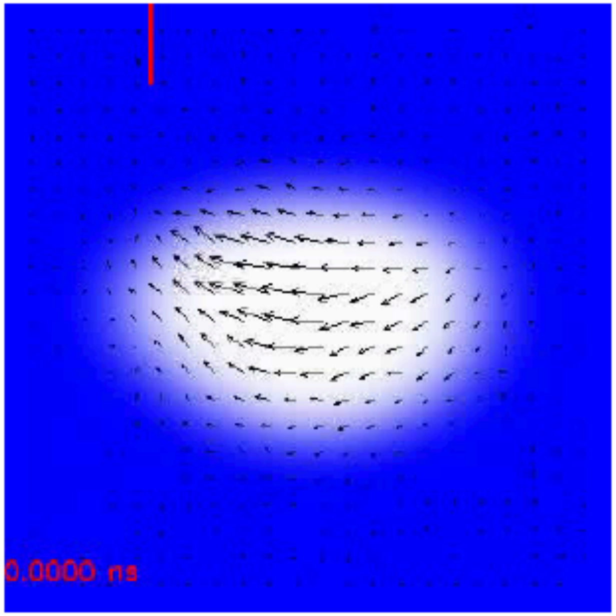
SESAME, Jordan……

SR Light Sources ~ 50 facilities worldwide
Tens of thousands of scientific and industrial user
The filed is still growing!

X-rays have come a long way......

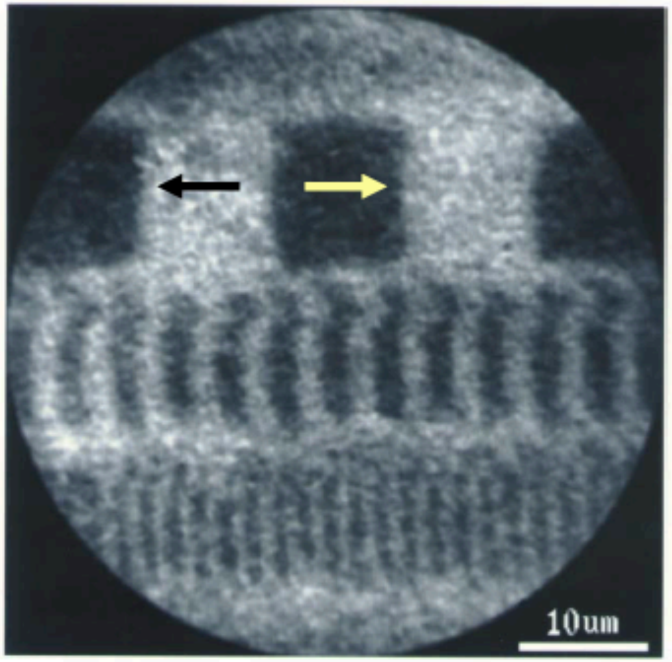
1895

10 cm



100 nm

1993

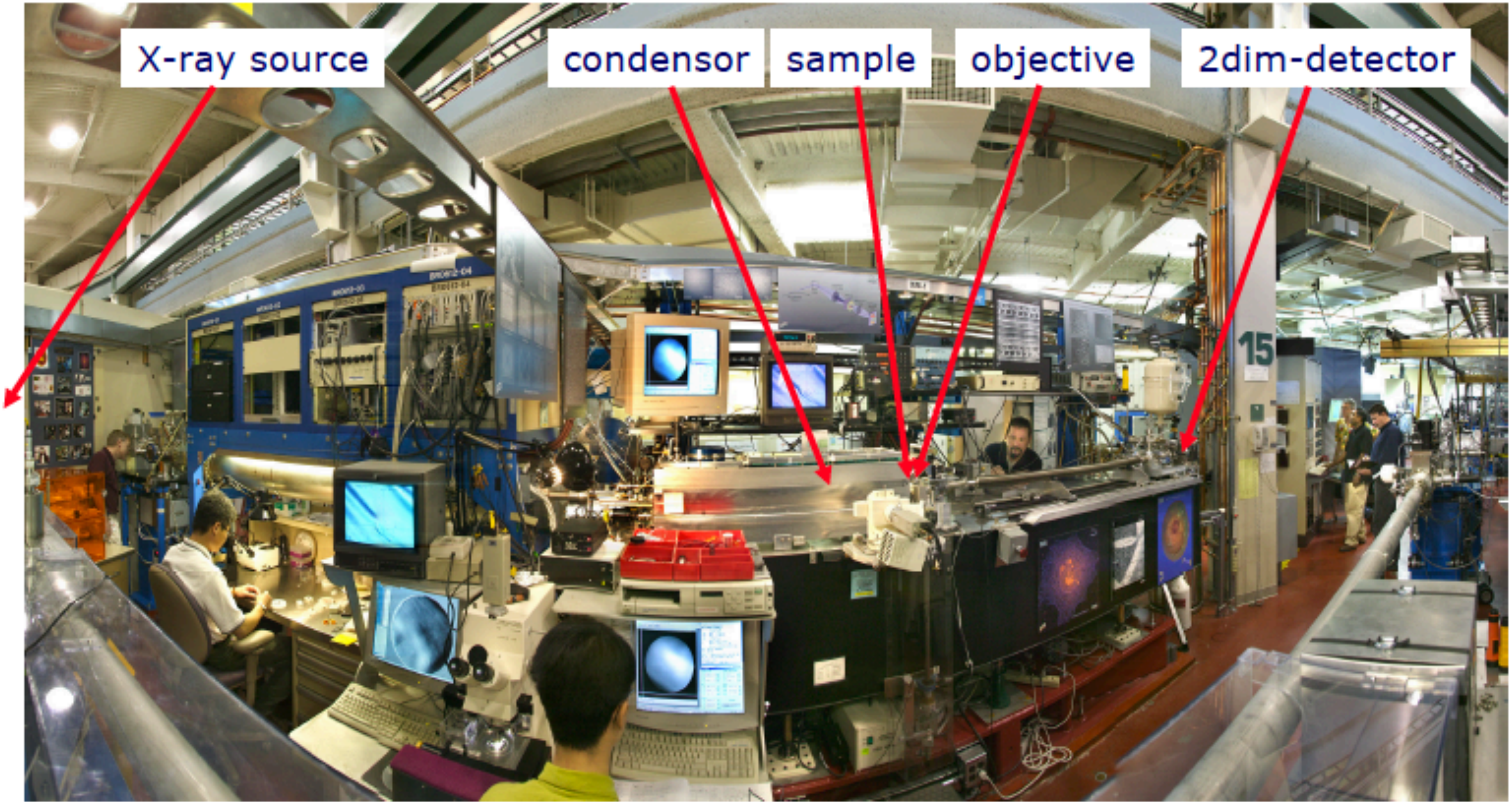


10 µm

Courtesy of J. Stöhr, SSRL

Soft X-ray Microscope XM-1 (BL 6.1.2 @ ALS)

http://www.cxro.lbl.gov/BL612/



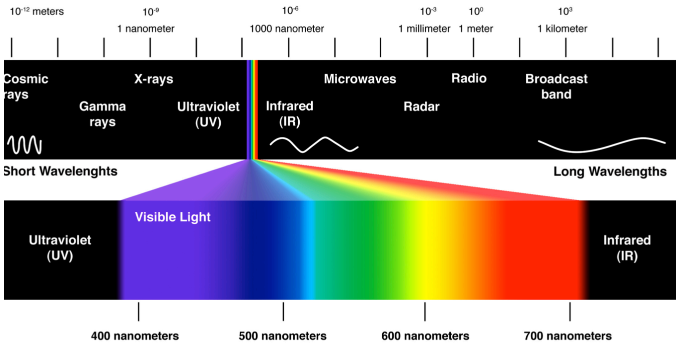
Courtesy of D. Attwood

What matters

- Rarely there is interest just a radiation power
- Typically people are interested in specific energy of photons (wavelength of radiation)

$$E_{ph} = \hbar\omega = \hbar c \frac{2\pi}{\lambda}$$

$$E_{ph}[keV] \approx \frac{12.4}{\lambda\left[\AA\right]};\; E_{ph}[eV] \approx \frac{1.24}{\lambda[\mu m]}$$



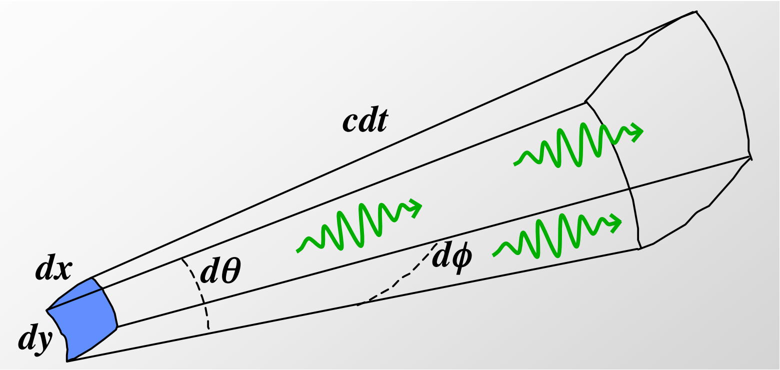
3 eV 2.5 eV 2 eV 1.75 eV

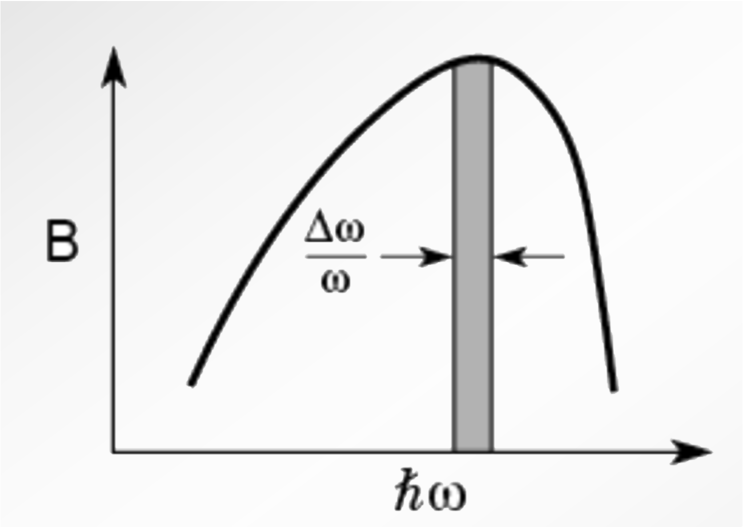
- 1 Å =10^{-10} m (0.1 nm or 100 pm), 12.4 keV photons

Figures of merit of light source

- Photon flux or spectral photon flux $\dot{N}_\omega = \frac{d^2 N}{dt\left(d\omega/\omega\right)}$

- Brightness of the source $B = \frac{d^4 N}{dt d\Omega dA\left(d\omega/\omega\right)}$





Few formulae

Spectral expansion of radiation field in an observation point $\vec{r}$ for a point particles moving on a given trajectory:

$$\vec{r}_o = \vec{r}_o(t); \vec{\mathrm{v}} = \vec{\mathrm{v}}_o(t) = \frac{d\vec{r}_o(t)}{dr};$$

can be easily calculated by applying Fourier transformation to Lienard-Wiechert 4-potential

$$\vec{A}(\vec{r},t) = \left.\frac{e\vec{\mathrm{v}}_o}{c\left(R - \frac{\vec{\mathrm{v}}_o}{c}\cdot\vec{R}\right)}\right|_{t'}; \varphi(\vec{r},t) = \left.\frac{e}{\left(R - \frac{\vec{\mathrm{v}}_o}{c}\cdot\vec{R}\right)}\right|_{t'}; t' + \frac{\vec{R}(t')}{c} = t; \vec{R}(t) = \vec{r} - \vec{r}_o(t).$$

with no problem of resolving retarded time problem:

$$\vec{A}_\omega(\vec{r},t) = \frac{e}{c}\int_{-\infty}^{\infty} e^{i\omega\left(t+\frac{R(t)}{c}\right)} \frac{\vec{\mathrm{v}}_o(t)}{R(t)}dt; \varphi(\vec{r},t) = \frac{e}{c}\int_{-\infty}^{\infty} e^{i\omega\left(t+\frac{R(t)}{c}\right)} \frac{dt}{R(t)}; \vec{R}(t) = \vec{r} - \vec{r}_o(t).$$

It means that given particle trajectory, you could always calculate 4-potential, at least numerically…

Few formulae

Frequently our instrument is located far from the radiation point and we are interested in radiated field at large distance:

$$\vec{R}(t)=\vec{r}-\vec{r}_o(t)\to\vec{R}(t)=\vec{R}_o+\vec{r}-\vec{r}_o(t);$$

$$\vec{R}_o=\vec{n}\left|\vec{R}_o\right|\equiv\vec{n}R_o;R_o>>\left|\vec{r}_o\right|\Rightarrow\vec{R}(t)\cong R_o-\vec{n}\vec{r}_o;\ k=\frac{\omega}{c};\vec{k}=\vec{n}k;$$

$$\mathbf{A}_\omega\cong\frac{e}{cR_o}e^{ikR_o}\int_{-\infty}^{\infty}\vec{\mathrm{v}}_o(t)e^{i(\omega t-\vec{k}\vec{r}_o)}dt\equiv\frac{e}{cR_o}e^{ikR_o}\int e^{i(\omega t-\vec{k}\vec{r}_o)}d\vec{r}_o$$

$$\mathbf{H}=\vec{\nabla}\times\mathbf{A}\to\mathbf{H}_\omega==i\left[\vec{k}\times\mathbf{A}_\omega\right];\ \frac{1}{c}\frac{\partial\mathbf{E}}{\partial t}=-\vec{\nabla}\times\mathbf{H}\to\mathbf{E}_\omega=\frac{ic}{\omega}\left[\vec{k}\times\left[\vec{k}\times\mathbf{A}_\omega\right]\right];$$

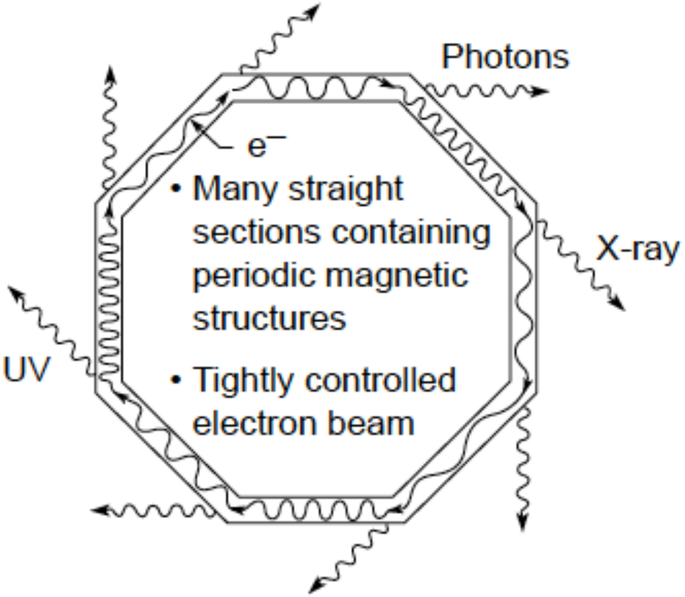
$$\left|\mathbf{E}_\omega\right|=\left|\mathbf{H}_\omega\right|;\mathbf{E}_\omega\cdot\mathbf{H}_\omega=0;\mathbf{E}_\omega\perp\mathbf{H}_\omega\perp\vec{n}.$$

$$\mathbf{S}_\omega=c\frac{\left[\mathbf{E}_\omega\times\mathbf{H}_\omega\right]}{4\pi}=c\vec{n}\frac{\mathrm{H}_\omega{}^2}{4\pi}=c\vec{n}\frac{\mathrm{E}_\omega{}^2}{4\pi}$$

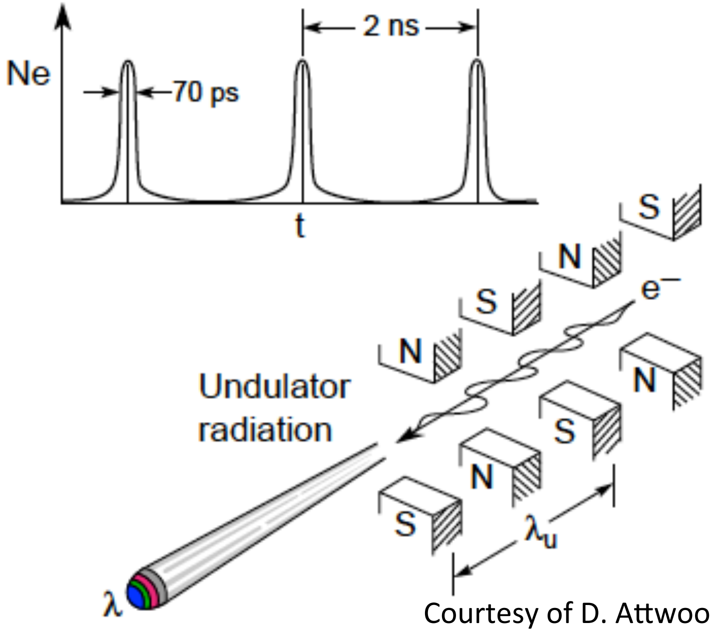
$$dE_\omega=\frac{c}{2\pi}\mathrm{H}_\omega{}^2R_o^2d\Omega\frac{d\omega}{2\pi}$$

It even simpler – it is reduced to integral along the particle trajectory.
Uing these formulae you can calculate anything, but math can be – and frequently is - messy!
You got accurate derivation of SR power and spectrum during last class.

Sources of Spontaneous Radiation



- Many straight sections containing periodic magnetic structures
- Tightly controlled electron beam



Courtesy of D. Attwood

Bending Magnet:

$$\hbar\omega_c = \frac{3e\hbar B\gamma^2}{2m} \qquad P = \frac{e^2 c}{6\pi\varepsilon_0}\frac{\gamma^4}{\rho^2}$$

Wiggler:

$$\hbar\omega_c = \frac{3e\hbar B\gamma^2}{2m}$$

$$n_c = \frac{3K}{4}\left(1 + \frac{K^2}{2}\right)$$

$$P_T = \frac{\pi e K^2\gamma^2 I N}{3\epsilon_0\lambda_u}$$

SI units

Undulator:

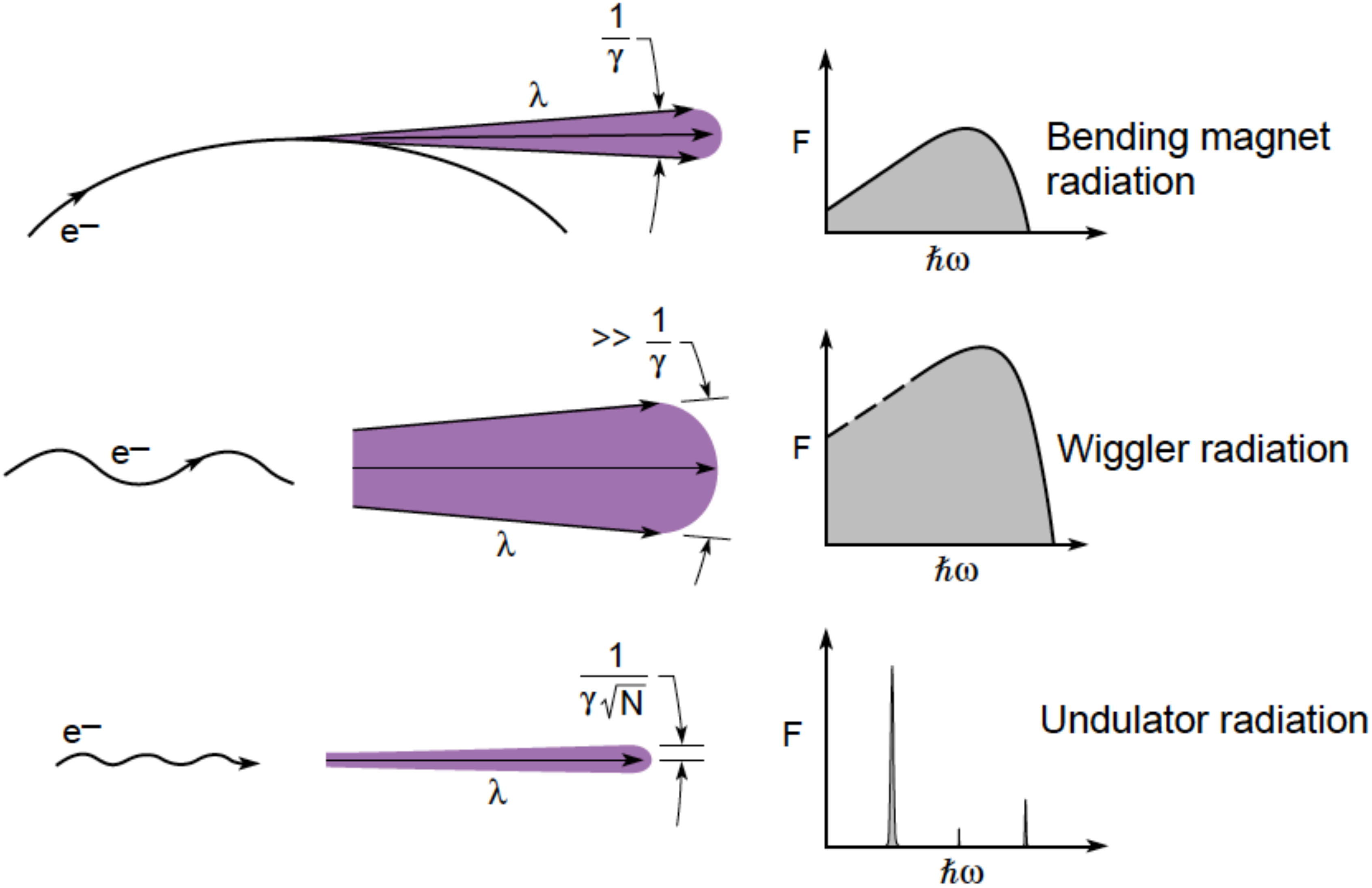
$$\lambda = \frac{\lambda_u}{2\gamma^2}\left(1 + \frac{K^2}{2} + \gamma^2\theta^2\right)$$

$$K = \frac{eB_0\lambda_u}{2\pi mc}$$

$$\theta_{\text{cen}} = \frac{1}{\gamma^*\sqrt{N}}$$

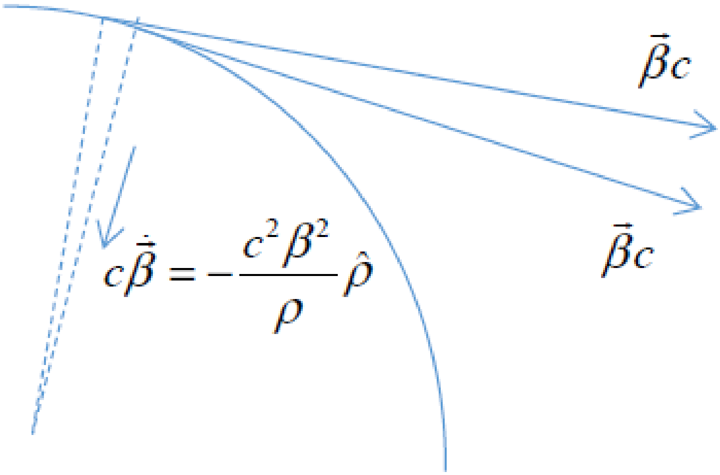
$$\left.\frac{\Delta\lambda}{\lambda}\right|_{\text{cen}} = \frac{1}{N}$$

$$\bar{P}_{\text{cen}} = \frac{\pi e\gamma^2 I}{\epsilon_0\lambda_u}\frac{K^2}{\left(1 + \frac{K^2}{2}\right)^2} f(K)$$



Courtesy of D. Attwood

Circular orbit



$$\dot{a} = -\frac{v^2}{\rho}\hat{\rho} \Rightarrow \dot{\vec{\beta}} = -\frac{\beta^2 c}{\rho}\hat{\rho}$$

$$P(t_r) = \frac{1}{4\pi\varepsilon_0}\frac{2}{3}\frac{e^2}{c}\gamma^6\dot{\beta}^2\left(1-\beta^2\right) = \frac{1}{4\pi\varepsilon_0}\frac{2e^2 c\beta^4\gamma^4}{3\rho^2}$$

For a storage ring, the energy loss per turn:

$$U_0 = \int_C P(t_r)\,dt = \frac{1}{\beta c}\int_C P(t_r)\,ds = \frac{1}{4\pi\varepsilon_0}\frac{2e^2\beta^3\gamma^4}{3}\int_C \frac{1}{\rho^2}ds$$

If all dipoles in the storage ring has the same bending radius (iso-magnetic case):

$$U_0 = \frac{1}{4\pi\varepsilon_0}\frac{2e^2\beta^3\gamma^4}{3}\frac{2\pi\rho}{\rho^2} = \frac{e^2\beta^3\gamma^4}{3\varepsilon_0\rho}$$

Power radiated by a beam of average current I_b:

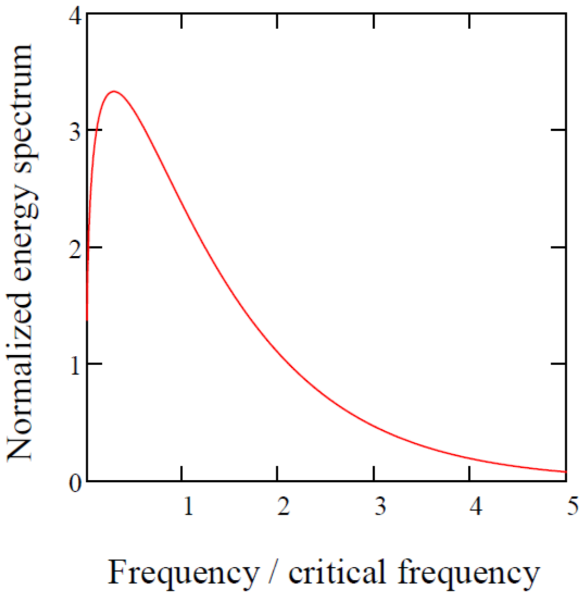
$$P_{beam} = U_0\frac{I_b}{e} = \frac{e\beta^3\gamma^4}{3\varepsilon_0\rho}I_b$$

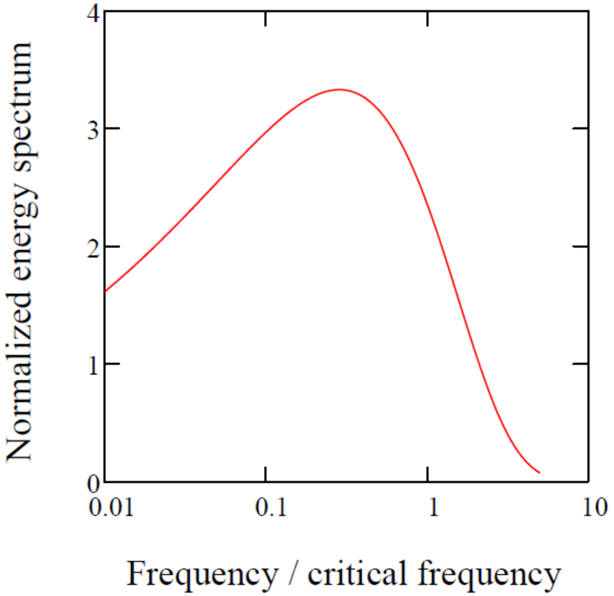
Energy spectrum V

- The total energy spectrum is obtained by integrating over the solid angle:

$$\frac{dW}{d\omega}=2\pi\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{d^2I(\omega)}{d\omega d\Omega}\cos\theta\,d\theta=\frac{2\pi}{\gamma}\int_{-\gamma\frac{\pi}{2}}^{\gamma\frac{\pi}{2}}\frac{d^2I(\omega)}{d\omega d\Omega}d(\gamma\theta)$$

$$\approx\frac{1}{4\pi\varepsilon_0}\frac{3e^2\gamma}{2\pi c}\frac{\omega^2}{\omega_c^2}\int_{-\infty}^{\infty}\left(1+y^2\right)^2\left\{\frac{y^2}{\left(1+y^2\right)}K_{\frac{1}{3}}{}^2\left(\frac{\omega}{2\omega_c}\left(1+y^2\right)^{\frac{3}{2}}\right)+K_{\frac{2}{3}}{}^2\left(\frac{\omega}{2\omega_c}\left(1+y^2\right)^{\frac{3}{2}}\right)\right\}dy$$

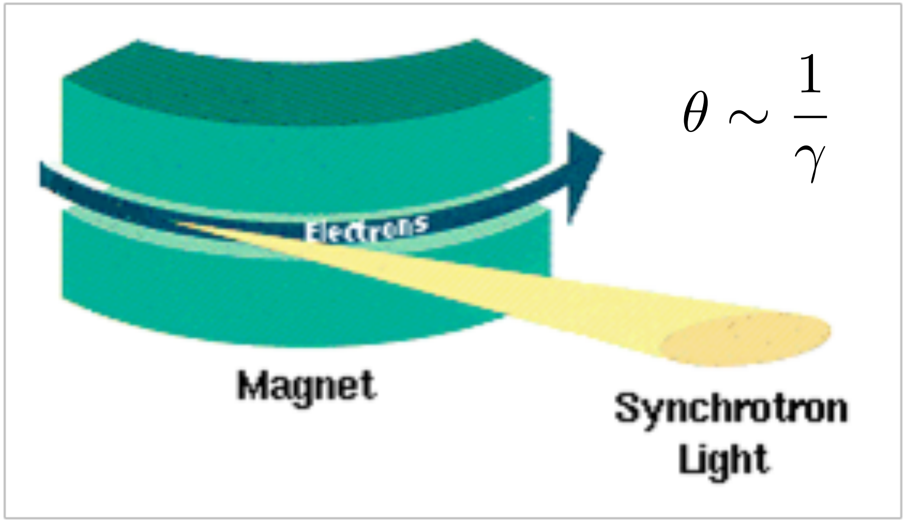




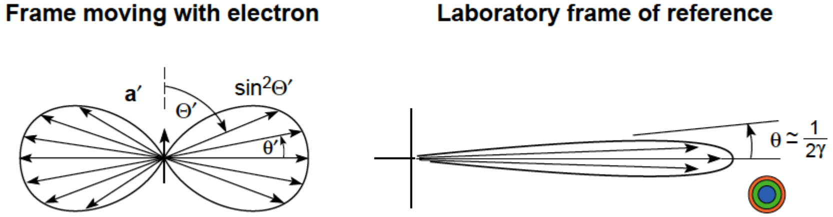
A more concise and popular expression for the energy spectrum:

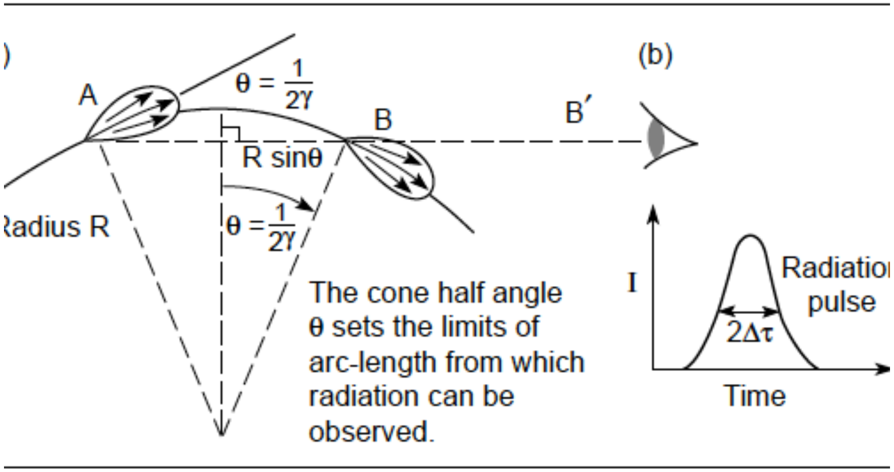
$$\frac{dW}{d\omega}=\frac{1}{4\pi\varepsilon_0}\sqrt{3}\frac{e^2\gamma}{c}\frac{\omega}{\omega_c}\int_{\omega/\omega_c}^{\infty}K_{\frac{5}{3}}(x)\,dx$$

SR from Bending Magnet simple considerations



$$\theta \sim \frac{1}{\gamma}$$





The cone half angle θ sets the limits of arc-length from which radiation can be observed.

$l_{rad} \propto \frac{\rho}{\gamma};$ Seen by observer

$l_{co-moving} = \frac{l_{rad}}{\gamma} \propto \frac{\rho}{\gamma^2};$ Lorenz contraction

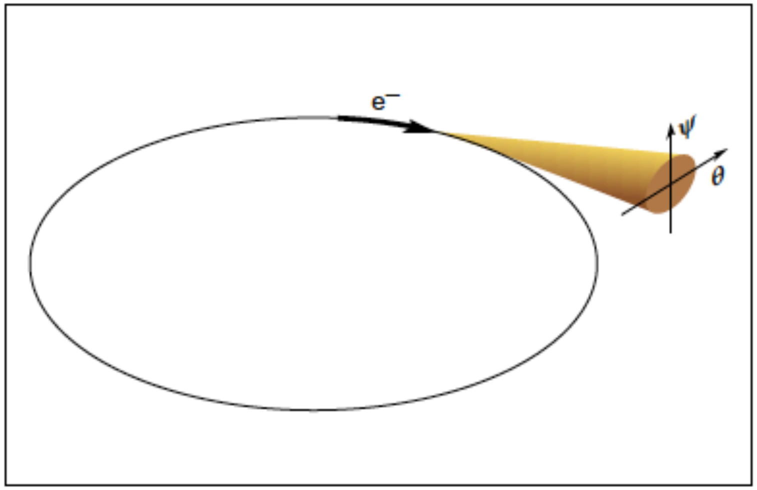
$l_{lab} = \frac{l_{co-moving}}{2\gamma} \propto \frac{\rho}{\gamma^3}$ Doppler boost

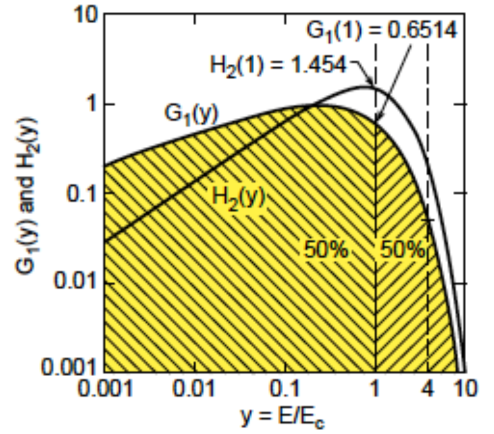
$\lambda_c = \frac{4\pi}{3}\frac{\rho}{\gamma^3}$ Critical wavelength

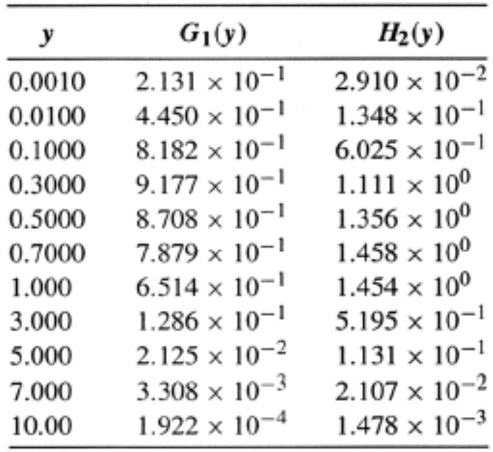
$$\epsilon_c \equiv \hbar\omega_c = \frac{3}{2}\frac{\hbar c\gamma^3}{\rho}$$

Courtesy of D. Attwood

SR from bending magnet (dipole magnet)







y	$G_1(y)$	$H_2(y)$
0.0010	2.131×10^{-1}	2.910×10^{-2}
0.0100	4.450×10^{-1}	1.348×10^{-1}
0.1000	8.182×10^{-1}	6.025×10^{-1}
0.3000	9.177×10^{-1}	1.111×10^{0}
0.5000	8.708×10^{-1}	1.356×10^{0}
0.7000	7.879×10^{-1}	1.458×10^{0}
1.000	6.514×10^{-1}	1.454×10^{0}
3.000	1.286×10^{-1}	5.195×10^{-1}
5.000	2.125×10^{-2}	1.131×10^{-1}
7.000	3.308×10^{-3}	2.107×10^{-2}
10.00	1.922×10^{-4}	1.478×10^{-3}

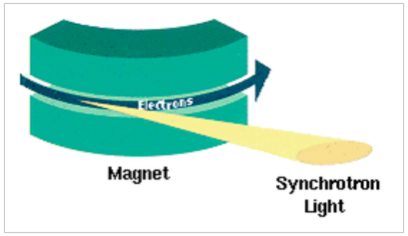
$$\boxed{E_c = \hbar\omega_c = \frac{3e\hbar B\gamma^2}{2m}} \qquad (5.7a)$$

$$E_c(\text{keV}) = 0.6650 E_e^2(\text{GeV}) B(\text{T}) \qquad (5.7b)$$

$$\gamma = \frac{E_e}{mc^2} = 1957 E_e(\text{GeV}) \qquad (5.5)$$

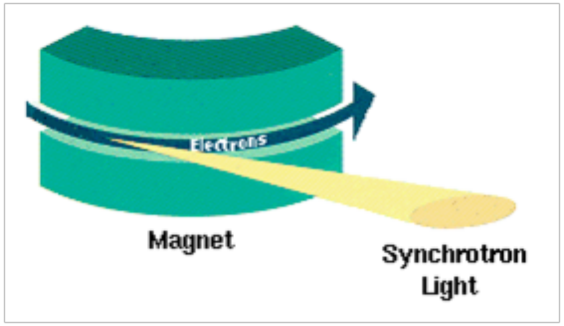
$$\left.\frac{d^3F_B}{d\theta\, d\psi\, d\omega/\omega}\right|_{\psi=0} = 1.33 \times 10^{13} E_e^2(\text{GeV}) I(\text{A}) H_2(E/E_c) \frac{\text{photons/s}}{\text{mrad}^2 \cdot (0.1\%\ \text{BW})} \qquad (5.6)$$

$$\frac{d^2F_B}{d\theta\, d\omega/\omega} = 2.46 \times 10^{13} E_e(\text{GeV}) I(\text{A}) G_1(E/E_c) \frac{\text{photons/s}}{\text{mrad} \cdot (0.1\%\text{BW})} \qquad (5.8)$$



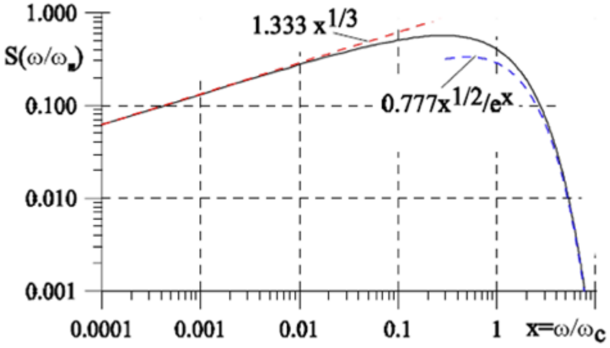
Courtesy of D. Attwood

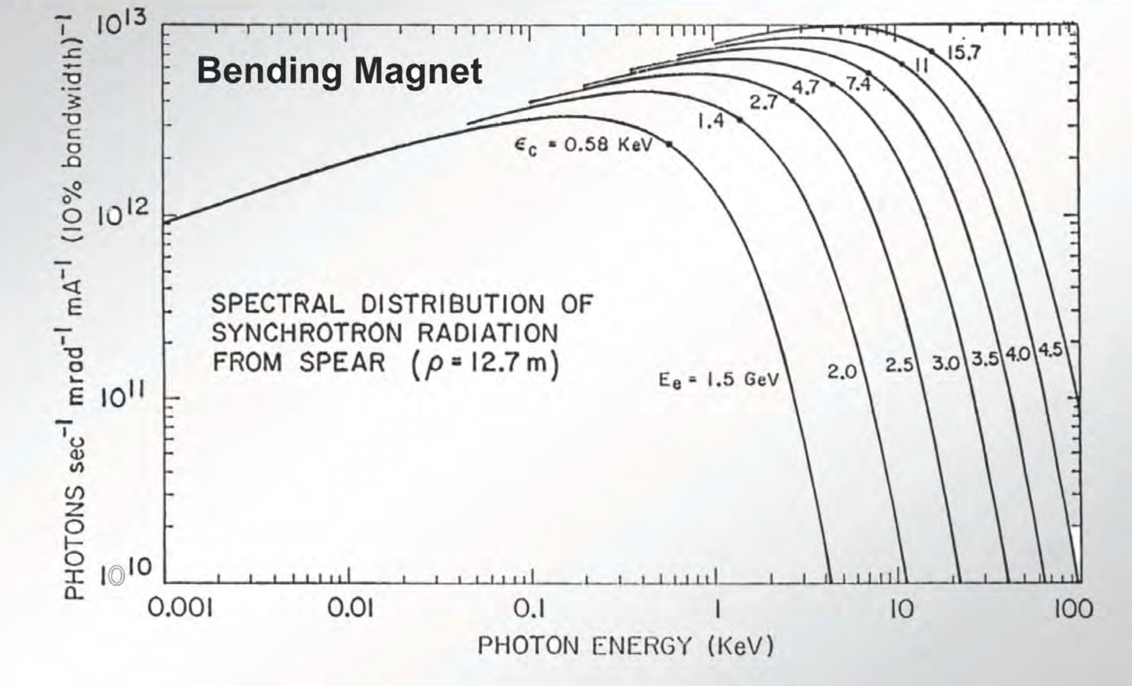
SR spectrum



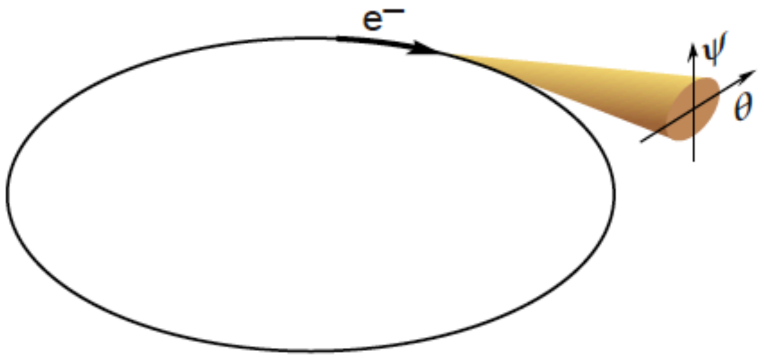
$$\epsilon_c \equiv \hbar\omega_c = \frac{3}{2}\frac{\hbar c \gamma^3}{\rho}$$

$$P = \frac{e^2 c}{6\pi\varepsilon_0}\frac{\gamma^4}{\rho^2}$$





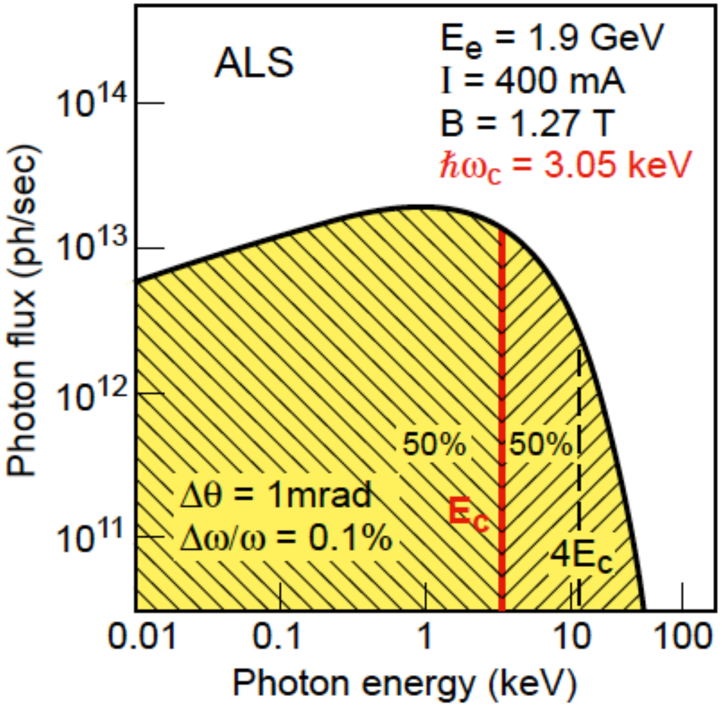
Bending Magnet Radiation Covers a Broad Region of the Spectrum, Including the Primary Absorption Edges of Most Elements



$$E_c = \hbar\omega_c = \frac{3e\hbar B\gamma^2}{2m} \tag{5.7a}$$

$$E_c(\text{keV}) = 0.6650E_e^2(\text{GeV})B(\text{T}) \tag{5.7b}$$

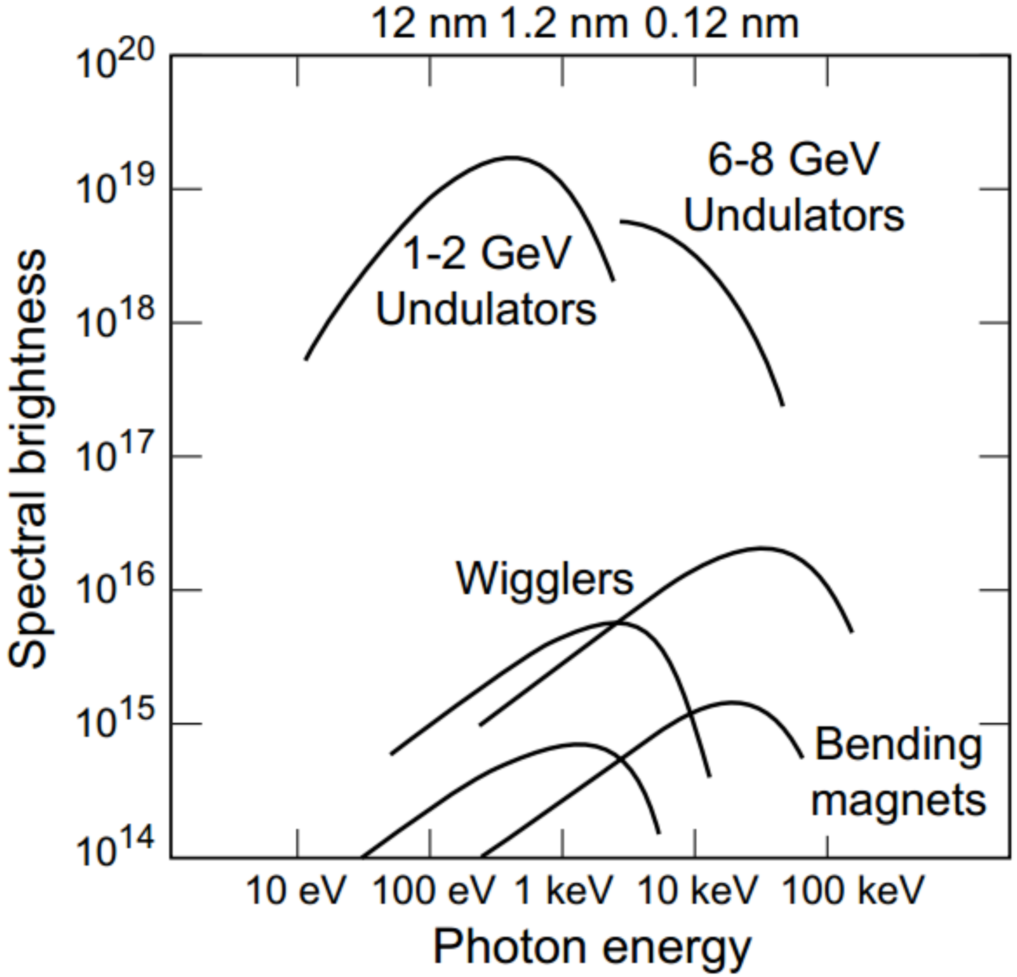
$$\frac{d^2F_B}{d\theta\, d\omega/\omega} = 2.46\times 10^{13} E_e(\text{GeV}) I(\text{A}) G_1(E/E_c) \frac{\text{photons/s}}{\text{mrad}\cdot(0.1\%\text{BW})} \tag{5.8}$$



<u>Advantages:</u>
- covers broad spectral range
- least expensive
- most accessable

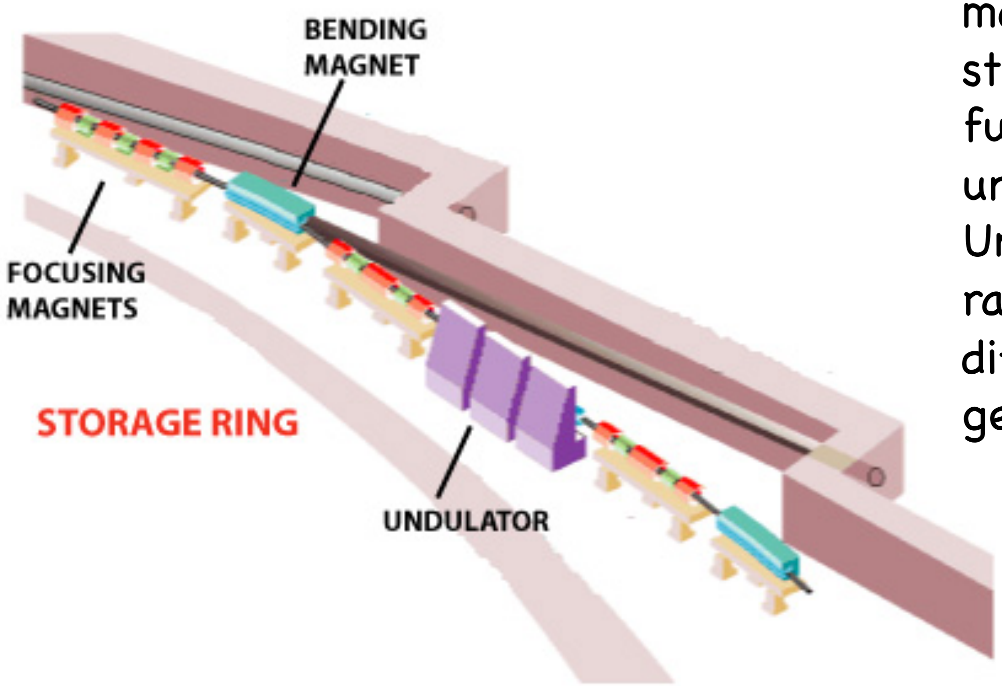
<u>Disadvantages:</u>
- limited coverage of hard x-rays
- not as bright as undulator

Brightness Comparison



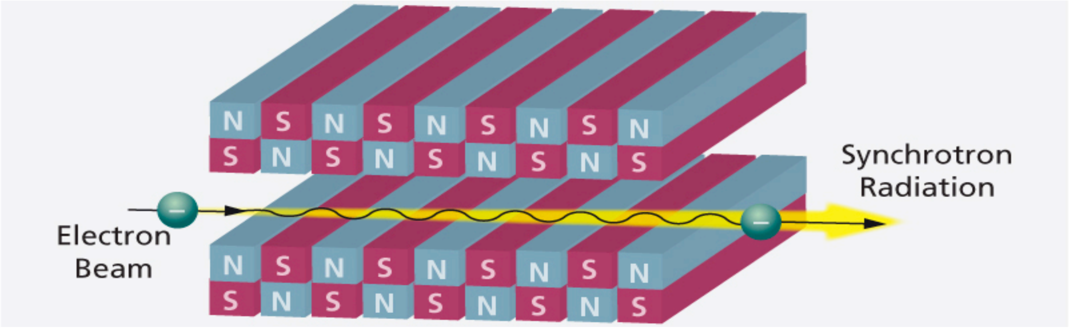
Courtesy of D. Attwood

Undulator/Wiggler

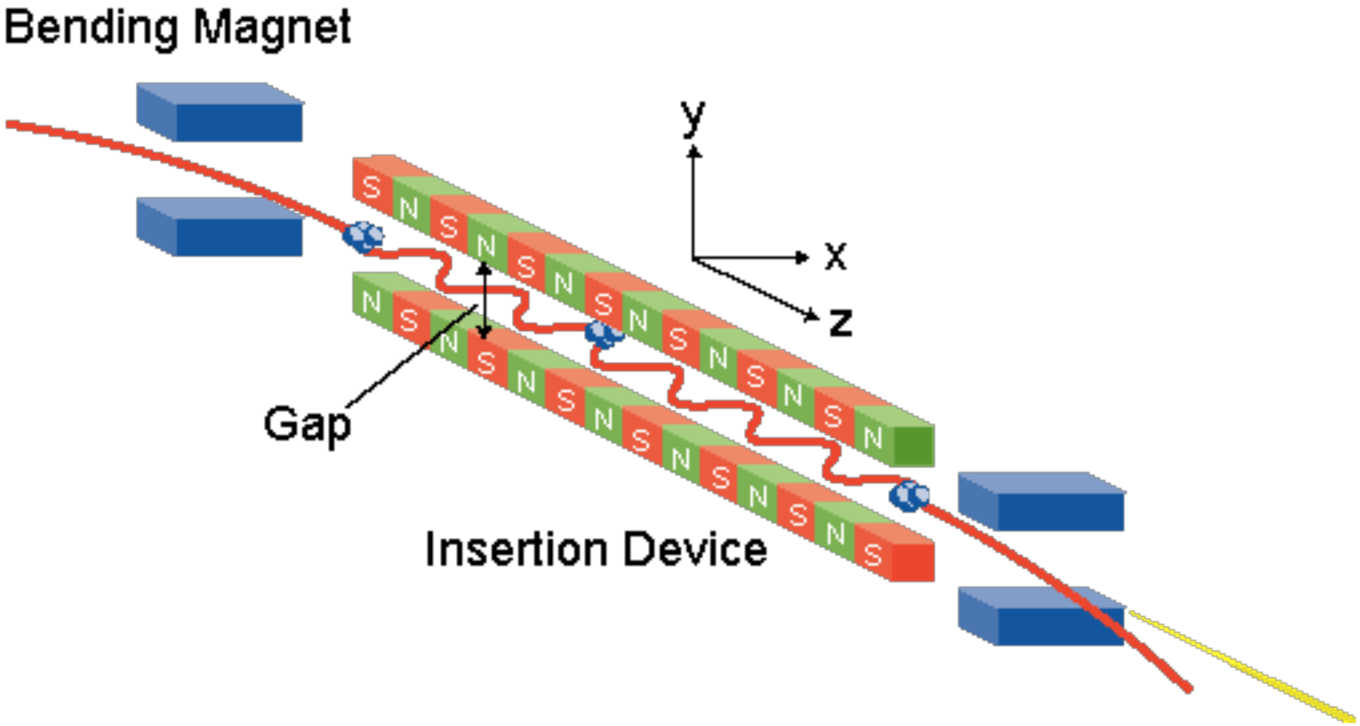


In addition to the SR from dipoles, modern light sources has many long straight section with zero dispersion function. They frequently used for undulators and wigglers.
Undulators and wigglers collect radiation from multiple poles: the difference is in coherence of generated radiation

Example: NSLS II
# of DBA cells: 30
# of 5m straights: 15
# of 8m straights: 15

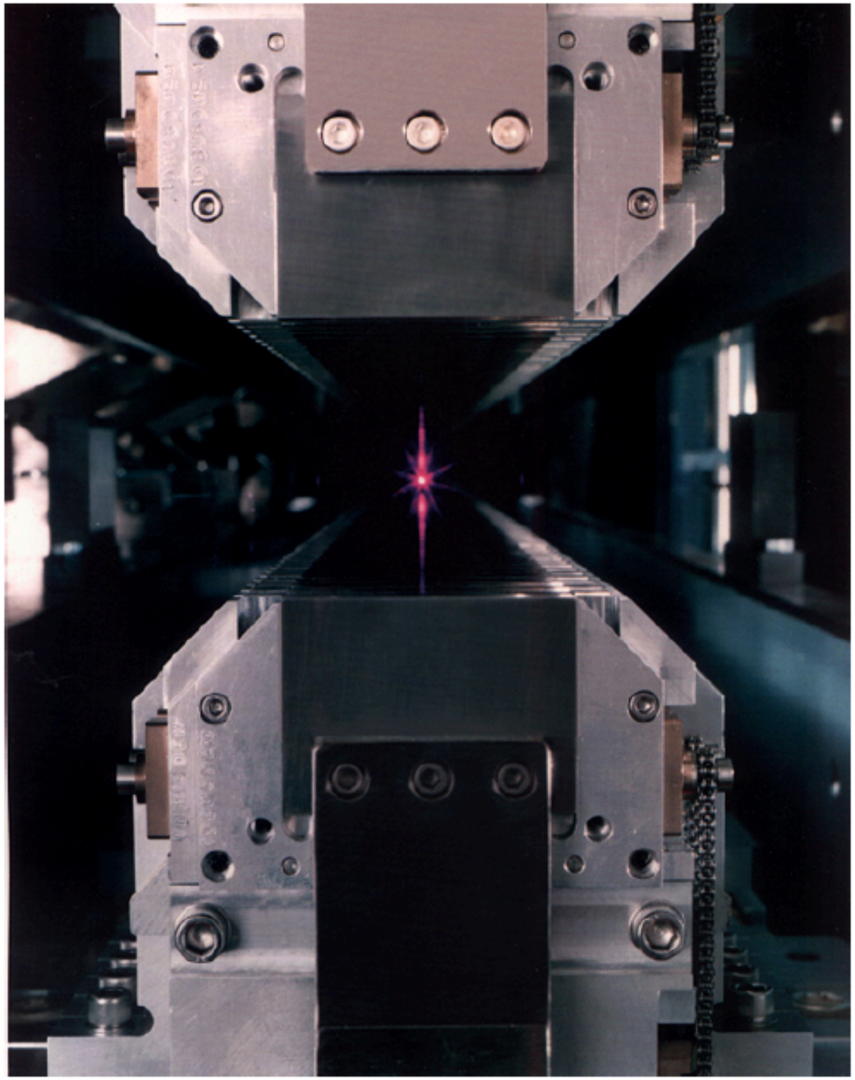


Radiation from Undulator/Wiggler



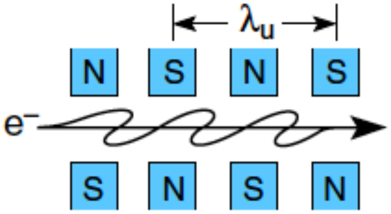
They are 'insertion devices' in straight sections. Modern accelerators provides many long straight sections.

An Undulator Up Close



ALS U5 undulator, beamline 7.0, N = 89, λ_u = 50 mm

Laboratory Frame of Reference

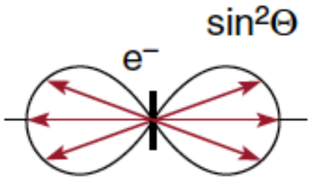


$E = \gamma mc^2$

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

N = # periods

Frame of Moving e⁻



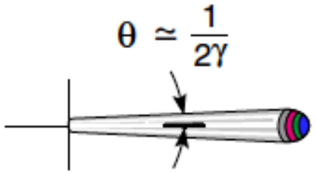
e⁻ radiates at the Lorentz contracted wavelength:

$$\lambda' = \frac{\lambda_u}{\gamma}$$

Bandwidth:

$$\frac{\lambda'}{\Delta\lambda'} \simeq N$$

Frame of Observer



Doppler shortened wavelength on axis:

$$\lambda = \lambda'\gamma(1 - \beta\cos\theta)$$

$$\lambda = \frac{\lambda_u}{2\gamma^2}(1 + \gamma^2\theta^2)$$

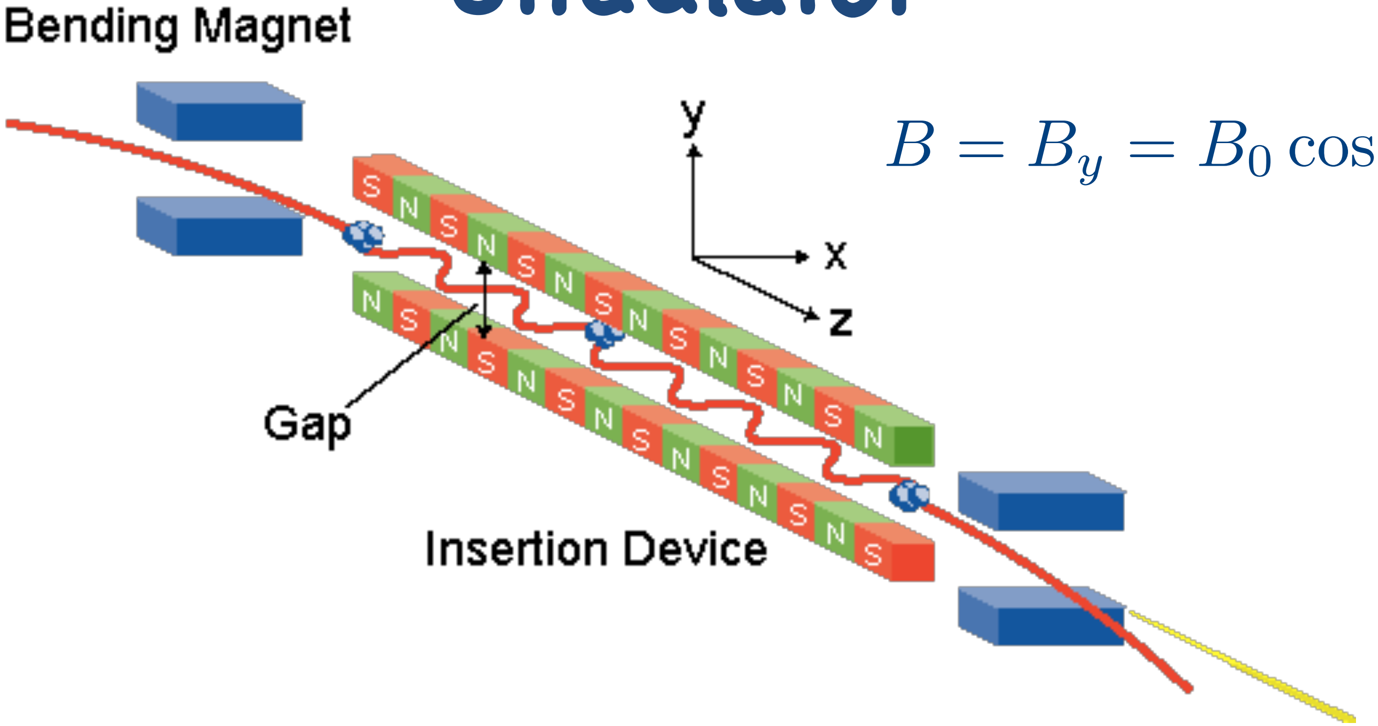
Accounting for transverse motion due to the periodic magnetic field:

$$\boxed{\lambda = \frac{\lambda_u}{2\gamma^2}\left(1 + \frac{K^2}{2} + \gamma^2\theta^2\right)}$$

where $K = eB_0\lambda_u/2\pi mc$

Courtesy of D. Attwood

Electron Motion inside planar Undulator



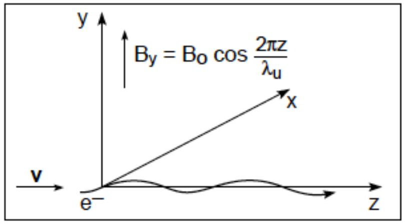
$$B = B_y = B_0 \cos \frac{2\pi z}{\lambda_u}$$

$$v_x = \frac{eB_0\lambda_u}{2\pi m\gamma} \sin \frac{2\pi z}{\lambda_u} \equiv \frac{Kc}{\gamma} \sin \frac{2\pi z}{\lambda_u}$$

$$K = \frac{eB_0\lambda_u}{2\pi mc} = 0.934 B_0[T]\lambda_u[cm]$$

Formulas (SI, D. Attwood)

Magnetic fields in the periodic undulator cause the electrons to oscillate and thus radiate. These magnetic fields also slow the electrons axial (z) velocity somewhat, reducing both the Lorentz contraction and the Doppler shift, so that the observed radiation wavelength is not quite so short. The force equation for an electron is



$$\frac{d\mathbf{p}}{dt} = -e(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \qquad (5.16)$$

where $\mathbf{p} = \gamma m\mathbf{v}$ is the momentum. The radiated fields are relatively weak so that

$$\frac{d\mathbf{p}}{dt} \simeq -e(\mathbf{v} \times \mathbf{B})$$

Taking to first order $v \simeq v_z$, motion in the x-direction is

$$m\gamma \frac{d\mathrm{v}_x}{dt} = +e\mathrm{v}_z B_y$$

$$m\gamma \frac{d\mathrm{v}_x}{dt} = e\frac{dz}{dt} \cdot B_0 \cos\left(\frac{2\pi z}{\lambda_u}\right) \qquad (0 \le z \le N\lambda_u)$$

$$m\gamma\, d\mathrm{v}_x = e\, dz\, B_0 \cos\left(\frac{2\pi z}{\lambda_u}\right)$$

$$m\gamma\, d\mathrm{v}_x = e\, dz\, B_0 \cos\left(\frac{2\pi z}{\lambda_u}\right)$$

integrating both sides

$$m\gamma \mathrm{v}_x = eB_0 \frac{\lambda_u}{2\pi} \int \cos\left(\frac{2\pi z}{\lambda_u}\right) \cdot d\left(\frac{2\pi z}{\lambda_u}\right)$$

$$m\gamma \mathrm{v}_x = \frac{eB_0\lambda_u}{2\pi} \sin\left(\frac{2\pi z}{\lambda_u}\right) \qquad (5.17)$$

$$\boxed{\mathrm{v}_x = \frac{Kc}{\gamma} \sin\left(\frac{2\pi z}{\lambda_u}\right)} \qquad (5.19)$$

$$\boxed{K \equiv \frac{eB_0\lambda_u}{2\pi mc} = 0.9337 B_0(\mathrm{T})\lambda_u(\mathrm{cm})} \qquad (5.18)$$

is the non-dimensional "magnetic deflection parameter."
The "deflection angle", θ, is

$$\theta = \frac{\mathrm{v}_x}{\mathrm{v}_z} \simeq \frac{\mathrm{v}_x}{c} = \frac{K}{\gamma} \sin k_u z$$

In a magnetic field γ is a constant; to first order the electron neither gains nor looses energy.

$$\gamma \equiv \frac{1}{\sqrt{1 - \frac{\mathrm{v}^2}{c^2}}} = \frac{1}{\sqrt{1 - \frac{\mathrm{v}_x^2 + \mathrm{v}_z^2}{c^2}}}$$

thus

$$\frac{\mathrm{v}_z^2}{c^2} = 1 - \frac{1}{\gamma^2} - \frac{\mathrm{v}_x^2}{c^2} \qquad (5.22)$$

$$\frac{\mathrm{v}_z^2}{c^2} = 1 - \frac{1}{\gamma^2} - \frac{K^2}{\gamma^2} \sin^2\left(\frac{2\pi z}{\lambda_u}\right)$$

Taking the square root, to first order in the small parameter K/γ

$$\frac{\mathrm{v}_z}{c} = 1 - \frac{1}{2\gamma^2} - \frac{K^2}{2\gamma^2} \sin^2\left(\frac{2\pi z}{\lambda_u}\right) \qquad (5.23a)$$

Using the double angle formula $\sin^2 k_u z = (1 - \cos 2k_u z)/2$, where $k_u = 2\pi/\lambda_u$,

$$\frac{\mathrm{v}_z}{c} = 1 \underbrace{- \frac{1 + K^2/2}{2\gamma^2}}_{\substack{\text{Reduced} \\ \text{axial velocity}}} + \underbrace{\frac{K^2}{4\gamma^2} \cos\left(2 \cdot \frac{2\pi z}{\lambda_u}\right)}_{\substack{\text{A double frequency} \\ \text{component of the motion}}}$$

The first two terms show the reduced axial velocity due to the finite magnetic field (K). The last term indicates the presence of harmonic motion, and thus harmonic frequencies of radiation.

Averaging the z-component of velocity over a full cycle (or N full cycles) gives

$$\frac{\bar{\mathrm{v}}_z}{c} = 1 - \frac{1 + K^2/2}{2\gamma^2} \qquad (5.25)$$

We can use this to define an effective Lorentz factor γ* in the axial direction

$$\gamma^* \equiv \frac{\gamma}{\sqrt{1 + K^2/2}} \qquad (5.26)$$

As a consequence, the observed wavelength in the laboratory frame of reference is modified from Eq. (5.12), taking the form

$$\lambda = \frac{\lambda_u}{2\gamma^{*2}}(1 + \gamma^{*2}\theta^2)$$

that is, the Lorentz contraction and relativistic Doppler shift now involve γ* rather than γ

$$\lambda = \frac{\lambda_u}{2\gamma^2}\left(1 + \frac{K^2}{2}\right)\left(1 + \frac{\gamma^2}{1 + K^2/2}\theta^2\right)$$

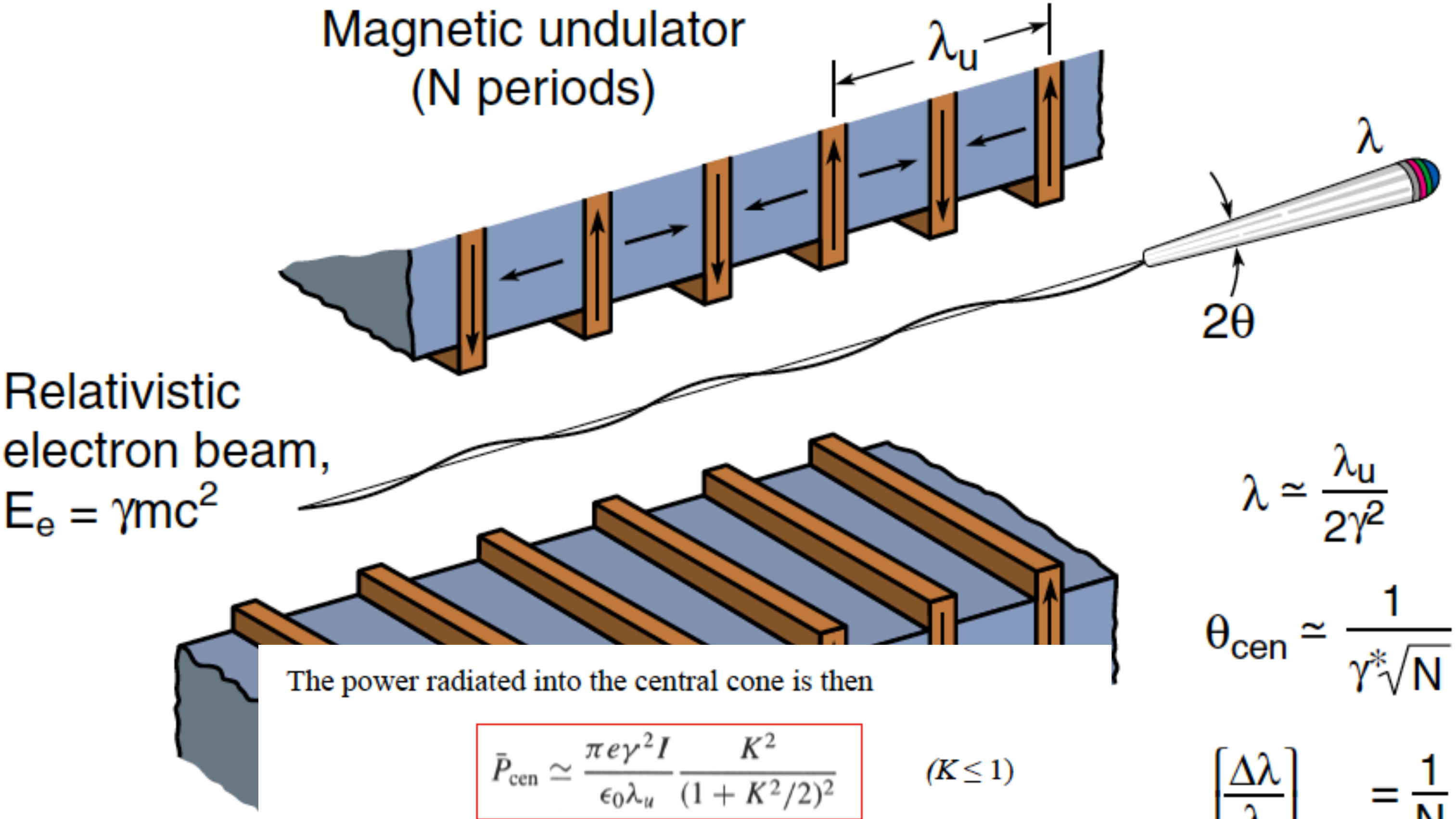
$$\boxed{\lambda = \frac{\lambda_u}{2\gamma^2}\left(1 + \frac{K^2}{2} + \gamma^2\theta^2\right)} \qquad (5.28)$$

where $K \equiv e B_o \lambda_u/2\pi mc$. This is the *undulator equation*, which describes the generation of short (x-ray) wavelength radiation by relativistic electrons traversing a periodic magnet structure, accounting for magnetic tuning (K) and off-axis (γθ) radiation. In practical units

$$\lambda(\mathrm{nm}) = \frac{1.306\lambda_u(\mathrm{cm})\left(1 + \frac{K^2}{2} + \gamma^2\theta^2\right)}{E_e^2(\mathrm{GeV})} \qquad (5.29a)$$

Courtesy of D. Attwood

Narrow Cone Undulator Radiation, Generated by Relativistic Electrons Traversing a Periodic Magnet Structure



$$\lambda \simeq \frac{\lambda_u}{2\gamma^2}$$

$$\theta_{cen} \simeq \frac{1}{\gamma^* \sqrt{N}}$$

$$\left[\frac{\Delta\lambda}{\lambda}\right]_{cen} = \frac{1}{N}$$

The power radiated into the central cone is then

$$\boxed{\bar{P}_{\text{cen}} \simeq \frac{\pi e \gamma^2 I}{\epsilon_0 \lambda_u} \frac{K^2}{(1 + K^2/2)^2}} \qquad (K \leq 1)$$

or

$$\bar{P}_{\text{cen}} = (5.69 \times 10^{-6}\ \text{W}) \frac{\gamma^2 I(\text{A})}{\lambda_u(\text{cm})} \frac{K^2}{(1 + K^2/2)^2}$$

Courtesy of D. Attwood

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Corrections to $\bar{P}_{\text{cen}}$ for Finite K

Our formula for calculated power in the central radiation cone ($\theta_{\text{cen}} = 1/\gamma^*\sqrt{N}$, $\Delta\lambda/\lambda = 1/N$)

$$\bar{P}_{\text{cen}} \simeq \frac{\pi e \gamma^2 I}{\epsilon_0 \lambda_u} \frac{K^2}{(1 + K^2/2)^2} \qquad (5.39)$$

is strictly valid for $K << 1$. This restriction is due to our neglect of K^2 terms in the axial velocity v_z. The $\bar{P}_{\text{cen}}$ formula, however, indicates a peak power at $K = \sqrt{2}$, suggesting that we explore extension of this very useful analytic result to somewhat higher K values. Kim* has studied undulator radiation for arbitrary K and finds an additional multiplicative factor, $f(K)$, which accounts for energy transfer to higher harmonics:

$$\boxed{\bar{P}_{\text{cen}} = \frac{\pi e \gamma^2 I}{\epsilon_0 \lambda_u} \frac{K^2}{(1 + K^2/2)^2} f(K)} \qquad (5.41a)$$

where

$$f(K) = [J_0(x) - J_1(x)]^2 \qquad (5.40a)$$

and

$$x = K^2/4(1 + K^2/2)$$

$$f(K) = 1 - x - \frac{x^2}{4} + \frac{3x^3}{8} + \cdots \qquad (5.40b)$$

K	x	$f(K)$
0	0	1.000
0.5	0.0556	0.944
1.0	0.1667	0.828
$\sqrt{2}$	0.2500	0.740
1.5	0.2647	0.725
2.0	0.3333	0.653
2.5	0.3788	0.606

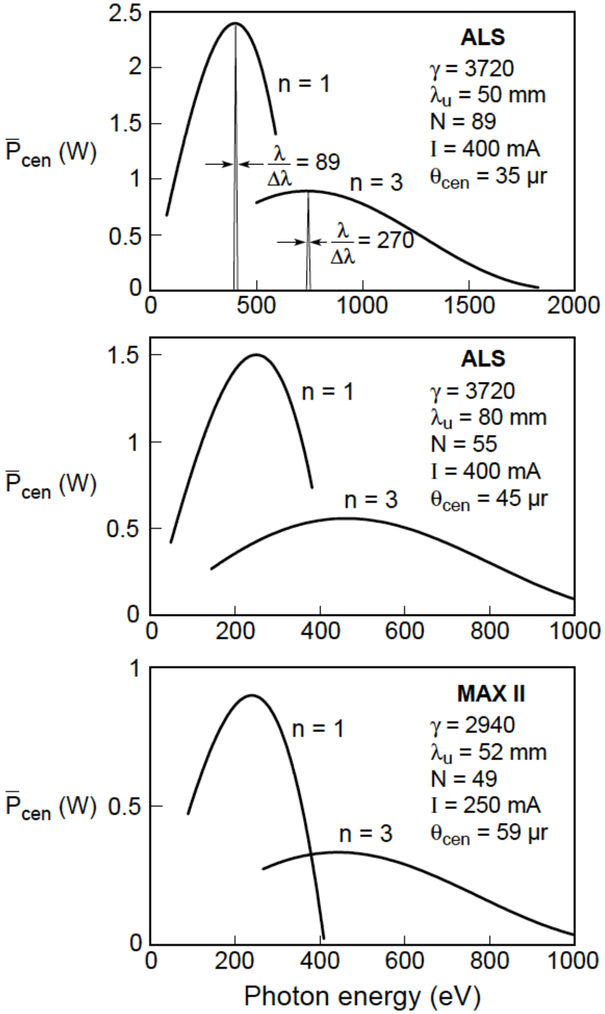
---

* K.-J. Kim, "Characteristics of Synchrotron Radiation", pp. 565-632 in *Physics of Particle Accelerators* (AIP, New York, 1989), M. Month and M. Dienes, Editors.

Also see: P.J. Duke, *Synchrotron Radiation* (Oxford Univ. Press, UK, 2000).

A. Hofmann, "The Physics of Synchrotron Radiation" (Cambridge Univ. Press, 2004).

Power in the Central Radiation Cone For Three Soft X-Ray Undulators



$$\theta_{cen} = \frac{1}{\gamma^* \sqrt{N}}$$

$$\left[\frac{\Delta\lambda}{\lambda}\right]_1 = \frac{1}{N}$$

$$\left[\frac{\Delta\lambda}{\lambda}\right]_3 = \frac{1}{3N}$$

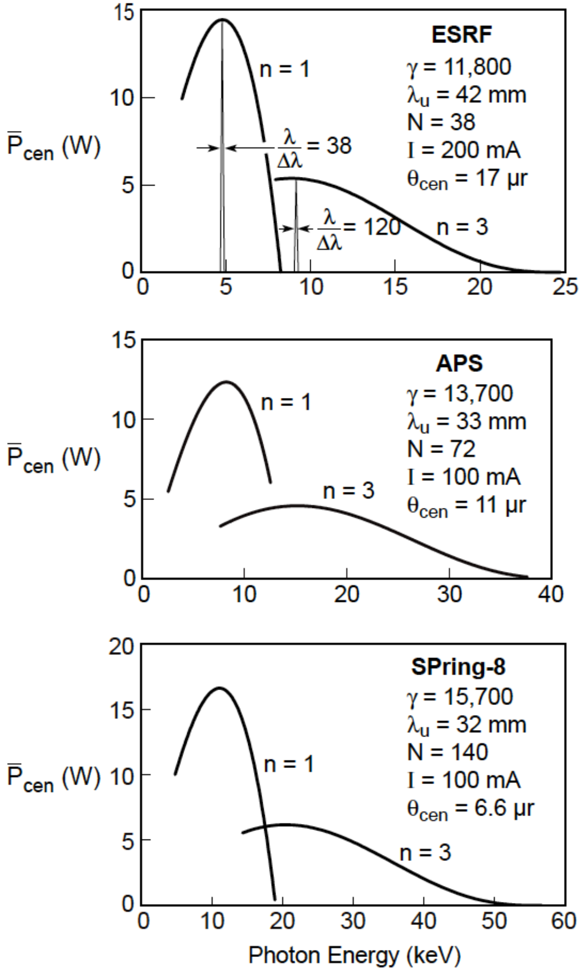
$$\hbar\omega_o = 4\pi\hbar\gamma^2 c / \lambda_u$$

$$\boxed{\bar{P}_{cen} = \frac{2\pi e\gamma^2 I}{\epsilon_0 \lambda_u} \cdot \frac{\hbar\omega}{\hbar\omega_0}\left(1 - \frac{\hbar\omega}{\hbar\omega_0}\right) f(\hbar\omega/\hbar\omega_0)}$$

$$\bar{P}_{cen} = (1.14 \times 10^{-5}\ \text{W})\frac{\gamma^2 I(\text{A})}{\lambda_u(\text{cm})} \cdot \frac{\hbar\omega}{\hbar\omega_0}\left(1 - \frac{\hbar\omega}{\hbar\omega_0}\right) f(\hbar\omega/\hbar\omega_0)$$

$$f(\hbar\omega/\hbar\omega_0) \simeq \frac{7}{16} + \frac{5}{8}\frac{\hbar\omega}{\hbar\omega_0} - \frac{1}{16}\left(\frac{\hbar\omega}{\hbar\omega_0}\right)^2 + \cdots$$

For Three X-Ray Undulators



$$\theta_{cen} = \frac{1}{\gamma^* \sqrt{N}}$$

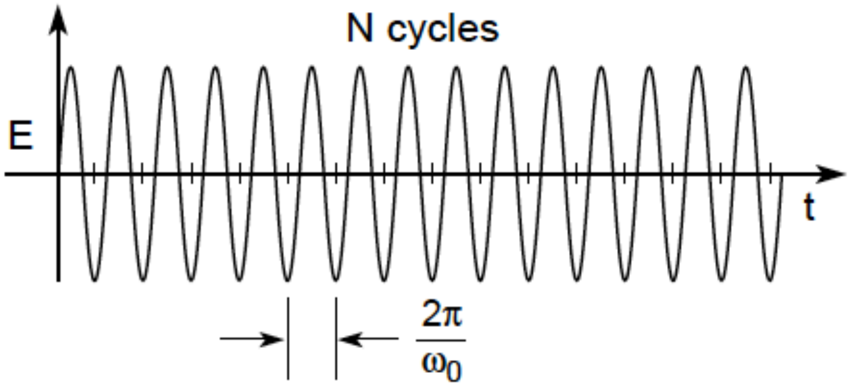
$$\left[\frac{\Delta\lambda}{\lambda}\right]_1 = \frac{1}{N}$$

$$\left[\frac{\Delta\lambda}{\lambda}\right]_3 = \frac{1}{3N}$$

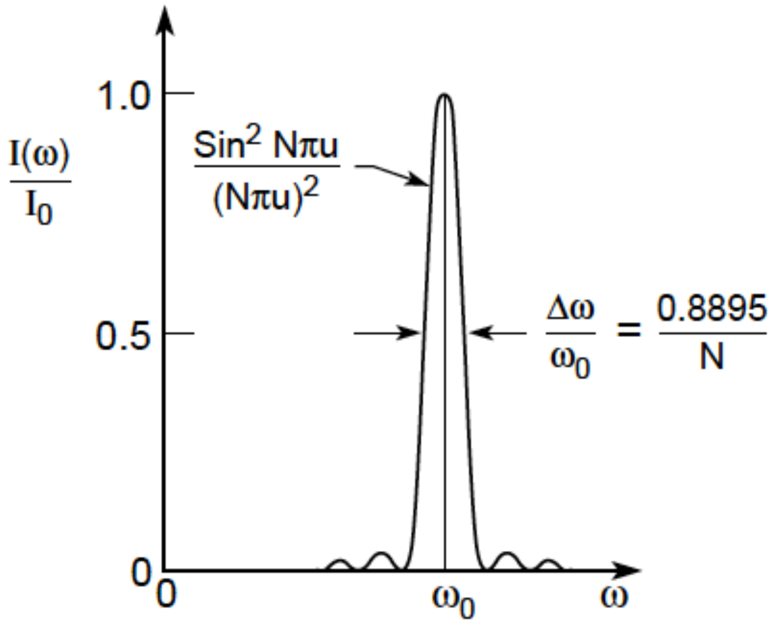
Courtesy of D. Attwood

Spectral Bandwidth of Undulator Radiation from a Single Electron

Radiated Wavetrain



Spectral Distribution



$$\lambda_x = \frac{\lambda_u}{2\gamma^2}\left(1 + \frac{K^2}{2} + \gamma^2\theta^2\right)$$

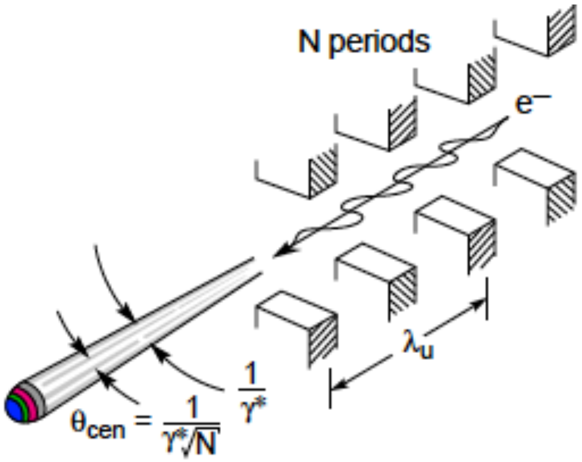
$$\bar{P}_{cen} = \frac{\pi e \gamma^2 I}{\epsilon_0 \lambda_u} \, \frac{K^2}{\left(1 + \frac{K^2}{2}\right)^2} \, f(K)$$

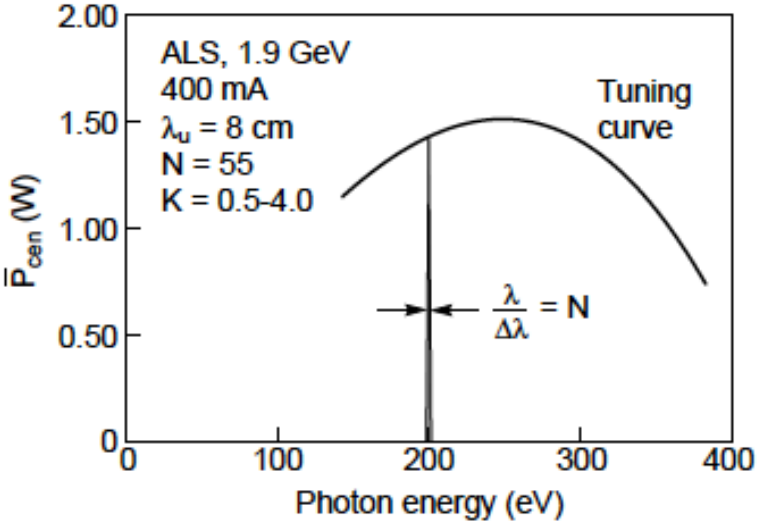
$$\theta_{cen} = \frac{1}{\gamma^* \sqrt{N}}$$

$$\left(\frac{\Delta\lambda}{\lambda}\right)_{cen} = \frac{1}{N}$$

$$K = \frac{e B_0 \lambda_u}{2\pi m_0 c}$$

$$\gamma^* = \gamma / \sqrt{1 + \frac{K^2}{2}}$$

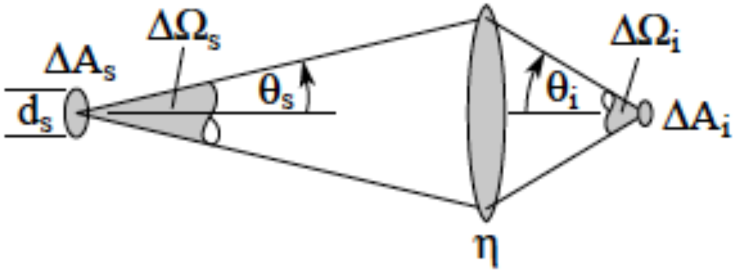




Brightness and Spectral Brightness

Brightness is defined as radiated power per unit area and per unit solid angle at the source:

$$\boxed{B = \frac{\Delta P}{\Delta A \cdot \Delta \Omega}} \qquad (5.57)$$

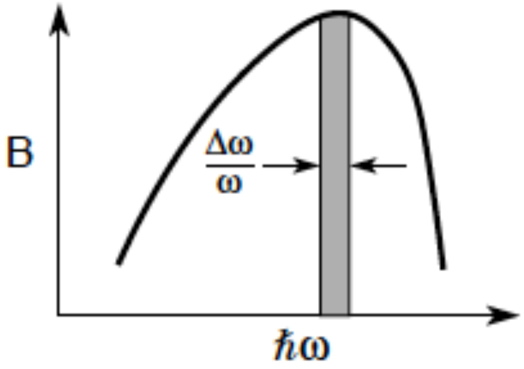


Brightness is a conserved quantity in perfect optical systems, and thus is useful in designing beamlines and synchrotron radiation experiments which involve focusing to small areas.

Perfect optical system:
$\Delta A_s \cdot \Delta\Omega_s = \Delta A_i \cdot \Delta\Omega_i$; $\eta = 100\%$

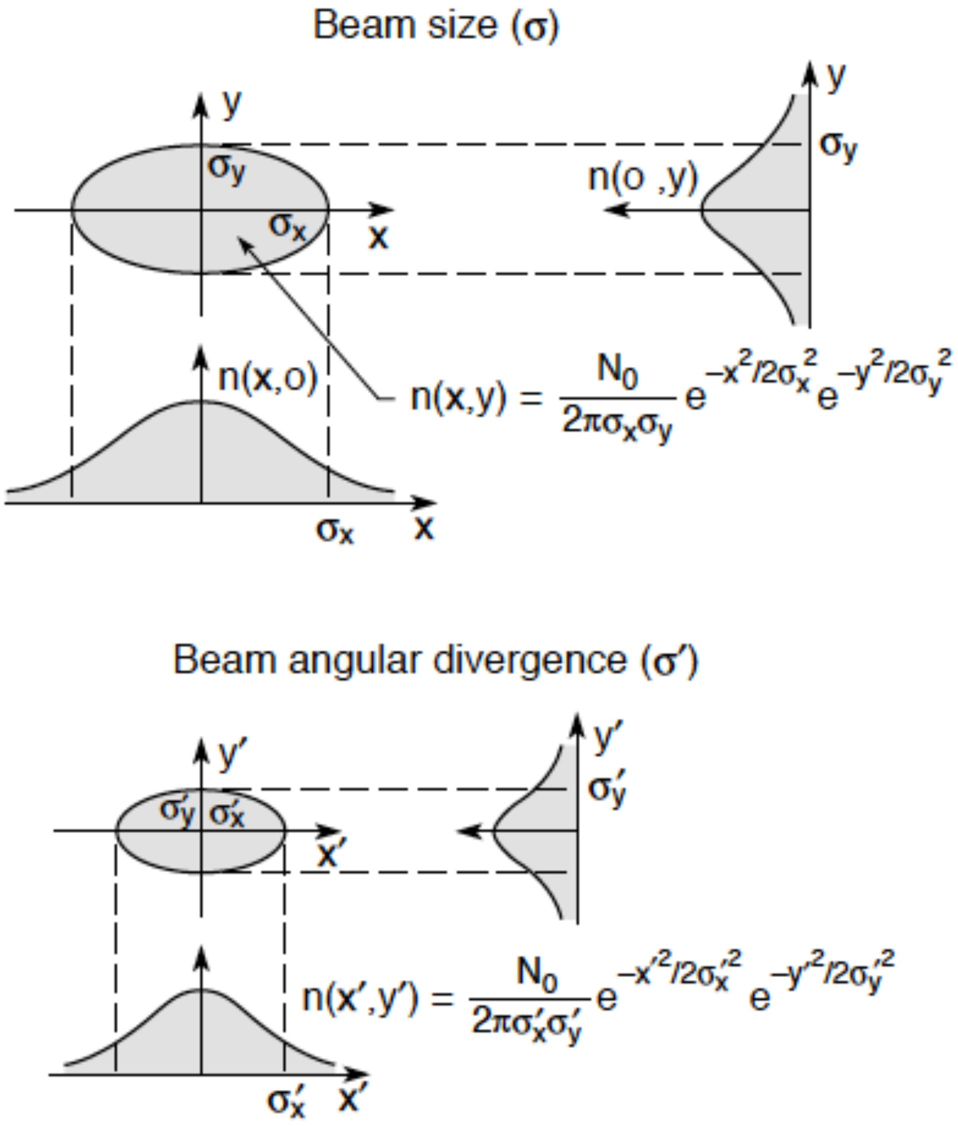
Spectral brightness is that portion of the brightness lying within a relative spectral bandwidth $\Delta\omega/\omega$:

$$\boxed{B_{\Delta\omega/\omega} = \frac{\Delta P}{\Delta A \cdot \Delta \Omega \cdot \Delta\omega/\omega}} \qquad (5.58)$$



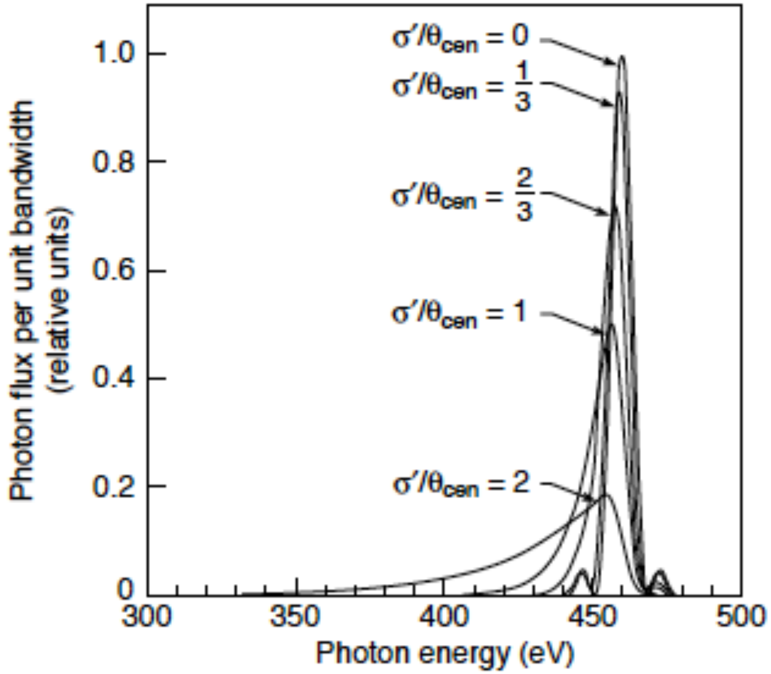
Courtesy of D. Attwood

What defines Brightness?



$$n(x,y) = \frac{N_0}{2\pi\sigma_x\sigma_y} e^{-x^2/2\sigma_x^2} e^{-y^2/2\sigma_y^2}$$

$$n(x',y') = \frac{N_0}{2\pi\sigma'_x\sigma'_y} e^{-x'^2/2\sigma'^2_x} e^{-y'^2/2\sigma'^2_y}$$



Preserving the spectral line shape of undulator radiation requires

$$\sigma'^2 \ll \theta_{\text{cen}}^2 \tag{5.55b}$$

Define effective, or total central cone half-angles

$$\theta_{Tx} = \sqrt{\theta_{\text{cen}}^2 + \sigma_x'^2} \quad \text{and} \quad \theta_{Ty} = \sqrt{\theta_{\text{cen}}^2 + \sigma_y'^2} \tag{5.56}$$

Courtesy of D. Attwood

Spectral Brightness of Undulator Radiation

The Synchrotron radiation community prefers to express spectral brightness in units of photons/sec, rather than power, and has standardized on a relative spectral bandwidth of $\Delta\omega/\omega = 10^{-3}$, or 0.1% BW. To obtain a relationship for spectral brightness of undulator radiation we can use our expression for $\bar{P}_{\text{cen}}$, radiated into a solid angle $\Delta\Omega = \pi\theta_{\text{cen}}^2 = \pi\theta_{Tx}\theta_{Ty}$, from an elliptically shaped source area of $\Delta A = \pi\sigma_x\sigma_y$, and within a relative spectral bandwidth $\Delta\omega/\omega = 1/N$. Defining the photon flux in the central radiation cone as

$$\bar{F}_{\text{cen}} = \frac{\bar{P}_{\text{cen}}}{\hbar\omega/\text{photon}} \tag{5.59}$$

$$\bar{B}_{\Delta\omega/\omega} = \frac{\bar{F}_{\text{cen}}}{\Delta A \cdot \Delta\Omega \cdot N^{-1}} = \frac{\bar{F}_{\text{cen}} \cdot (N/1000)}{\Delta A \cdot \Delta\Omega \cdot (0.1\%\text{BW})} \tag{5.60}$$

on-axis

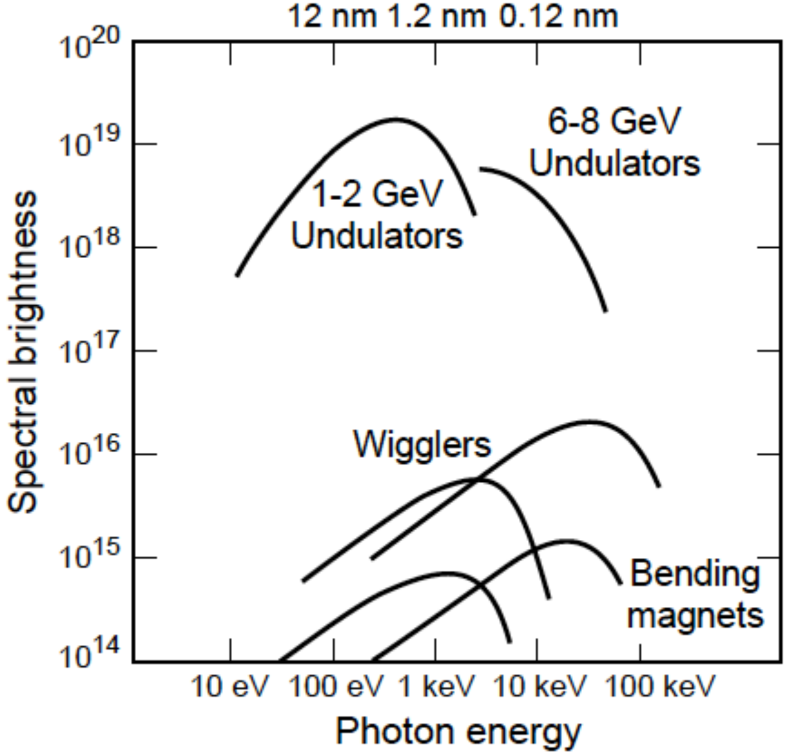
$$\bar{B}_{\Delta\omega/\omega}(0) = \frac{\bar{F}_{\text{cen}} \cdot (N/1000)}{2\pi^2\sigma_x\sigma_y\theta_{Tx}\theta_{Ty}(0.1\%\text{BW})} \tag{5.64}$$

or

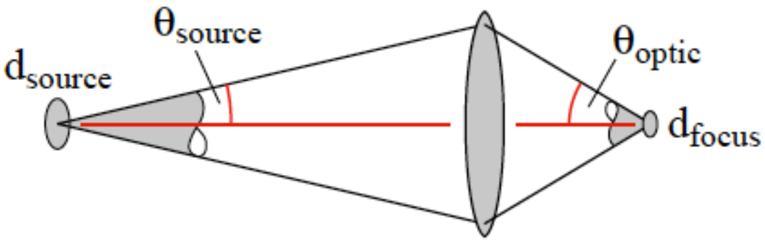
$$\boxed{\bar{B}_{\Delta\omega/\omega}(0) = \frac{7.25 \times 10^6\gamma^2 N^2 I(\text{A})}{\sigma_x(\text{mm})\sigma_y(\text{mm})\left(1 + \dfrac{\sigma_x'^2}{\theta_{\text{cen}}^2}\right)^{1/2}\left(1 + \dfrac{\sigma_y'^2}{\theta_{\text{cen}}^2}\right)^{1/2}} \cdot \frac{K^2 f(K)}{\left(1 + K^2/2\right)^2}\ \frac{\text{photons/s}}{\text{mm}^2\text{mrad}^2(0.1\%\text{BW})} \quad (5.65)}$$

Assumes $\sigma'^2 \ll \theta_{\text{cen}}^2$. Note the N^2 factor.

Spectral Brightness is Useful for Experiments that Involve Spatially Resolved Studies



- Brightness is conserved (in lossless optical systems)



$$d_{source} \cdot \theta_{source} = d_{focus} \cdot \theta_{optic}$$

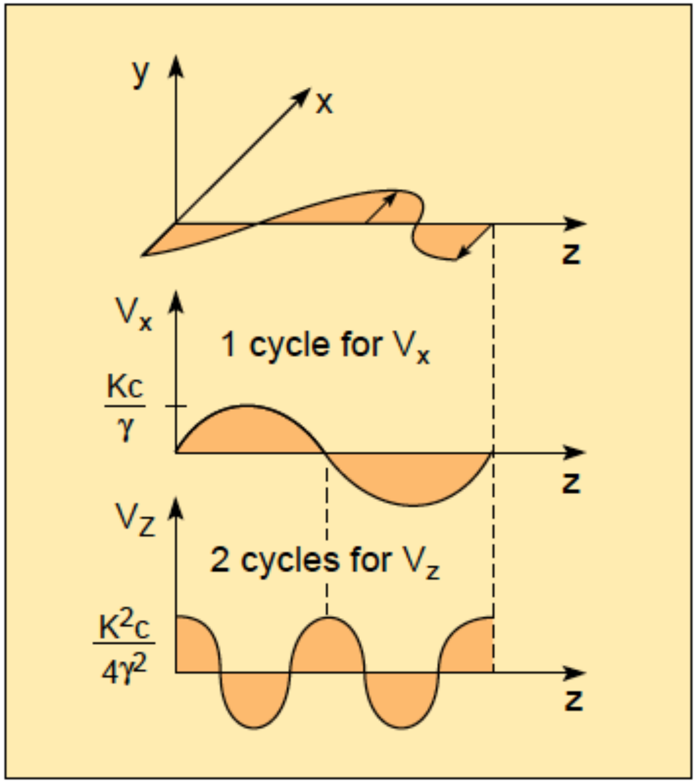
Smaller after focus

Large in a focusing optic

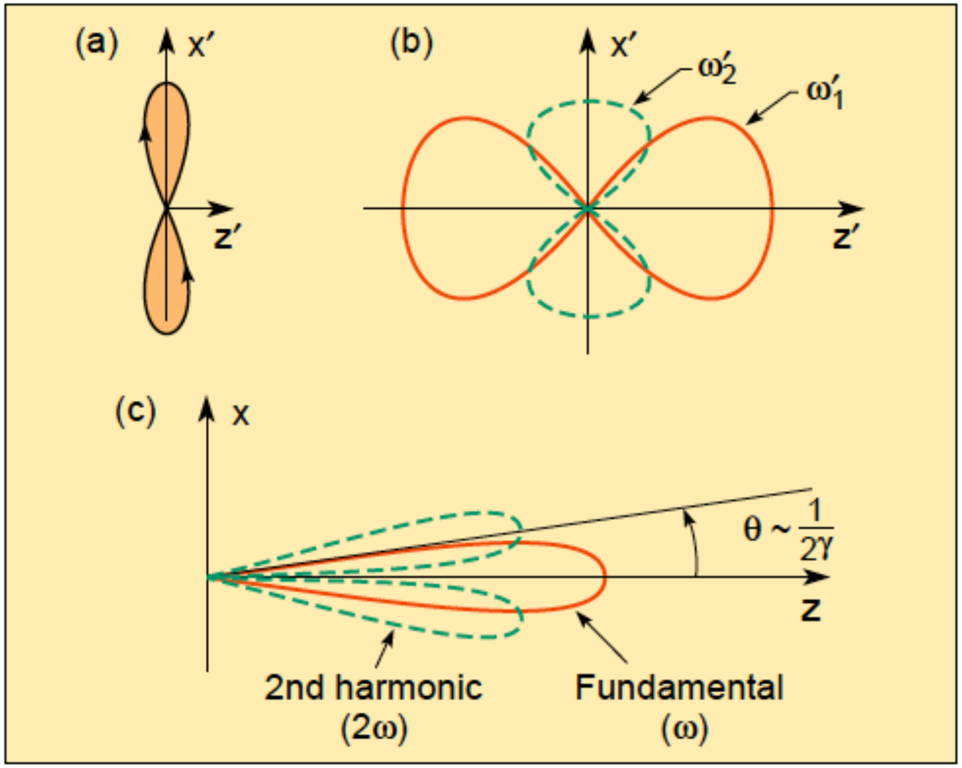
- Starting with many photons in a small source area and solid angle, permits high photon flux in an even smaller area

Comments on Undulator Harmonics

First and second harmonic motions



Radiation patterns in the electron and laboratory frames



$$\lambda_n = \frac{\lambda_u}{2\gamma^2 n}\left(1 + \frac{K^2}{2} + \gamma^2\theta^2\right) \tag{5.30}$$

$$\left(\frac{\Delta\lambda}{\lambda}\right)_n = \frac{1}{nN} \tag{5.31}$$

Recall that the axial velocity has a double frequency component

$$v_z = c\left[1 - \frac{1+K^2/2}{2\gamma^2} + \frac{K^2}{4\gamma^2}\cos(2k_u z)\right]$$

which in the frame of reference moving with the electrons, gives

$$z'(t') \simeq \frac{K^2}{8k'_u}\sin 2\omega'_u t' \tag{5.70}$$

where $k'_u = \gamma^* k_u$ and $\omega'_u = \gamma^* \omega_u$. The transverse motion in this frame is

$$x'(t') \simeq -\frac{K}{k_u \gamma}\cos \omega_u \gamma^* \left(t' + \frac{z'}{c}\right)$$

To a higher degree of accuracy, we now keep the z'/c term

$$x'(t') \simeq -\frac{K}{k'_u}\cos\left(\omega'_u t' + \frac{K^2}{8}\sin 2\omega'_u t'\right) \tag{5.71}$$

for small K

$$x'(t') \simeq -\frac{1}{k'_u}\left[\enclose{circle}{K}\cos \enclose{circle}{\omega'_u} t' + \frac{\enclose{circle}{K^3}}{16}\cos \enclose{circle}{3\omega'_u} t'\right] \tag{5.72}$$

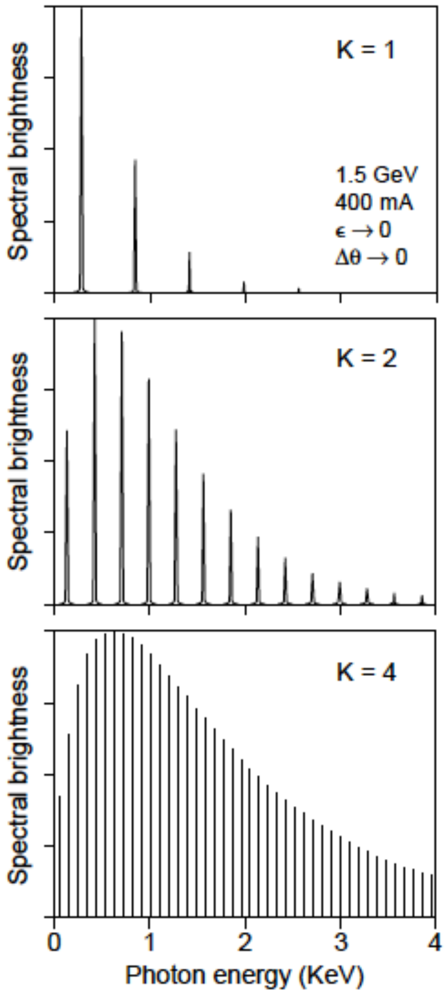
Taking second derivatives to find acceleration, and squaring $|a'(t')|^2$

$$\boxed{\frac{dP'}{d\Omega'} \propto n^4 K^{2n}}$$

Thus harmonics grow very rapidly for K > 1.

The Transition from Undulator Radiation ($K \le 1$) to Wiggler Radiation ($K >> 1$)

$\lambda_u = 5$ cm, $N = 89$



(Courtesy of K.-J. Kim)

Undulator radiation ($K \lesssim 1$)

- Narrow spectral lines
- High spectral brightness
- Partial coherence

$$\lambda = \frac{\lambda_u}{2\gamma^2}\left(1 + \frac{K^2}{2} + \gamma^2\theta^2\right)$$

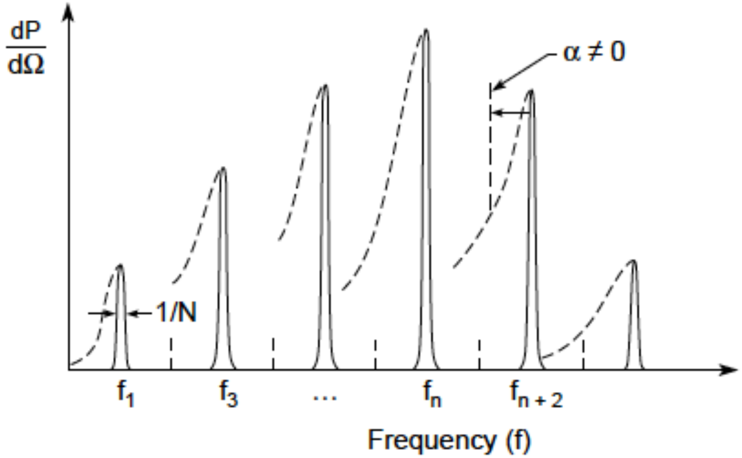
$$K = \frac{eB_o\lambda_u}{2\pi mc}$$

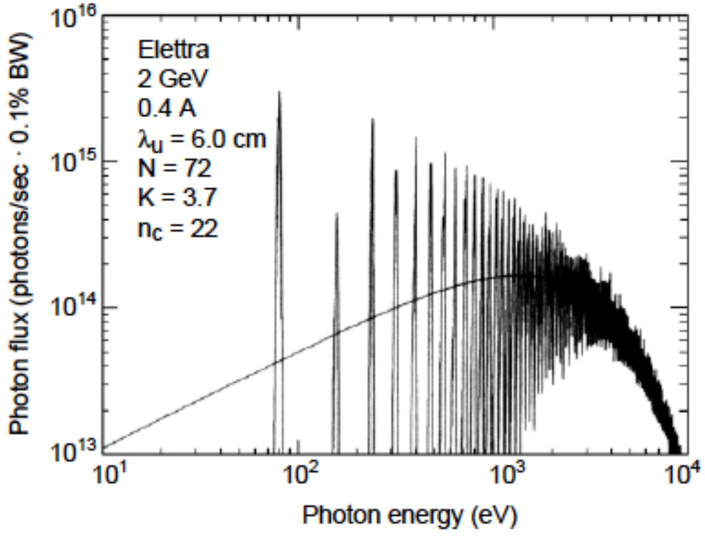
Wiggler radiation ($K >> 1$)

- Higher photon energies
- Spectral continuum
- Higher photon flux (2N)

$$\hbar\omega_c = \frac{3}{2}\,\frac{\hbar\gamma^2 eB_o}{m}$$

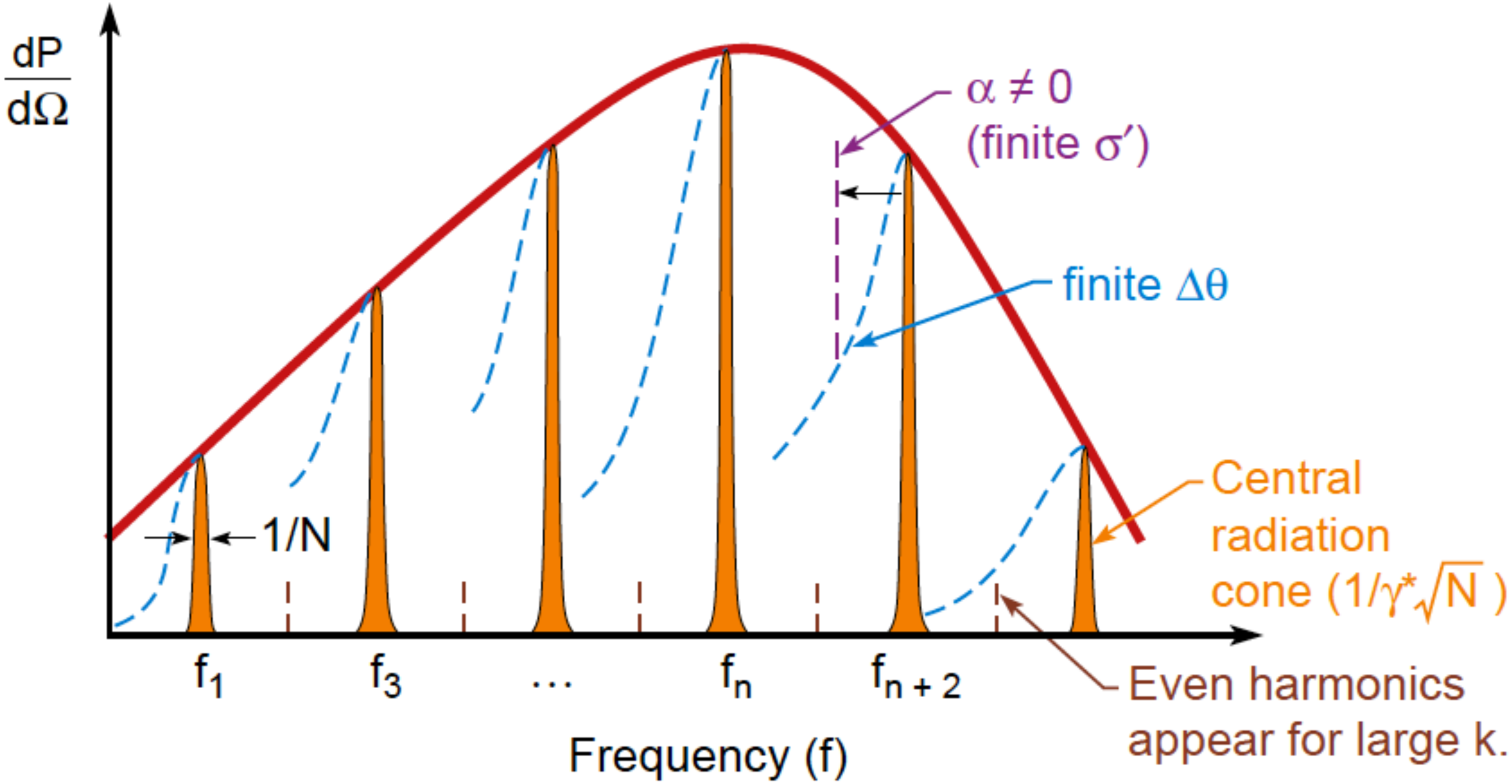
$$n_c = \frac{3K}{4}\left(1 + \frac{K^2}{2}\right)$$





(Courtesy of R.P. Walker and B. Diviacco)

For Very Large K >> 1, and Large Dq, a Continuum Emerges



$\alpha \neq 0$ (finite σ')

finite $\Delta\theta$

Central radiation cone ($1/\gamma^*\sqrt{N}$)

Even harmonics appear for large k.

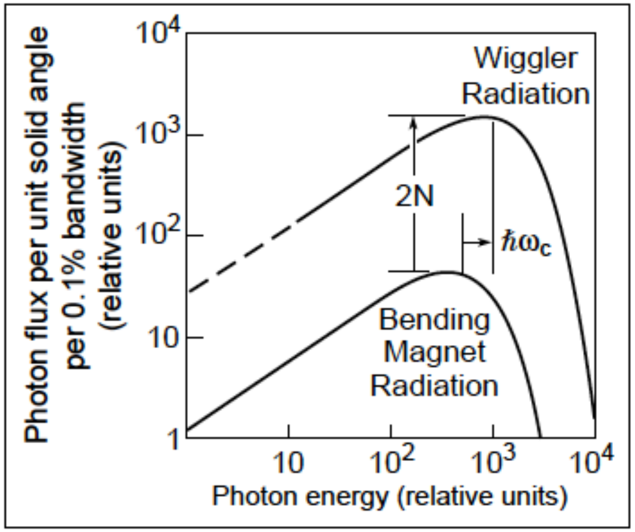
Wiggler Radiation

At very high K >> 1, the radiated energy appears in very high harmonics, and at rather large horizontal angles $\theta \simeq \pm K/\gamma$ (eq. 5.21). Because the emission angles are large, one tends to use larger collection angles, which tends to spectrally merge nearby harmonics. The result is a continuum at very high photon energies, similar to that of bending magnet radiation, but increased by 2N (the number of magnet pole pieces).

$$E_c = \hbar\omega_c = \frac{3e\hbar B\gamma^2}{2m} \quad ; \quad n_c = \frac{3K}{4}\left(1+\frac{K^2}{2}\right) \qquad \text{(5.7a \& 82)}$$

$$\left.\frac{d^2F}{d\theta\, d\Psi\, d\omega/\omega}\right|_0 = 2.65\times 10^{13} N E_e^2(\text{GeV}) I(\text{A}) H_2(E/E_c) \frac{\text{photons/s}}{\text{mrad}^2(0.1\%\text{BW})} \qquad (5.86)$$

$$\frac{d^2F}{d\theta\, d\omega/\omega} = 4.92\times 10^{13} N E_e(\text{GeV}) I(\text{A}) G_1(E/E_c) \frac{\text{photons/s}}{\text{mrad}\cdot(0.1\%\text{BW})} \qquad (5.87)$$



Typical Parameters for Synchrotron Radiation

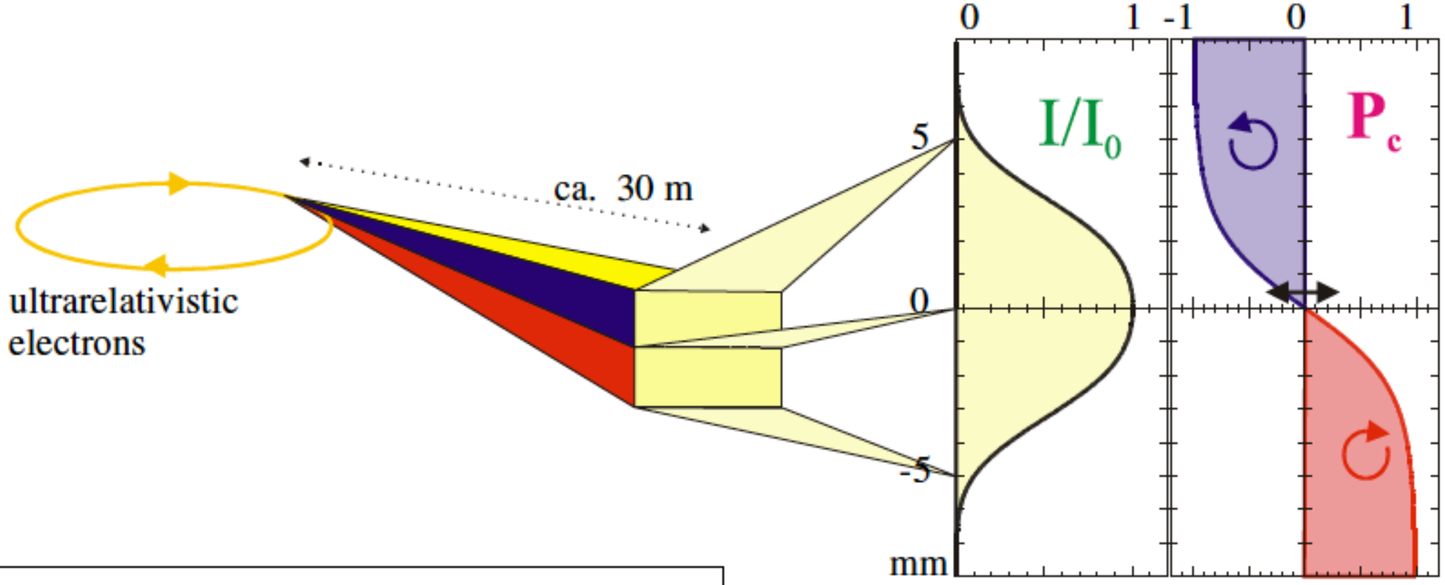
Facility	ALS	ELETTRA	Australian Synchrotron	APS
Electron energy	1.90 GeV	2.0 GeV	3.0 GeV	7.00 GeV
γ	3720	3910	5871	13,700
Current (mA)	400	300	200	100
Circumference (m)	197	259	216	1100
RF frequency (MHz)	500	500	500	352
Pulse duration (FWHM) (ps)	35-70	37	~100	100
Bending Magnet Radiation:				
Bending magnet field (T)	1.27	1.2	1.31	0.599
Critical photon energy (keV)	3.05	3.2	7.84	19.5
Critical photon wavelength	0.407 nm	0.39 nm	1.58 Å	0.636 Å
Bending magnet sources	24	12	28	35
Undulator Radiation:				
Number of straight sections	12	12	14	40
Undulator period (typical) (cm)	5.00	5.6	22.0	3.30
Number of periods	89	81	80	72
Photon energy ($K = 1$, $n = 1$)	457 eV	452 eV	2.59 keV	9.40 keV
Photon wavelength ($K = 1$, $n = 1$)	2.71 nm	2.74 nm	0.478 nm	1.32 Å
Tuning range ($n = 1$)	230-620 eV	2.0-6.7 nm	0.319-0.835 nm	3.5-12 keV
Tuning range ($n = 3$)	690-1800 eV	0.68-2.2 nm	0.106-0.278 nm	10-38 keV
Central cone half-angle ($K = 1$)	35 μrad	35 μrad	23 μrad	11 μrad
Power in central cone ($K = 1$, $n = 1$) (W)	2.3	1.7	6.6	12
Flux in central cone (photons/s)	3.1×10^{16}	2.3×10^{16}	1.6×10^{16}	7.9×10^{15}
σ_x, σ_y (μm)	260, 16	255, 23	320, 16	320, 50
σ'_x, σ'_y (μrad)	23, 3.9	31, 9	34, 6	23, 7
Brightness ($K = 1$, $n = 1$)[a] [(photons/s)/mm^2 · mrad^2 · (0.1%BW)]	2.3×10^{19}	9.9×10^{18}	1.3×10^{19}	5.9×10^{18}
Total power ($K = 1$, all n, all θ) (W)	83	126	476	350
Other undulator periods (cm)	3.65, 8.00, 10.0	8.0, 12.5	6.8, 18.3	2.70, 5.50, 12.8
Wiggler Radiation:				
Wiggler period (typical) (cm)	16.0	14.0	6.1	8.5
Number of periods	19	30	30	28
Magnetic field (maximum) (T)	2.1	1.5	1.9	1.0
K (maximum)	32	19.6	12	7.9
Critical photon energy (keV)	5.1	4.0	11.4 keV	33
Critical photon wavelength	0.24 nm	0.31 nm	0.11 nm	0.38 Å
Total power (max. K) (kW)	13	7.2	9.3	7.4

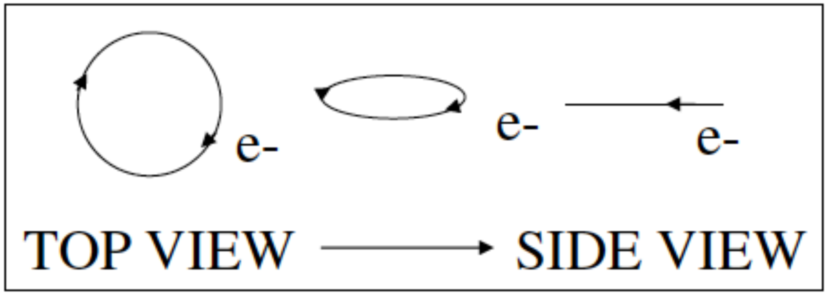
[a] Using Eq. (5.65). See comments following Eq. (5.64) for the case where $\sigma'_{x,y} \simeq \theta_{cen}$.

SR is polarized light

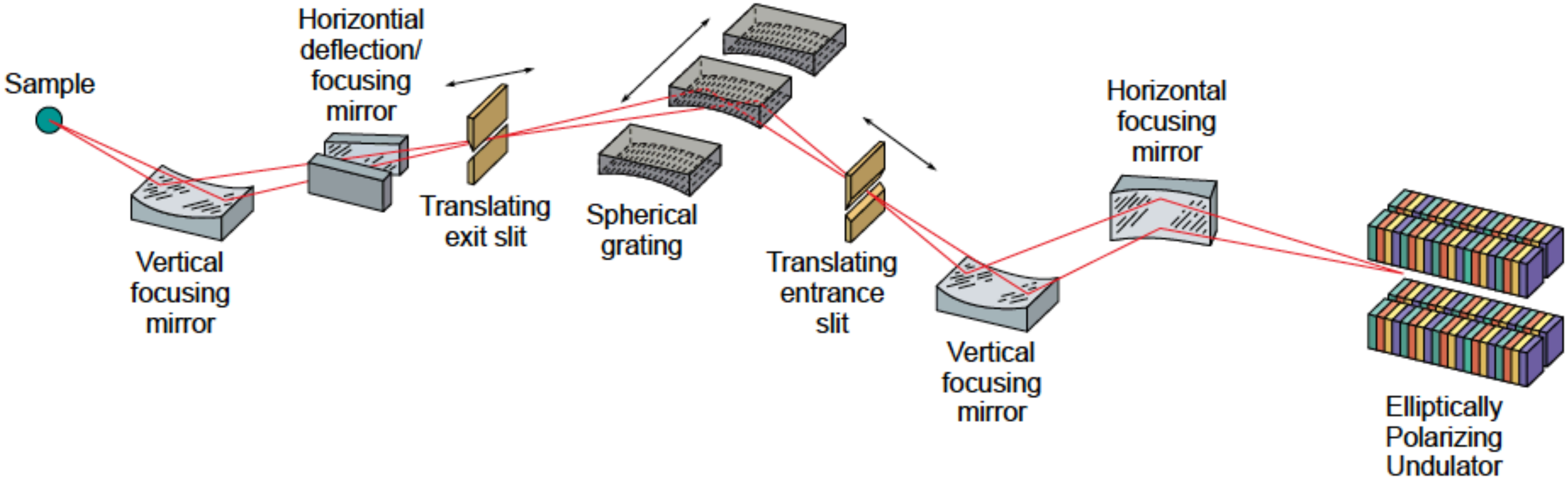
But we simply do not have time to discuss this in detail

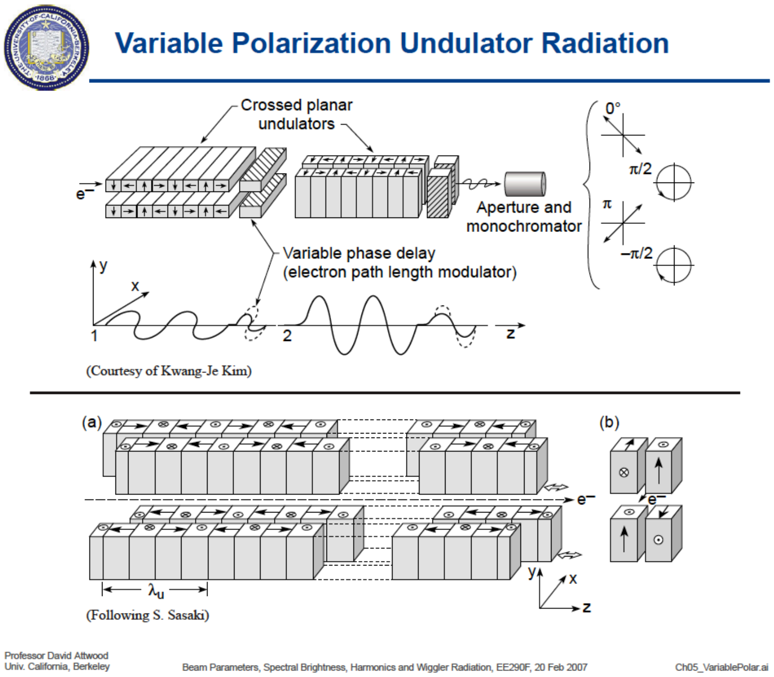
Polarization properties of SR





Gain in polarization
⇔ Loss in intensity




Variable Polarization Undulator Radiation

(Courtesy of Kwang-Je Kim)

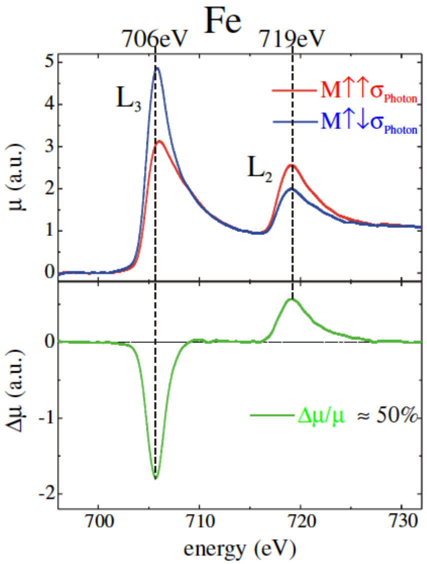
(Following S. Sasaki)

Professor David Attwood
Univ. California, Berkeley

Beam Parameters, Spectral Brightness, Harmonics and Wiggler Radiation, EE290F, 20 Feb 2007

Ch05_VariablePolar.ai

X-ray Magnetic Circular Dichroism (XMCD)



A typical XMCD result @ Fe $L_{3,2}$ absorption edges

- element specific
- huge magnetic contrast
- $\underline{M}\cdot\underline{\sigma}_{\text{Photon}}$
- Quantitative probe of spin and orbital moments

M=magnetization
= magnetic moment per volume

What are the Relative Merits?

Bending magnet radiation

- Broad spectrum
- Good photon flux
- No heat load
- Less expensive
- Easier access

Wiggler radiation

- Higher photon energies
- More photon flux
- Expensive magnet structure
- Expensive cooled optics
- Less access

Undulator radiation

- Brighter radiation
- Smaller spot size
- Partial coherence
- Expensive
- Less access

What we learned?

- SR has a wide variety of applications
- Light sources are mostly storage ring based
- Bending magnet SR is broad band, high power, but not very bright when compared to
- Undulator radiation – which is brightest between sources: its spectral brightness is proportional to number of poles square of the number of periods
- Undulators can produce also very bright radiation on harmonics
- Wiggler is an undulator with very large field whose harmonics are overlapped (because of the electron beam parameters!) and it power and brightness is proportional to number of periods
- Ultimately, electron beam parameters (beam current, emittances and energy spread) are determining performance of the light sources
- Polarization plays critical role in studies of magnetic materials
- There is a drive for so-called diffraction limited light sources where transverse emittances of the beam are below $\lambda/4\pi$. In this case the diffraction of the light itself determines spatial resolution/coherence of the beam, while brightness is simply proportional to the flux.

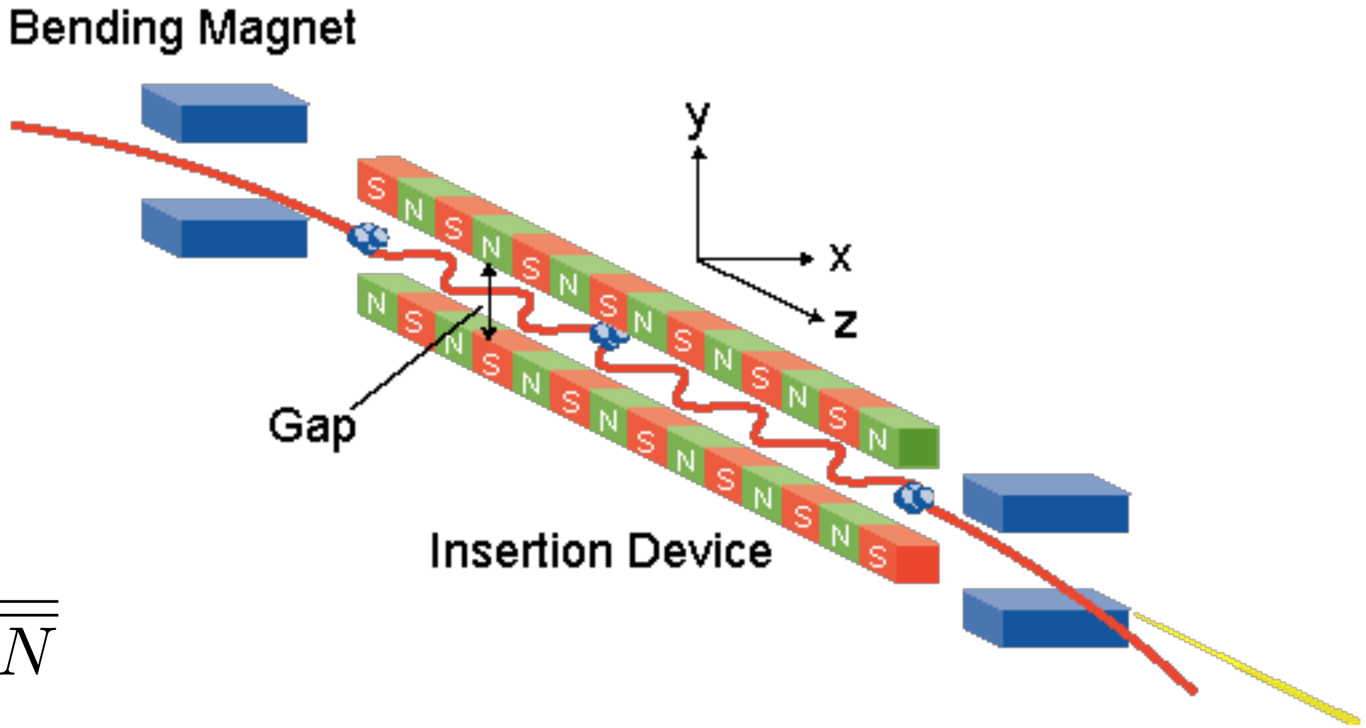
Radiation for K<1

Radiation Bandwidth:

$$\frac{\Delta\lambda}{\lambda} = \frac{1}{N}$$

Angle for central cone

$$\theta_{cen} \sim \frac{1}{\gamma\sqrt{N}}$$



Courtesy of W. Barletta

Power:

$$P_{cen} = \frac{\pi e \gamma^2 I}{\epsilon_0 \lambda_u} \left(\frac{K}{1 + K^2/2} \right)^2 f(K)$$