Complex Analysis Refresher

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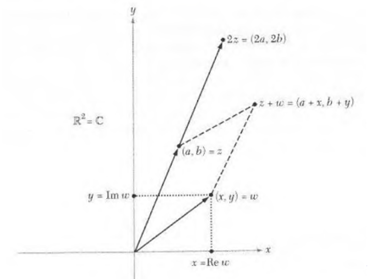
Septermber 28, 2023

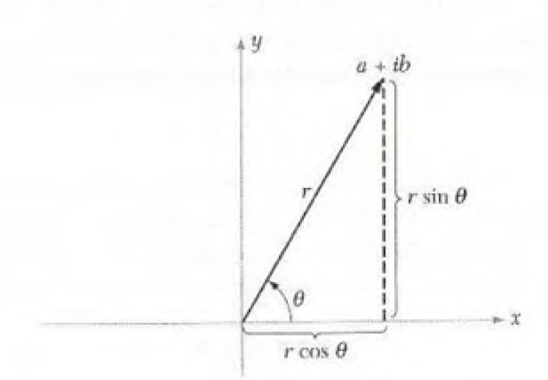
Definition

$$
\begin{aligned}
i &= \sqrt{-1} \\
w &= x + iy = (x, y), w \in \mathbb{C}, (x, y) \in \mathbb{R}^2 \\
x &= \textbf{Re } w, y = \textbf{Im } w \\
(a + ib) + (c + id) &= (a + c) + i(b + d) \\
k(a + ib) &= ka + ikb \\
(a + ib)(c + id) &= ac + iad + ibc + i^2 bd = (ac - bd) + i(ad + bc)
\end{aligned}
$$



Polar Representation of Complex Numbers

$$
\begin{aligned}
z &= a + ib = r\cos\theta + i(r\sin\theta) = r(\cos\theta + i\sin\theta) \\
|z| &= r = \sqrt{a^2+b^2} \text{ is called norm, or modulus, or absolute value of } z. \\
\theta &= \arg z \text{ is called the argument of } z.
\end{aligned}
$$



Multiplication of Complex Numbers in Polar Representation

$$\begin{aligned}
z_1 &= r_1(\cos\theta_1 + i\sin\theta_1) \\
z_2 &= r_2(\cos\theta_2 + i\sin\theta_2) \\
z_1 z_2 &= r_1 r_2[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\cos\theta_1\sin\theta_2 + \cos\theta_2\sin\theta_1)] \\
&= r_1 r_2[\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)] \\
|z_1 z_2| &= |z_1|\cdot|z_2|, \arg(z_1 z_2) = \arg z_1 + \arg z_2 \pmod{2\pi} \\
z^n &= r^n(\cos n\theta + i\sin n\theta), n \text{ is a positive integer} \\
w^n &= z = r(\cos\theta + i\sin\theta), \\
w &= \sqrt[n]{r}\left[\cos\left(\frac{\theta}{n} + \frac{k}{n}2\pi\right) + i\sin\left(\frac{\theta}{n} + \frac{k}{n}2\pi\right)\right] \\
& k = 0, 1, 2, \cdots, n-1
\end{aligned}$$

Complex Conjugation

$$
\begin{aligned}
z &= a + ib, \bar{z} = a - ib \\
\overline{z + z'} &= \bar{z} + \bar{z'} \\
\overline{zz'} &= \bar{z}\bar{z'} \\
\overline{z/z'} &= \bar{z}/\bar{z'} \text{ for } z' \neq 0 \\
z\bar{z} &= |z|^2 \\
z &= \bar{z} \text{ if and only if } z \text{ is real} \\
\textbf{Re } z &= (z + \bar{z})/2, \textbf{Im } z = (z - \bar{z})/2i \\
\bar{\bar{z}} &= z
\end{aligned}
$$

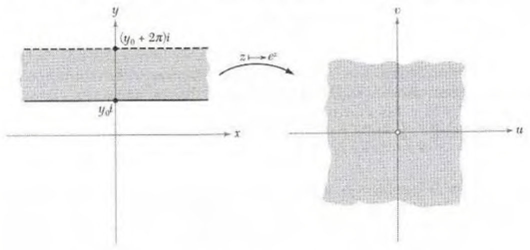
Some Elementary Functions

$$
\begin{aligned}
\sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \\
\cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots \\
e^x &= 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \\
e^{iy} &= 1 + \frac{(iy)}{1!} + \frac{(iy)^2}{2!} + \frac{(iy)^3}{3!} + \cdots \\
&= \left(1 - \frac{y^2}{2!} + \frac{y^4}{4!} - \cdots\right) + i\left(y - \frac{y^3}{3!} + \frac{y^5}{5!} - \cdots\right) \\
&= \cos y + i \sin y
\end{aligned}
$$

Some Elementary Functions

$$
\begin{aligned}
e^z &= e^{x+iy} = e^x(\cos y + i \sin y) \\
e^{z+w} &= e^z e^w \\
e^z & \quad \text{is never zero} \\
|e^{x+iy}| &= e^x \\
e^{i\pi/2} &= i, e^{i\pi} = -1, e^{i3\pi/2} = -i, e^{i2\pi} = 1 \\
e^z & \quad \text{is periodic, each period has the form } i2\pi n, \text{ for some integer } n \\
e^z &= 1 \text{ if and only if } z = i2\pi n \\
\sin z &= \frac{e^{iz} - e^{-iz}}{2i}, \cos z = \frac{e^{iz} + e^{-iz}}{2}
\end{aligned}
$$

$A_{y_0} = \{x + iy | x \in \mathbb{R}, y_0 \leq y < y_0 + 2\pi\}$, e^z maps A_{y_0} in a one-to-one manner onto the set $\mathbb{C} \backslash \{0\}$



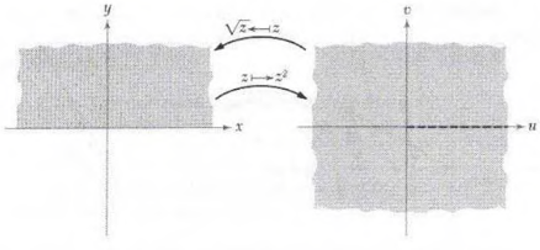
The function $\log : \mathbb{C} \backslash \{0\} \to \mathbb{C}$, with range $y_0 \leq \text{Im } \log z < y_0 + 2\pi$, is defined by

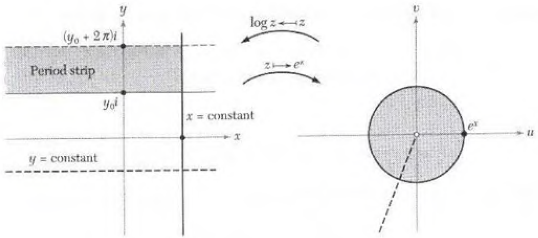
$$\log z = \log |z| + i \arg z$$

where $\arg z$ takes values in the interval $[y_0, y_0 + 2\pi)$.

$$\sqrt[n]{z} = z^{1/n} = e^{(\log z)/n}$$

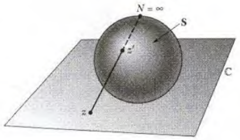
Geometry of some functions





Functions

- Open sets, closed sets, connected sets, compact sets, point at infinity, Riemann sphere **S**, $\bar{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$



- Functions on $\mathbb{C} \to \mathbb{C}$ can be written as $f(z) = f(x+iy) = u(x,y) + iv(x,y)$, where $u(x,y) = \textbf{Re}\ f(z)$ and $v(x,y) = \textbf{Im}\ f(z)$.
- Limit, continuity, uniform continuity
- Extreme value theorem: Continuous function on a compact set attains finite maximum and minimum values.

Analytic Functions

Function f is said to be differentiable (in the complex sense) at $z_0 \in A (A \subset \mathbb{C})$ if

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists. f is said to be analytic (or holomorphic) on A if f is complex differentiable at each $z_0 \in A$.

$$\begin{aligned} (af + bg)'(z) &= af'(z) + bg'(z) \\ (fg)'(z) &= f'(z)g(z) + f(z)g'(z) \\ \left(\frac{f}{g}\right)'(z) &= \frac{f'(z)g(z) - g'(z)f(z)}{[g(z)]^2} \\ \frac{d}{dz}(g \circ f)(z) &= g'(f(z)) \cdot f'(z) \text{ (Chain Rule)} \end{aligned}$$

Cauchy-Riemann Theorem: $f'(z_0)$ exists if and only if f is real differentiable

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Analytic Functions

Inverse Function Theorem:

$$\frac{d}{dw}f^{-1}(w) = \frac{1}{f'(z)} \text{ where } w = f(z)$$

Harmonic:

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

If $f = u + iv$ is analytic, then u and v are hormonic, u and v are harmonic conjugates. For example,

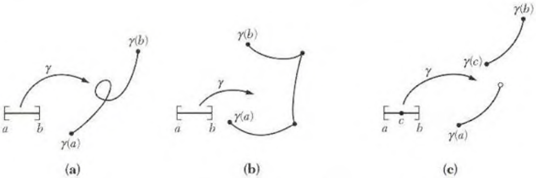
$$u(x, y) = x^2 - y^2, v(x, y) = 2xy$$

Differentiation of Elementary Functions

$$
\begin{aligned}
\frac{de^z}{dz} &= e^z \\
\frac{d}{dz}\log z &= \frac{1}{z} \\
\frac{d}{dz}\sin z &= \cos z \\
\frac{d}{dz}\cos z &= -\sin z \\
\frac{d}{dz}a^z &= (\log a)a^2 \\
\frac{d}{dz}z^b &= bz^{b-1}
\end{aligned}
$$

Contour Integrals

A curve or contour $\gamma : [a, b] \to \mathbb{C}$, smooth, piecewise C^1, discontinuous.



$\int_\gamma f = \int_\gamma f(z)dz = \sum_{i=0}^{n-1} \int_{a_i}^{a_{i+1}} f(\gamma(t))\gamma'(t)dt$ is called the integral of f along γ.

$$\begin{aligned} f(z) &= u(x,y) + iv(x,y) \\ f(\gamma(t)\gamma'(t)) &= [u(x(t),y(t))x'(t) - v(x(t),y(t)y'(t)] \\ &\quad +i[v(x(t),y(t))x'(t) + u(x(t),y(t))y'(t)] \\ f(z)dz &= (u+iv)(dx+idy) = udx - vdy + i(vdx+udy) \\ |\int_\gamma f| &\leq \int_\gamma |f||dz| \end{aligned}$$

Fundamental Theorem of Calculus

$\gamma : [0, 1] \to \mathbb{C}$ is a piecewise smooth curve.
F is a function defined and analytic on an open set G containing γ.

$$\int_\gamma F'(z)dz = F(\gamma(1)) - F(\gamma(0))$$
$$\int_\gamma F'(z)dz = 0 \text{ if } \gamma(0) = \gamma(1)$$

Integrals are path-independent.
Integrals around closed curves are 0.
An examle: let γ be the circle of radius r around $a \in \mathbb{C}$, n is an integer,

$$\int_\gamma (z-a)^n dz = \begin{matrix} 0, n \neq -1 \\ 2\pi i, n = -1 \end{matrix}$$

Cauchy's Theorem

γ is a simple closed curve (γ intersects itself only at its endpoint). f is analytic on and inside γ

$$\int_\gamma f = 0$$

Green's Theorem

$$\int_\gamma P(x,y)dx + Q(x,y)dy = \int\int_A \left[\frac{\partial Q}{\partial x}(x,y) - \frac{\partial P}{\partial y}(x,y)\right] dxdy$$

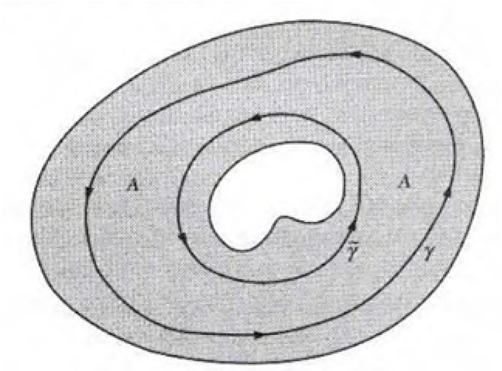
Suppose f is analytic on and inside a simple closed curve γ

$$\begin{aligned}
\int_\gamma f &= \int_\gamma f(z)dz = \int_\gamma (u+iv)(dx+idy) \\
&= \int_\gamma (udx - vdy) + i\int_\gamma (udy + vdx) \\
&= \int\int_A \left[-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right] dxdy + i\int\int_A \left[\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right] dxdy \\
&= 0 \text{ (Cauchy-Riemann equations)}
\end{aligned}$$

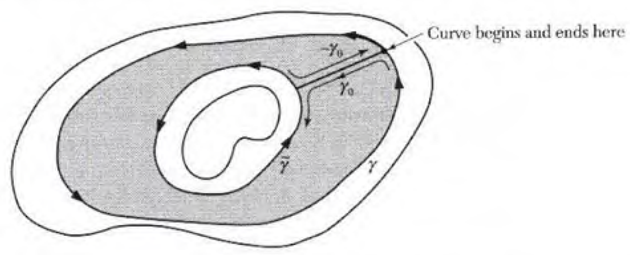
Deformation Theorem

f is analytic on a region A, γ is a simple closed curve in A. If γ can be continuously deformed to another simple closed curve $\tilde{\gamma}$ without passing outside the region A (γ is nomotopic to $\tilde{\gamma}$ in A), then

$$\int_{\gamma} f = \int_{\tilde{\gamma}} f$$



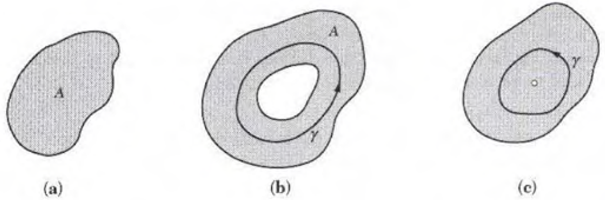
Deformation Theorem



$$0 = \int_{\gamma+\gamma_0-\tilde{\gamma}-\gamma_0} f = \int_{\gamma} f + \int_{\gamma_0} f - \int_{\tilde{\gamma}} f - \int_{\gamma_0} f = \int_{\gamma} f - \int_{\tilde{\gamma}} f$$

Simply Connected Regions

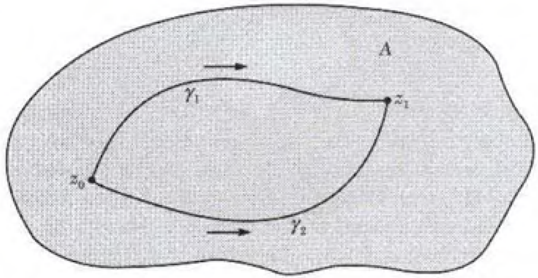
A is called simply connected if A is connected and every closed curve γ in A can be deformed in A to some constant curve $\tilde{\gamma} = z_0$.



$\int_\gamma f = 0$, if f is analytic on a simply connected region G and γ is a closed curve in G.

f is analytic on simply connected region A, γ_1 and γ_2 are two curves joining two points z_0 and z_1 in A,

$$\int_{\gamma_1} f = \int_{\gamma_2} f$$



Antiderivative Theorem

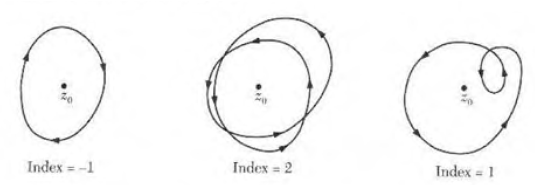
f is analytic on simply connected region A. Then there is an analytic function F defined on A that is unique up to an additive constant, such that $F'(z) = f(z)$, F is the antiderivative of f on A.

$$\begin{aligned} F_0(z) &= F(z) + C \\ (F_0 - F)'(z) &= F_0'(z) - F'(z) = f(z) - f(z) = 0 \end{aligned}$$

A is a simply connected region and $0 \notin A$. Then there is an analytic function F(z), unique up to the addition of multiples of $2\pi i$, such that $e^{F(z)} = z$. We write $F(z) = \log z$ and call F a branch of the logarithm function.

$$\begin{aligned} e^{F(z)} &= z, e^{G(z)} = z \\ e^{F(z)-G(z)} &= 1 \\ F(z_0) - G(z_0) &= 2\pi n i \text{ for some integer } n \text{ at fixed } z_0 \end{aligned}$$

Index of a Closed Path



$$I(\gamma; z_0) = \frac{1}{2\pi i}\int_\gamma \frac{dz}{z - z_0}$$

$I(\gamma; z_0)$ is the index of γ with respect of z_0. We say that γ winds around z_0, $I(\gamma; z_0)$ times.
$\gamma(t) = z_0 + re^{it}, 0 \le t \le 2\pi n$ has index n with respect to z_0,
$-\gamma(t) = z_0 + re^{-it}, 0 \le t \le 2\pi n$ has index $-n$ with respect to z_0.
γ and $\tilde{\gamma}$ are homotopic in $\mathbb{C}\backslash\{z_0\}$, z_0 does not lie on either γ or $\tilde{\gamma}$, then $I(\gamma; z_0) = I(\tilde{\gamma}; z_0)$

Cauthy's Integral Formula

$$f(z_0) \cdot I(\gamma; z_0) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z - z_0} dz$$

When γ is a simple closed curve and $I(\gamma; z_0) = 1$, then

$$f(z_0) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z - z_0} dz$$

Example: $\int_\gamma \frac{e^z}{z} dz = 2\pi i \cdot e^0 = 2\pi i$ by choosing $f(z) = e^z$ and $z_0 = 0$.
Cauthy's Integral Formula for Derivatives:

$$f^{(k)}(z_0) \cdot I(\gamma; z_0) = \frac{k!}{2\pi i} \int_\gamma \frac{f(z)}{(z - z_0)^{k+1}} dz, k = 1, 2, 3, \ldots$$

Cauchy's Inequalities and Liouville's Theorem

Cauchy's Inequalities: f is analytic on a region A, γ is a circle with radius R and center z_0. Suppose $|f(z) \leq M$ for all z on γ, then for $k = 0, 1, 2, ...$

$$|f^{(k)}(z_0)| \leq \frac{k!}{R^k}M$$

Liouville's Theorem: f is entire and $|f(z)| \leq M$ for all $z \in \mathbb{C}$, then f is constant.

Cauchy Inequalities with $k = 1$, $|f'(z_0)| \leq M/R$. Hold z_0 and let $R \to \infty$, we have $|f'(z_0)| = 0$ and therefore $f'(z_0) = 0$ for every $z_0 \in \mathbb{C}$.

Mean Value Property

$$f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + re^{i\theta})d\theta$$

$\gamma(\theta) = z_0 + re^{i\theta}, 0 \leq \theta \leq 2\pi$

$$\begin{aligned} f(z_0) &= \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z - z_0}dz \\ &= \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(z_0 + re^{i\theta})}{re^{i\theta}} re^{i\theta} d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + re^{i\theta})d\theta \end{aligned}$$

Poisson's Formula

Mean Value Property for Harmonic Functions:

$$u(x_0, y_0) = \frac{1}{2\pi}\int_0^{2\pi} u(z_0 + re^{i\theta})d\theta$$

Poisson's Formula real form:

$$u(\rho e^{i\phi}) = \frac{r^2 - \rho^2}{2\pi}\int_0^{2\pi} \frac{u(re^{i\phi})}{r^2 - 2r\rho\cos(\phi - \theta) + \rho^2}d\theta$$

Poisson's Formula complex form:

$$u(z) = \frac{1}{2\pi}\int_0^{2\pi} u(re^{i\theta})\frac{r^2 - |z|^2}{|re^{i\theta} - z|^2}d\theta$$

Series Representation

Taylor's series:

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n$$

Laurent series:

$$\begin{aligned}
f(z) &= \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n} \\
a_n &= \frac{1}{2\pi i} \int_{\gamma} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta, n = 0, 1, 2, \ldots \\
b_n &= \frac{1}{2\pi i} \int_{\gamma} f(\zeta)(\zeta - z_0)^{n+1} d\zeta, n = 1, 2, \ldots
\end{aligned}$$

Isolated Singularities

$$f(z) = \ldots + \frac{b_n}{(z-z_0)^n} + \ldots + \frac{b_1}{z-z_0} + a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \ldots$$

If f is analytic on $\{z | 0 < |z - z_0| < r\}$, z_0 is an isolated singularity of f

- A finite number of b_n are zero, z_0 is called a pole of f. k is the highest integer such that $b_k \neq 0$, z_0 is a pole of order k. A first-order pole is a simple pole.
- An infinite number of b_k are nonzero, z_0 is an essential singularity.
- b_1 is the residue of f at z_0.
- All the b_k are zero, z_0 is a removable singularity.

$$\int_\gamma f(z)dz = b_1 \cdot 2\pi i$$

Isolated Singularities

z_0 is a removable singularity if and only if one of the following conditions holds:

- f is bounded in a deleted neighborhood of z_0
- $\lim_{z \to z_0} f(z)$ exists
- $\lim_{z \to z_0}(z - z_0)f(z) = 0$

z_0 is a simple hole if and only if $\lim_{z \to z_0}(z - z_0)f(z)$ exists and is unequal to zero. This limit equals the residue of f at z_0.
f and g have zeros of order n and k respectively at z_0. $h(z) = f(z)/g(z)$:

- if $k > n$, h has a pole of order $k - n$ at z_0.
- if $k = n$, h has a removable singularity with nonzero limit at z_0.
- if $k < n$, h has a zero of order $n - k$ at z_0.

Residues

$$f(z) = \ldots + \frac{b_2}{(z-z_0)^2} + \frac{b_1}{z-z_0} + a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \ldots$$
$$b_1 = Res(f; z_0)$$

$f(z) = g(z)/h(z)$: If g and h have zeros at z_0 of the same order, f has a removable singularity at z_0, $Res(f; z_0) = 0$.
If $g(z_0) \neq 0, h(z_0) = 0, h'(z_0) \neq 0$, f has a simple pole at z_0, $Res(f; z_0) = \frac{g(z_0)}{h'(z_0)}$.
If g has a zero of order k at z_0 and h has a zero of order $k+1$, then f has a simple pole, $Res(f; z_0) = (k+1)\frac{g^{(k)}(z_0)}{h^{k+1}(z_0)}$.
If $g(z_0) \neq 0, h(z_0) = 0, h'(z_0) = 0, h''(z_0) \neq 0$, f has a second-order pole at z_0, $Res(f; z_0) = 2\frac{g'(z_0)}{h''(z_0)} - \frac{2}{3}\frac{g(z_0)h'''(z_0)}{[h''(z_0)]^2}$.

High-Order Poles

k is the smallest integer ≥ 0 such that $\lim_{z \to z_0}(z - z_0)^k f(z)$ exists. Then f has a pole of order k. Let $\phi(z) = (z - z_0)^k f(z)$

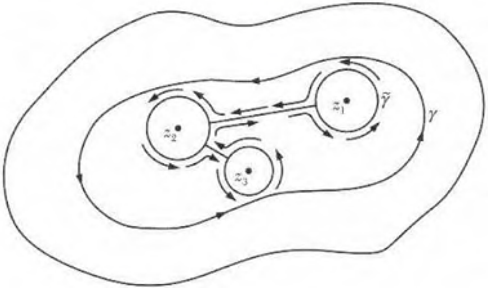
$$Res(f; z_0) = \frac{\phi^{(k-1)}(z_0)}{(k-1)!}$$

$g(z_0) \neq 0, h(z_0) = 0 = \ldots = h^{(k-1)}(z_0), h^{(k)}(z_0) \neq 0$, then g/h has a pole of order k

$$\mathrm{Res}(g/h; z_0) = \left[\frac{k!}{h^{(k)}(z_0)}\right]^k \times$$

$$\begin{vmatrix} \frac{h^{(k)}(z_0)}{k!} & 0 & 0 & \cdots & 0 & g(z_0) \\ \frac{h^{(k+1)}(z_0)}{(k+1)!} & \frac{h^{(k)}(z_0)}{k!} & 0 & \cdots & 0 & g^{(1)}(z_0) \\ \frac{h^{(k+2)}(z_0)}{(k+2)!} & \frac{h^{(k+1)}(z_0)}{(k+1)!} & \frac{h^{(k)}(z_0)}{k!} & \cdots & 0 & \frac{g^{(2)}(z_0)}{2!} \\ \vdots & \vdots & \vdots & & & \vdots \\ \frac{h^{(2k-1)}(z_0)}{(2k-1)!} & \frac{h^{(2k-2)}(z_0)}{(2k-2)!} & \frac{h^{(2k-3)}(z_0)}{(2k-3)!} & \cdots & \frac{h^{(k+1)}(z_0)}{(k+1)!} & \frac{g^{(k-1)}(z_0)}{(k-1)!} \end{vmatrix}.$$

$$\int_{\gamma} f(z)dz = 2\pi i \sum_{i=1}^{n} [Res(f; z_i)] I(\gamma; z_i)$$



Residue at Infinity

$$F(z) \quad = \quad f(1/z)$$

f has a pole of order k at ∞ if F has a pole of order k at 0;
f has a zero of order k at ∞ if F has a zero of order k at 0.
Define $Res(f;\infty) = -Res((1/z^2)F(z);0)$.

$$\int_\gamma f \quad = \quad -2\pi i \sum\{\text{residues of } f \text{ outside } \gamma \text{ including at } \infty\}$$

Evalution of Definite Integrals

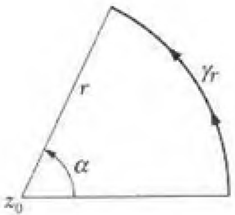
$$
\begin{aligned}
\sin\theta &= \frac{e^{i\theta}-e^{-i\theta}}{2i} = \frac{z-1/z}{2i} \\
\cos\theta &= \frac{e^{i\theta}+e^{-i\theta}}{2} = \frac{z+1/z}{2} \\
dz &= ie^{i\theta}d\theta = izd\theta \\
\int_0^{2\pi} R(\cos\theta,\sin\theta) &= -2\pi i \sum \{\text{residues of } f(z) \text{ inside the uni circle}\} \\
f(z) &= \frac{1}{iz} R\left(\frac{1}{2}(z+\frac{1}{z}), \frac{1}{2i}(z-\frac{1}{z})\right)
\end{aligned}
$$

Integrals on the Whole Real Line

$$\int_{-\infty}^{\infty} f(x)dx = 2\pi i \sum \{\text{residues of } f \text{ in } \mathcal{H}\}$$
$$\mathcal{H} = \{z \in \mathbb{C} | Im(z) \geq 0\}$$
$$\int_{-\infty}^{\infty} f(x)dx = -2\pi i \sum \{\text{residues of } f \text{ in } \mathcal{L}\}$$
$$\mathcal{L} = \{z \in \mathbb{C} | Im(z) \leq 0\}$$

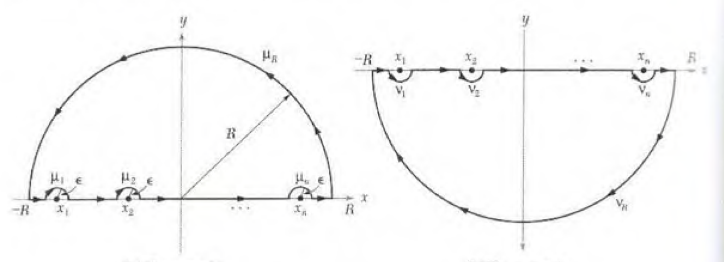
Cauchy Principal Value

The regular integral $\int_{-\infty}^{\infty} f(x)dx = \lim_{A,B\to\infty} \int_{-A}^{B} f(x)dx$ may not be convergent.
The Cauchy principal value may exist, with $A, B \to \infty$ using the symmetric way: $P.V. \int_{-\infty}^{\infty} f(x)dx = \lim_{R\to\infty} \int_{-R}^{R} f(x)dx$



$$\lim_{r\to 0} \int_{\gamma_r} f \quad = \quad \alpha i Res(f; z_0)$$

Singularities on Real Axie



$$(H)_0 = \{z \in \mathbb{C} | Im(z) > 0\}$$

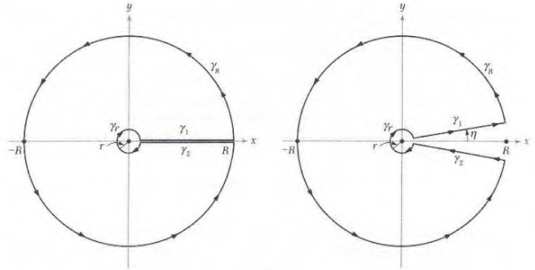
$$P.V. \int_{-\infty}^{\infty} f(z)dz = 2\pi i \sum \{\text{residues of } f \text{ in } \mathcal{H}_0\} + \pi i \sum_{j=1}^{n} Res(f; x_j)$$

$$(L)_0 = \{z \in \mathbb{C} | Im(z) < 0\}$$

$$P.V. \int_{-\infty}^{\infty} f(z)dz = -2\pi i \sum \{\text{residues of } f \text{ in } \mathcal{L}_0\} - \pi i \sum_{j=1}^{n} Res(f; x_j)$$

An example

Evaluate $\int_0^\infty \frac{\sqrt[3]{x}}{1+x^2}$



$$
\begin{aligned}
g(z) &= \sqrt[3]{z}/(1+z^2), 0 < r < 1, R > 1, r \to 0, R \to \infty \\
\Sigma &= Res(g; i) + Res(g; -i) = \frac{\sqrt[3]{i}}{2i} + \frac{\sqrt[3]{-i}}{-2i} = -e^{\pi i/3}/2 \\
2\pi i \Sigma &= \int_{\gamma_1} g + \int_{\gamma_R} g + \int_{\gamma_2} g + \int_{\gamma_r} g
\end{aligned}
$$

An example

$$
\begin{aligned}
\left|\int_{\gamma_R} g\right| &\leq \frac{R^{1/3} 2\pi R}{R^2 - 1} \to 0 \\
\left|\int_{\gamma_r} g\right| &\leq \frac{r^{1/3} 2\pi r}{1 - r^2} \to 0 \\
\int_{\gamma_1} g &= \int_r^R \frac{x^{1/3}}{1 + x^2} dx \to \int_0^\infty \frac{\sqrt[3]{x}}{1 + x^2} dx \\
\int_{\gamma_2} g &= \int_R^r \frac{x^{1/3} e^{2\pi i/3}}{1 + x^2 e^{4\pi i}} e^{2\pi i} dx \to -e^{2\pi i/3} \int_0^\infty \frac{\sqrt[3]{x}}{1 + x^2} dx \\
\int_0^\infty \frac{\sqrt[3]{x}}{1 + x^2} dx &= \frac{\pi}{\sqrt{3}}
\end{aligned}
$$

Laplace Transforms

$f : [0, \infty) \to \mathbb{C}$ or $\mathbb{R}$ is of exponential order:$|f(t)| \leq Ae^{tB}$ for all $t \leq 0$ with constants $A > 0, B \in \mathbb{R}$.

$$\tilde{f}(z) = \int_0^\infty e^{-zt} f(t) dt$$

$$\frac{d}{dz}\tilde{f}(z) = -\int_0^\infty te^{-zt} f(t) dt$$

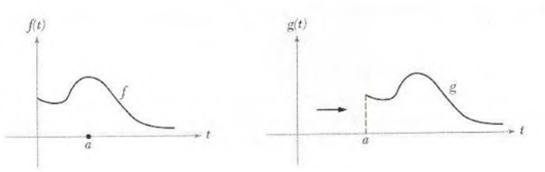
There exists a unique number $\sigma, -\infty \leq \sigma < \infty$, such that the integral converges if Re $z > \sigma$ and diverges if Re $z < \sigma$. σ is called the absciaa of convergence.

Define $\rho = \inf\{B \in \mathbb{R}|$ there exists an A>0 such that $|f(t)| \leq Ae^{Bt}\}$, then $\sigma \leq \rho$.

$\tilde{f}(z) = \tilde{h}(z)$ for Re $z > \gamma_0$ for some γ_0. Then $f(t) = h(t)$ for all $t \in [0, \infty)$.

$$\begin{aligned}
\widetilde{\left(\frac{df}{dt}\right)}(z) &= z\tilde{f} - f(0) \\
\widetilde{\left(\frac{d^2f}{dt^2}\right)}(z) &= z^2\tilde{f} - zf(0) - \frac{df}{dt}(0) \\
\tilde{g}(z) &= d\tilde{f}(z)/dz, \text{ where } g(t) = -tf(t) \\
g(t) &= \int_0^t f(\tau)d\tau, \text{ then } \tilde{g}(z) = \frac{\tilde{f}(z)}{z} \\
g(t) &= e^{-at}f(t), \text{ then } \tilde{g}(z) = \tilde{f}(z+a) \text{ (First Shifting Theorem)}
\end{aligned}$$

$H(t) = 0$ if $t < 0$ and $H(t) = 1$ if $t \geq 0$, $a \geq 0$, $g(t) = f(t-a)H(t-a)$.
$g(t) = 0$ if $t < a$ and $g(t) = f(t-a)$ if $t \geq a$,



$$\tilde{g}(z) \quad = \quad e^{-az}\tilde{f}(z)$$

If $g(t) = f(t)H(t-a)$, then $\tilde{g}(z) = e^{-az}\tilde{F}(z)$ where $F(t) = f(t+a)$

Convolution

$$
\begin{aligned}
(f * g)(t) &= \int_0^\infty f(t-\tau)\cdot g(\tau)d\tau \\
\widetilde{(f * g)}(z) &= \tilde{f}(z)\cdot\tilde{g}(z) \\
\widetilde{(f * g)}(z) &= \int_0^\infty e^{-zt}\left[\int_0^\infty f(t-\tau)\cdot g(\tau)d\tau\right]dt \\
&= \int_0^\infty \left[\int_0^\infty e^{-z\tau}e^{-z(t-\tau)}f(t-\tau)g(\tau)d\tau\right]dt \\
&= \int_0^\infty e^{-z\tau}\left[\int_0^\infty e^{-z(t-\tau)}f(t-\tau)dt\right]g(\tau)d\tau \\
&= \int_0^\infty e^{-z\tau}\tilde{f}(z)g(\tau)d\tau \\
&= \tilde{f}(z)\cdot\tilde{g}(z)
\end{aligned}
$$

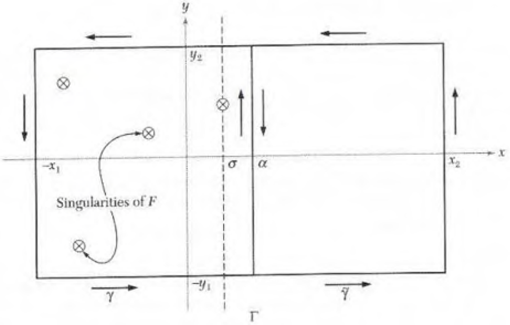
Some Common Laplace Transforms

If $f(t) = e^{-at}$, then $\tilde{f}(z) = \frac{1}{z+a}$ and $\sigma(f) = -\text{Re } a$
If $f(t) = \cos at$, then $\tilde{f}(z) = \frac{z}{z^2+a^2}$ and $\sigma(f) = |\text{Im } a|$
If $f(t) = \sin at$, then $\tilde{f}(z) = \frac{a}{z^2+a^2}$ and $\sigma(f) = |\text{Im } a|$
If $f(t) = t^a, a > -1$, then $\tilde{f}(z) = \frac{\Gamma(a+1)}{z^{a+1}}$ and $\sigma(f) = 0$
$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt$, $\Gamma(n) = (n-1)!$
If $f(t) = 1$, then $\tilde{f}(z) = \frac{1}{z}$ and $\sigma(f) = 0$

Complex Inversion Formula

$$f(t) = \sum \{\text{residues of } e^{zt}F(z) \text{ at each of its singularities in } \mathbb{C}\}$$

Then $\tilde{f}(z) = F(z)$ for Re $z > \sigma$.
Split Γ into a sum of two rectangular paths γ and $\tilde{\gamma}$ by a vertical line through Re $z = \alpha > \sigma$.



Complex Inversion Formula

$$\begin{aligned}
\int_\gamma e^{zt}F(z)dz &= 2\pi i\sum\{\text{residues of } e^{zt}F(z)\} = 2\pi i f(t) \\
2\pi i\tilde{f}(z) &= \lim_{r\to\infty}\int_0^r e^{-zt}\left[\int_\gamma e^{\zeta t}F(\zeta)d\zeta\right]dt \\
&= \lim_{r\to\infty}\int_\gamma\int_0^r e^{(\zeta-z)t}F(\zeta)dtd\zeta \\
&= \lim_{r\to\infty}\int_\gamma\left(e^{(\zeta-z)r}-1\right)\frac{F(\zeta)}{\zeta-z}d\zeta \\
&= -\int_\gamma\frac{F(\zeta)}{\zeta-z}d\zeta = \int_{\tilde{\gamma}}\frac{F(\zeta)}{\zeta-z}d\zeta - \int_\Gamma\frac{F(\zeta)}{\zeta-z}d\zeta \\
&= 2\pi i F(z)
\end{aligned}$$

Examples

If $\tilde{f}(z) = 1/(z-3)$, $f(t) = e^{3t}$.
If $\tilde{f}(z) = \log(z^2+z)$,

$$\begin{aligned}
\tilde{g}(z) &= -\frac{d}{dz}\tilde{f}(z) = -\frac{2z+1}{z^2+z} = -\frac{1}{z} - \frac{1}{z+1} \\
g(t) &= -1 - e^{-t} \\
f(t) &= -\frac{1}{t}(1+e^{-t})
\end{aligned}$$

$\tilde{f}(z) = F(z) = \frac{z}{(z+1)^2(z^2+3z-10)}$

$$\begin{aligned}
f(t) &= \sum\{\text{residues of } \frac{e^{zt}z}{(z+1)^2(z^2+3z-10)} = \frac{e^{zt}z}{(z+1)^2(z+5)(z-2)}\} \\
f(t) &= Res(e^{zt}F(z); -1) + Res(e^{zt}F(z); -5) + Res(e^{zt}F(z); 2) \\
&= \frac{1}{12}\left(te^{-t} - e^{-t} + \frac{e^{-t}}{12}\right) + \frac{5e^{-5t}}{112} + \frac{2e^{2t}}{63}
\end{aligned}$$

Fourier Transforms

$f : \mathbb{R} \to \mathbb{R}$, the Fourier transform is defined as

$$\hat{f} \;:=\; (\mathcal{F} \cdot f)(\omega) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx$$

The factor of $1/\sqrt{2\pi}$ may be missing and the exponent may be $-2\pi i \omega x$.

$$\begin{aligned}
\int_{-\infty}^{\infty} f(x) \cos(\omega x) dx &= \text{Re } (\hat{f}(\omega)) \\
\int_{-\infty}^{\infty} f(x) \sin(\omega x) dx &= -\text{Im } (\hat{f}(\omega)) \\
\hat{f}(\omega) &= \int_{-\infty}^{\infty} dx f(x) e^{-i2\pi\omega x} \\
f(x) &= \int_{-\infty}^{\infty} d\omega \hat{f}(\omega) e^{-i2\pi\omega x}
\end{aligned}$$

Fourier Transforms

$\omega < 0$, $|e^{-i\omega z}| = e^{\omega y} \leq 1$ in the upper half plane $\mathcal{H}$

$$\int_{-\infty}^{\infty} f(x)e^{-i\omega x}dx = 2\pi i \sum \{\text{residues of } f(z)e^{-i\omega z} in \mathcal{H}\}$$

$\omega > 0$, $|e^{-i\omega z}| = e^{\omega y} \leq 1$ in the lower half plane $\mathcal{L}$

$$\int_{-\infty}^{\infty} f(x)e^{-i\omega x}dx = -2\pi i \sum \{\text{residues of } f(z)e^{-i\omega z} in \mathcal{L}\}$$