

## Homework 7

### Problem 1. 10 points. FODO cell.

Consider a general FODO cell comprised of two quadrupoles F and D separated by two drift sections, e.g. the structure below:

$$F: K_F = \frac{e}{pc} \frac{\partial B_y}{\partial x}, l_F;$$

$$O1: l_1$$

$$D: K_D = \frac{e}{pc} \frac{\partial B_y}{\partial x}, l_D;$$

$$O2: l_2$$

(a) **2 points:** write matrix (both x and y or 4x4) of general FODO cell (not assuming any limitations on K F,D).

(b) **3 points:** write stability criteria (for x and y) for periodic lattice built of this FOD cell. Hint – do not try to solve it!

(c,d) make transition to short lens approximation and assume equal strength of

$$l_F K_F = -K_D l_D = \frac{1}{f} = \text{const}, l_{F,D} \rightarrow 0$$

$$l = l_1 = l_2$$

and

(c) **3 points:** show that both x and y motion can be stable (e.g. prove so called strong focusing: combination of focusing and defocusing length can provide focusing in both directions);

(d) **2 points:** define (e.g solve) the stability criteria for such cell.

**Problem 2. 2x5 points.** Find not-trivial solution for building an unit 2x2 transport matrix out of repeating cells:

$$M^4 = I; M \neq I$$

(a) **5 points:** show that one of the solutions  $\text{trace}(M) = 0$ ; Hint: used  $M^2 = -I$ ;

(b) **5 points:** for a “symmetric” FODO cell and finite length equally strong quadrupoles  $K_F = -K_D = K; l_F = l_D = L; l_1 = l_2 = l$  write the condition that  $M_x^4 = M_y^4 = I$ , e.g. the 4x4 transport matrix is unit.

### Solutions: Problem 1:

(a) We know already matrices of all these elements and need just multiply them in correct order

$$M_x = O_2 D_x O_1 F_x = \begin{bmatrix} 1 & l_2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \cosh \varphi_D & \frac{\sinh \varphi_D}{\omega_D} \\ \omega_D \sinh \varphi_D & \cosh \varphi_D \end{bmatrix} \begin{bmatrix} 1 & l_1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \cos \varphi_F & \frac{\sin \varphi_F}{\omega_F} \\ -\omega_F \sin \varphi_F & \cos \varphi_F \end{bmatrix}$$

$$M_y = O_2 D_y O_1 F_y = \begin{bmatrix} 1 & l_2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \cos \varphi_D & \frac{\sin \varphi_D}{\omega_D} \\ -\omega_D \sin \varphi_D & \cos \varphi_D \end{bmatrix} \begin{bmatrix} 1 & l_1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \cosh \varphi_F & \frac{\sinh \varphi_F}{\omega_F} \\ \omega_F \sinh \varphi_F & \cosh \varphi_F \end{bmatrix}$$

$$\omega_F = \sqrt{|K_F|}; \varphi_F = \omega_F l_F; \omega_D = \sqrt{|K_D|}; \varphi_D = \omega_D l_D$$

$$M_x = O_2 D_x O_1 F_x = \begin{bmatrix} \text{ch}_D + l_2 \omega_D \text{sh} \varphi_D & \frac{\text{sh}_D}{\omega_D} + l_2 \text{ch}_D \\ \omega_D \text{sh}_D & \text{ch}_D \end{bmatrix} \begin{bmatrix} \text{cs}_F - l_1 \omega_F \text{sn}_F & \frac{\text{sn}_F}{\omega_F} + l_1 \text{cs}_F \\ -\omega_F \text{sn}_F & \text{cs}_F \end{bmatrix}$$

$$= \begin{bmatrix} (\text{ch}_D + l_2 \omega_D \text{sh}_D)(\text{cs}_F - l_1 \omega_F \text{sn}_F) - \omega_F \text{sn}_F \left( \frac{\text{sh}_D}{\omega_D} + l_2 \text{ch}_D \right) & (\text{ch}_D + l_2 \omega_D \text{sh}_D) \left( \frac{\text{sn}_F}{\omega_F} + l_1 \text{cs}_F \right) + \text{cs}_F \left( \frac{\text{sh}_D}{\omega_D} + l_2 \text{ch}_D \right) \\ \omega_D \text{sh}_D (\text{cs}_F - l_1 \omega_F \text{sn}_F) - \omega_F \text{sn}_F \text{ch}_D & \omega_D \text{sh}_D \left( \frac{\text{sn}_F}{\omega_F} + l_1 \text{cs}_F \right) + \text{cs}_F \text{ch}_D \end{bmatrix}$$

and similarly ugly expression for vertical matrix,

$$M_y = O_2 D_y O_1 F_y =$$

$$= \begin{bmatrix} (\text{cs}_D - l_2 \omega_D \text{sn}_D)(\text{ch}_F + l_1 \omega_F \text{sh}_F) + \omega_F \text{sh}_F \left( \frac{\text{sn}_D}{\omega_D} + l_2 \text{cs}_D \right) & (\text{cs}_D + l_2 \omega_D \text{sn}_D) \left( \frac{\text{sh}_F}{\omega_F} + l_1 \text{ch}_F \right) + \text{ch}_F \left( \frac{\text{sn}_D}{\omega_D} + l_2 \text{cs}_D \right) \\ -\omega_D \text{sn}_D (\text{ch}_F + l_1 \omega_F \text{sh}_F) + \omega_F \text{sh}_F \text{cs}_D & -\omega_D \text{sn}_D \left( \frac{\text{sh}_F}{\omega_F} + l_1 \text{ch}_F \right) + \text{ch}_F \text{cs}_D \end{bmatrix}$$

(b) stability criteria are:

$$|\text{Trace}[M_{x,y}]| < 2$$

$$-2 < (\text{ch}_D + l_2 \omega_D \text{sh}_D)(\text{cs}_F - l_1 \omega_F \text{sn}_F) - \omega_F \text{sn}_F \left( \frac{\text{sh}_D}{\omega_D} + l_2 \text{ch}_D \right) + \omega_D \text{sh}_D \left( \frac{\text{sn}_F}{\omega_F} + l_1 \text{cs}_F \right) + \text{cs}_F \text{ch}_D < 2$$

$$-2 < (\text{ch}_F + l_1 \omega_F \text{sh}_F)(\text{cs}_D - l_2 \omega_D \text{sn}_D) - \omega_D \text{sn}_D \left( \frac{\text{sh}_F}{\omega_F} + l_1 \text{ch}_F \right) + \omega_F \text{sh}_F \left( \frac{\text{sn}_D}{\omega_D} + l_2 \text{cs}_D \right) + \text{cs}_D \text{ch}_F < 2$$

(c,d)

$$\begin{aligned}
& \left[ \begin{array}{cc} \cosh \varphi & \frac{\sinh \varphi}{\omega} \\ \omega \sinh \varphi & \cosh \varphi \end{array} \right] \rightarrow \left[ \begin{array}{cc} 1 & 0 \\ \frac{1}{f} & 1 \end{array} \right]; \left[ \begin{array}{cc} \cos \varphi & \frac{\sin \varphi}{\omega} \\ -\omega \sin \varphi & \cos \varphi \end{array} \right] \rightarrow \left[ \begin{array}{cc} 1 & 0 \\ -\frac{1}{f} & 1 \end{array} \right] \\
M_x = \left[ \begin{array}{cc} 1 & l \\ 0 & 1 \end{array} \right] \left[ \begin{array}{cc} 1 & 0 \\ \frac{1}{f} & 1 \end{array} \right] \left[ \begin{array}{cc} 1 & l \\ 0 & 1 \end{array} \right] \left[ \begin{array}{cc} 1 & 0 \\ -\frac{1}{f} & 1 \end{array} \right] = \left[ \begin{array}{cc} 1+\frac{l}{f} & l \\ \frac{1}{f} & 1 \end{array} \right] \left[ \begin{array}{cc} 1-\frac{l}{f} & l \\ -\frac{1}{f} & 1 \end{array} \right] = \left[ \begin{array}{cc} 1-\left(\frac{l}{f}\right)^2 -\frac{l}{f} & 2l+\frac{l^2}{f} \\ -\frac{l}{f^2} & 1+\frac{l}{f} \end{array} \right] \\
M_y = \left[ \begin{array}{cc} 1 & l \\ 0 & 1 \end{array} \right] \left[ \begin{array}{cc} 1 & 0 \\ -\frac{1}{f} & 1 \end{array} \right] \left[ \begin{array}{cc} 1 & l \\ 0 & 1 \end{array} \right] \left[ \begin{array}{cc} 1 & 0 \\ \frac{1}{f} & 1 \end{array} \right] = \left[ \begin{array}{cc} 1-\left(\frac{l}{f}\right)^2 +\frac{l}{f} & 2l-\frac{l^2}{f} \\ -\frac{l}{f^2} & 1-\frac{l}{f} \end{array} \right]
\end{aligned}$$

Stability criteria is

$$\begin{aligned}
|\text{Trace}[M_{x,y}]| < 2 & \rightarrow -2 < 2 - \left(\frac{l}{f}\right)^2 < 2 \\
\left(\frac{l}{f}\right)^2 < 4; \quad \left|\frac{l}{f}\right| < 2
\end{aligned}$$

can be satisfied for both directions.

**Problem 2.** Ignoring trivial solution and complications imposed by  $M^2 = I$

$$M^4 = I \rightarrow M^2 = \pm I \text{ pick } M^2 = -I; \quad ad - bc = 1$$

$$\left[ \begin{array}{cc} a & b \\ c & d \end{array} \right] \left[ \begin{array}{cc} a & b \\ c & d \end{array} \right] = \left[ \begin{array}{cc} a^2 + bc & b(a+d) \\ c(a+d) & a^2 + bc \end{array} \right] \left[ \begin{array}{cc} a(a+d)-1 & b(a+d) \\ c(a+d) & d(a+d)-1 \end{array} \right] = \left[ \begin{array}{cc} -1 & 0 \\ 0 & -1 \end{array} \right]$$

has obvious solution

$$\text{Trace}M = a + d = 0$$

Using previous problem, we can write

$$\text{Trace}[M_{x,y}] = 0$$

$$\begin{aligned}
& (\cosh \varphi + l\omega \sinh \varphi)(\cos \varphi - l\omega \sin \varphi) - \omega \sin \varphi \left( \frac{\sinh \varphi}{\omega} + l \cosh \varphi \right) \\
& + \omega \sinh \varphi \left( \frac{\sin \varphi}{\omega} + l \cos \varphi \right) + \cos \varphi \cosh \varphi = 0
\end{aligned}$$

This a transcendental equation, which has solution which can be found numerically.