Homework 18. Due November 16

Problem 1. 15 points. Turning the beam around – ultimate storage rings

Let's consider that we build a storage ring (magnets only), where ultra-relativistic charged particles traveling in circle of constant radius ***R*** while radiating synchrotron radiation. It means that the magnetic field is adjusted to the loss of its energy.

(a) Find the energy of the particle as function of the traveled distance ***s*** or angle ***s/R***;
(b) Find the distance when the particle's energy is reduced by a factor 2.
(c) Loosing half of the energy is considered to be "dead-end" for recirculating the beams – than linear accelerators have to do the job. For R being 6,371 kilometers – that of the Earth, find critical energy of electrons, muons and protons when particles are loosing ½ of the energy in a single turn.

Problem 2. 10 points. Circulating particle in magnetic field

Consider ultra-relativistic charged particle with initial energy circulating in an uniform constant magnetic field $\mathbf{B_y}$.

(a) Find energy of the particle as function of time.
(b) What will be its trajectory?

Note: Neglect non-relativistic effects

Solution:

Problem 1: since we are considering ultra-relativistic particles, we can assume that $s=ct$, e.g. neglect $(1-\beta) <<< 1$. (a)Losses for radiation with fixed radius are

$$\frac{d\mathsf{E}_{SR}}{ds} = -mc^2\frac{d\gamma}{ds} \cong \frac{2}{3}\gamma^4\frac{e^2}{R^2}; \qquad (22\text{-}12)$$

where we used obvious: $E = \gamma mc^2; dE = -d\mathsf{E}_{SR}$. Solution is straightforward:

$$-\frac{d\gamma}{\gamma^4} = \frac{2}{3}\frac{r_c}{R^2}ds;\ \ r_c = \frac{e^2}{mc^2};\ \ \frac{\gamma^{-3}-\gamma_o^{-3}}{3} = \frac{2}{3}\frac{r_c}{R^2}s = \frac{2}{3}\frac{r_c}{R}\theta;\theta = \frac{s}{R};$$

$$\gamma = \frac{\gamma_o}{\sqrt[3]{1+2\gamma_o^{\ 3}\frac{r_c}{R^2}s}} = \frac{\gamma_o}{\sqrt[3]{1+2\gamma_o^{\ 3}\frac{r_c}{R}\theta}};$$

(b) $\gamma = \gamma_o/2$ means

$$\sqrt[3]{1+2\gamma_o^{\ 3}\frac{r_c}{R^2}s} = 2 \rightarrow s_{1/2} = \frac{7}{2}\frac{R^2}{\gamma_o^{\ 3}r_c}.$$

(c) with R= 6.371x10^6 m one turn is s=2πR and we have the relativistic factor of a particle loosing ½ of its energy in one turn around the Earth:

(d) $s_{1/2} = 2\pi R = \frac{7}{2}\frac{R^2}{\gamma_{cr}^{\ 3}r_c} \rightarrow \gamma_{cr} = \sqrt[3]{\frac{7}{4\pi}\frac{R}{r_c}}$

Classical radius of the electron is 2.82E-15 m we get critical γ_{cr} = 2.72x10^7. The rest energy of electron is $m_e c^2$=0.511x10^6 eV (0.511 MeV), it means that the dead-end energy of electron storage ring at Earth is

$$E_{cr\,e} = 2\gamma_{cr} m_e c^2 = 13.9 \cdot 10^{12}\, eV = 13.9\, TeV$$

Rest energy of a muon is $m_\mu c^2$=1.057 x10^8 eV (106 MeV), classical radius of 1.36E-17 m, γ_{cr} = 1.61x10^8 and

$$E_{cr\,\mu} = 2\gamma_{cr} m_\mu c^2 = 1.70 \cdot 10^{16}\, eV = 17{,}002\ TeV$$

For proton with $m_p c^2$=1.057 x10^8 eV (106 MeV), classical radius of 1.53E-18 m, γ_{cr} = 3.33x10^8 and

$$E_{cr\,p} = 2\gamma_{cr} m_p c^2 = 3.13 \cdot 10^{17}\, eV = 3.13 \cdot 10^5\ TeV$$

Note, that the later will require average bending magnetic field of 164 T, which is not within reach of current technology.

Problem 2. 10 points. Circulating particle in magnetic field

The losses of ultra-relativistic charge particle circulating is constant magnetic fields is
(a)

$$\frac{1}{R} = \frac{eB}{pc} \cong \frac{eB}{E} = \frac{eB}{\gamma mc^2} = \frac{1}{\gamma}\frac{1}{\rho_m}; \frac{d\gamma}{dt} \cong \frac{1}{c}\frac{d\gamma}{ds};\ \rho_m = \frac{mc^2}{eB};$$

$$\frac{d\gamma}{dt} \cong -\frac{2}{3c}\gamma^2 \frac{r_c}{\rho_m{}^2} \to \frac{d\gamma}{\gamma^2} = \frac{2}{3c}\frac{r_c}{\rho_m{}^2} dt = \frac{2}{3}\frac{r_c}{\rho_m{}^2} ds;$$

$$\gamma^{-1} - \gamma_o{}^{-1} = \frac{2}{3}\frac{r_c}{\rho_m{}^2} s = \frac{2}{3c}\frac{r_c}{\rho_m{}^2} t \to \gamma(t) = \frac{\gamma_o}{1 + \frac{2}{3c}\frac{r_c}{\rho_m{}^2}\gamma_o t}$$

$$\gamma(s) = \frac{\gamma_o}{1 + \frac{2\gamma_o}{3}\frac{r_c}{\rho_m{}^2} s}$$

(b) The easiest is to describe it are radius dependence on the bending angle in parametric form:

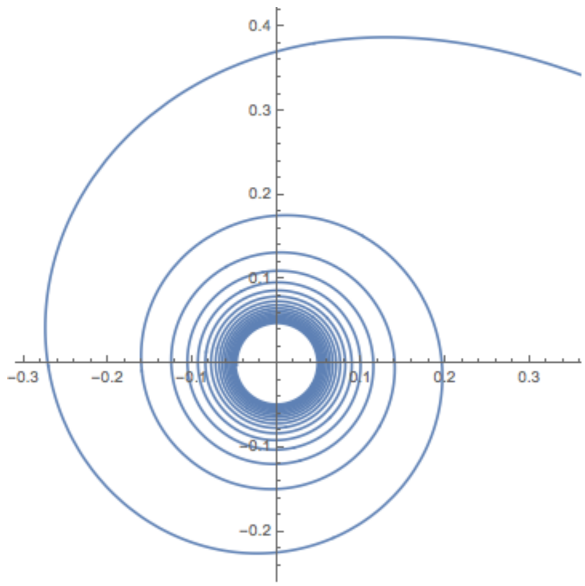
$$d\theta = \frac{ds}{R(s)} = \frac{1 + \frac{2\gamma_o}{3}\frac{r_c}{\rho_m{}^2} s}{\gamma_o \rho_m} ds;$$

$$R(s) = \gamma \rho_m = \frac{\gamma_o \rho_m}{1 + \frac{2\gamma_o}{3}\frac{r_c}{\rho_m{}^2} s}; \theta = \frac{s + \frac{\gamma_o}{3}\frac{r_c}{\rho_m{}^2} s^2}{\gamma_o \rho_m}.$$

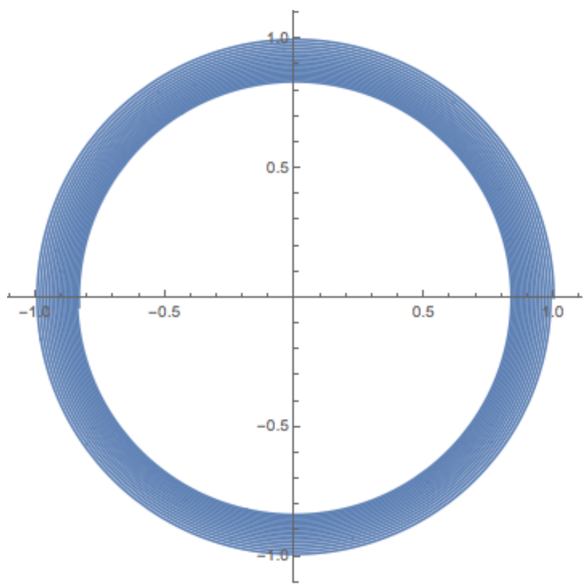
In dimensionless form it will be

$$\frac{R(s)}{\gamma_o \rho_m} = \frac{R(s)}{R_o} = \frac{1}{1+2\zeta}; \theta = \alpha\zeta(1+\zeta);\ \zeta = \frac{\gamma_o}{3}\frac{r_c}{\rho_m{}^2}s; \alpha = \frac{3\rho_m}{\gamma_o{}^2 r_c};$$

For $\alpha = 1$:



For $\alpha = 1000$:



Since energy is lost, it is always a collapsing