PHY 564
Advanced Accelerator Physics
Lectures 4
Choosing the EM Gauge and expanding Accelerator Hamiltonian

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Choosing a specific gauge

This allows us to express 4-potential as explicit function of electric and magnetic fields – it is useful when you explain what you need to build to engineers…

The equations (114) and (115) are the general form of the single-particle Hamiltonian equation in an accelerator. It undoubtedly is nonlinear (the square root signifies relativistic mechanics), and cannot be solved analytically in general. Only few specific cases allow such solutions.

The only additional option we have is to choose a gauge for the 4-potential. One good choice (my preference) is to make the vector potential equal to zero at the reference trajectory. Two other auxiliary conditions will allow us to express the components of the 4-vector potential in a form of the Taylor series:

$$a)\ \vec{A}(s,0,0,t)=0;\ b)\ \partial_x^n A_1\Big|_{s,0,0,t}=\partial_y^n A_3\Big|_{s,0,0,t}=0;\ c)\ \frac{\partial A_1}{\partial y}\bigg|_{s,0,0,t}+\frac{\partial A_3}{\partial x}\bigg|_{s,0,0,t}=0 \qquad (116)$$

that can be achieved by gauge transformation

$$\vec{A} = \tilde{\vec{A}} - \vec{\nabla} f; \varphi = \varphi + \frac{1}{c}\frac{\partial f}{\partial t}; f = f_a = f_b + f_c$$

$$f_a = \int_0^s \tilde{A}_2(s_1,0,0,t)\,ds_1 + \tilde{A}_1(s_1,0,0,t)\cdot x + \tilde{A}_3(s_1,0,0,t)\cdot y$$

$$f_b = \sum_{n=1}\left(\partial_x^n \tilde{A}_1\Big|_{s,0,0,t}\frac{x^{n+1}}{(n+1)!} + \partial_y^n \tilde{A}_3\Big|_{s,0,0,t}\frac{y^{n+1}}{(n+1)!}\right) \tag{117}$$

$$f_c = \frac{1}{2}\sum_{n,k=0}\partial_x^n \partial_y^k\left(\partial_y \tilde{A}_1 + \partial_x \tilde{A}_3\right)\frac{x^{n+1}}{(n+1)!}\frac{y^{k+1}}{(n+1)!}$$

$$a)\ \vec{A}(s,0,0,t) = 0;\ b)\ \partial_x^n A_1\Big|_{s,0,0,t} = \partial_y^n A_3\Big|_{s,0,0,t} = 0;\ c)\ \frac{\partial A_1}{\partial y}\Bigg|_{s,0,0,t} + \frac{\partial A_3}{\partial x}\Bigg|_{s,0,0,t} = 0$$

$$a)\ \vec{A}(s,0,0,t)=0;\ \ b)\ \partial_x^n A_1\Big|_{s,0,0,t}=\partial_y^n A_3\Big|_{s,0,0,t}=0;\ \ c)\ \frac{\partial A_1}{\partial x}\Big|_{s,0,0,t}+\frac{\partial A_3}{\partial y}\Big|_{s,0,0,t}=0$$

Conditions (116) have following important consequences:

$$a)\,\partial_s^k\partial_t^l\vec{A}(s,0,0,t)\equiv 0;\ b)A_1(s,x,0,t)\equiv 0;$$

$$A_3(s,0,y,t)\equiv 0;\ c)\,\partial_s^k\partial_t^l\partial_x^m\partial_y^n\left(\frac{\partial A_1}{\partial x}+\frac{\partial A_3}{\partial y}\right)\equiv 0 \qquad (116{+}{+})$$

We do not assume any additional restrictions on electric and magnetic field, except that they satisfy Maxwell equations:

$$\vec{B} = curl\vec{A},\ \vec{E} = -\frac{1}{c}\cdot\frac{\partial\vec{A}}{\partial t} - \vec{\nabla}\varphi \Rightarrow div\vec{B} = 0;\ \ curl\vec{E} = -\frac{1}{c}\cdot\frac{\partial\vec{B}}{\partial t},$$

and we do not apply any additional (such as free space) conditions on second pair of Maxwell equations:

$$div\vec{E} = 4\pi\rho;\ \ curl\vec{B} = \frac{1}{c}\cdot\frac{\partial\vec{E}}{\partial t} + 4\pi\vec{j},$$

i.e. we include so-called collective effects (space charge and wake-field induced by our beam in the surrounding environment) into the equations of motions. When we discuss a single particle motion in vacuum, we can neglect its influence on macroscopic field of the accelerator and we will use homogenous second pair of Maxwell equations with $\rho = 0;\ \vec{j} = 0$. But this conditions will be specifically stated when we are using them. Here we do not apply any limitations for $\rho,\ \vec{j}$.

One more important definition of the EM field components:

$$\vec{E}(s,x,y,t) = \vec{E}(s,o,o,t) + \sum_{n+m>0} \partial_x^n \partial_y^m \vec{E}\Big|_{x=y=0} \cdot \frac{x^n}{n!}\cdot\frac{y^m}{m!} \equiv \vec{E}\Big|_{ro} + \sum_{n+m>0} \partial_x^n \partial_y^m \vec{E}\Big|_{ro} \cdot \frac{x^n}{n!}\cdot\frac{y^m}{m!};$$

$$\vec{B}(s,x,y,t) = \vec{B}(s,o,o,t) + \sum_{n+m>0} \partial_x^n \partial_y^m \vec{B}\Big|_{x=y=0} \cdot \frac{x^n}{n!}\cdot\frac{y^m}{m!} \equiv \vec{B}_{ro} + \sum_{n+m>0} \partial_x^n \partial_y^m \vec{B}\Big|_{ro} \cdot \frac{x^n}{n!}\cdot\frac{y^m}{m!};$$

$$\vec{E}(s,x,y,t) = \vec{\tau}(s)\cdot E_s(s,x,y,t) + \vec{n}(s)\cdot E_x(s,x,y,t) + \vec{b}(s)\cdot E_y(s,x,y,t);$$

$$\vec{B}(s,x,y,t) = \vec{\tau}(s)\cdot B_s(s,x,y,t) + \vec{n}(s)\cdot B_x(s,x,y,t) + \vec{b}(s)\cdot B_y(s,x,y,t).$$

After a one-page-long exercise, using the first pair of Maxwell equations and conditions (116), one can express the 4-potential in this gauge though the components of the magnetic- and electric- fields, in other words, make an unique vector potential:

$$A_1 = \frac{1}{2}\sum_{n.k=0}^{\infty} \partial_x^k \partial_y^n B_s\Big|_{ro} \frac{x^k}{k!}\frac{y^{n+1}}{(n+1)!};\ A_3 = -\frac{1}{2}\sum_{n.k=0}^{\infty} \partial_x^k \partial_y^n B_s\Big|_{ro} \frac{x^{k+1}}{(k+1)!}\frac{y^n}{n!}$$

$$A_2 = \sum_{n=1}^{\infty}\left\{\partial_x^{n-1}\left(\left(1+Kx\right)B_y + \kappa x B_s\right)_{ro}\frac{x^n}{n!} - \partial_y^{n-1}\left(\left(1+Kx\right)B_x - \kappa y B_s\right)_{ro}\frac{y^n}{n!}\right\} +$$

$$+\frac{1}{2}\sum_{n.k=1}^{\infty}\left\{\partial_x^{n-1}\partial_y^k\left(\left(1+Kx\right)B_y + \kappa x B_s\right)_{ro}\frac{x^n}{n!}\frac{y^k}{k!} - \partial_x^n\partial_y^{k-1}\left(\left(1+Kx\right)B_x - \kappa y B_s\right)_{ro}\frac{x^n}{n!}\frac{y^k}{k!}\right\};$$

$$\varphi = \varphi_o\left(s,t\right) - \sum_{n=1}^{\infty}\partial_x^{n-1}E_x\Big|_{ro}\frac{x^n}{n!} - \sum_{n=1}^{\infty}\partial_y^{n-1}E_y\Big|_{ro}\frac{y^n}{n!} - \frac{1}{2}\sum_{n.k=1}^{\infty}\left(\partial_x^{n-1}\partial_y^k E_x\Big|_{ro} + \partial_x^n\partial_y^{k-1}E_y\Big|_{ro}\right)\frac{x^n}{n!}\frac{y^k}{k!};$$

(118)

where $f\big|_{ro}$; $\left(f\right)_{ro}$ denotes that the value of the function $\boldsymbol{f}$ is taken at the reference orbit $r_o(s)$: i.e., at $x = 0$; $y = 0$, but in an arbitrary moment of time $\boldsymbol{t}$.

Repeat from Lecture 3: Most General Form of the Accelerator Hamiltonian

$$h^* = -(1+Kx)\sqrt{\frac{(H-e\varphi)^2}{c^2} - m^2c^2 - \left(P_1 - \frac{e}{c}A_1\right)^2 - \left(P_3 - \frac{e}{c}A_3\right)^2}$$

$$-\frac{e}{c}A_2 + \kappa x\left(P_3 - \frac{e}{c}A_3\right) - \kappa y\left(P_1 - \frac{e}{c}A_1\right)$$

$$x' = \frac{dx}{ds} = \frac{\partial h^*}{\partial P_1}; \quad \frac{dP_1}{ds} = -\frac{\partial h^*}{\partial x}; \qquad y' = \frac{dy}{ds} = \frac{\partial h^*}{\partial P_3}; \quad \frac{dP_3}{ds} = -\frac{\partial h^*}{\partial y}$$

$$t' = \frac{dt}{ds} = \frac{\partial h^*}{\partial P_t}; \; -\frac{\partial h^*}{\partial H}; \quad \frac{dP_t}{ds} = -\frac{\partial h^*}{\partial t}; \quad \frac{dH}{ds} = \frac{\partial h^*}{\partial t}$$

We always have a choice of the reference orbit (e.g. K and κ) as well as of the gauge of 4-potential. We can use this flexibility for our benefit!
We will use a specific gauge to express components of 4-potentail as explicit functions of electric and magnetic fields

We reserve the notions $f\big|_{ref}$; $(f)_{ref}$ for values taken at the reference trajectory $\dot{r} = \dot{r}_o(s)$ at the reference time $t = t_o(s)$. It is noteworthy that the value of our new Hamiltonian for the reference particle is the full particle's momentum with the minus sign:

$$h^*\big|_{ref} = -p_o(s) \tag{119}$$

We should note that $\varphi_o(s,t)$ is determined with the accuracy of an arbitrary constant, which can be eliminated by requesting $\varphi_o\left(s_o, t_o(s_o)\right) = 0$ at some point along the reference trajectory. The coefficients in (118) can be expanded further using a trivial time series

$$f(t) = f(t_o(s)) + \sum_{n=1}^{\infty} \frac{d^n f}{dt^n}\bigg|_{t=t_o(s)} \frac{(t - t_o(s))^n}{n!}.$$

One important feature of this expansion that no conditions in the EM field are assumed; thus, it can be in free-space field (typical for single-particle dynamics) or a field with sources (for example, charges and currents of beam are examples). Hence, the expansion is applicable to any arbitrary accelerator problem.

An equilibrium particle and a reference trajectory.

A particle that follows the reference trajectory is called an equilibrium (or reference) one:

$$\vec{r} = \vec{r}_o(s); \quad t = t_o(s); \quad H = H_o(s) = E_o(s) + e\varphi_o(s, t_o(s)), \tag{120}$$

with $x \equiv 0; \; y \equiv 0; \; p_x \equiv 0; p_y \equiv 0$. This is where condition (L2.20a) $\vec{A}\big|_{ref} = 0$ is useful, i.e., for

$$x\big|_{ref} = 0; \; y\big|_{ref} = 0; \; P_1\big|_{ref} = p_x\big|_{ref} + \frac{e}{c}A_1\big|_{ref} \equiv 0; \quad P_3\big|_{ref} = p_y\big|_{ref} + \frac{e}{c}A_3\big|_{ref} \equiv 0. \tag{121}$$

The differential form of (121)

$$\begin{aligned} &\frac{dx}{ds}\bigg|_{ref} = \frac{\partial h^*}{d\,P_1}\bigg|_{ref} = 0; \quad \frac{dy}{ds}\bigg|_{ref} = \frac{\partial h^*}{dP_3}\bigg|_{ref} = 0; \\ &\frac{dP_1}{ds}\bigg|_{ref} = -\frac{\partial h^*}{dx}\bigg|_{ref} = 0; \quad \frac{dP_3}{ds}\bigg|_{ref} = -\frac{\partial h^*}{dy}\bigg|_{ref} = 0; \end{aligned} \tag{122}$$

should be combined with the expression for the Hamiltonian (L2.19). The two first equations in (122) give us the already known conditions, viz., that of the zero transverse component of momentum. The following two equations are not as trivial; they set the two conditions at the reference orbit.

Completing a trivial differentiation on ***x*** (where most of the terms are turned into zero at the reference orbit, except $\square_x \varphi$ and $\square_x A_2$) we have

$$-\frac{\partial h^*}{\partial x}\bigg|_{ref} = K\sqrt{G}\Big|_{ref} - \left(1+Kx\right)_{ref} \frac{\left(\left[\frac{eE}{c^2}\frac{\partial \varphi}{\partial x}\right] + p_x \frac{e}{c}\frac{\partial A_1}{\partial x} + p_y \frac{e}{c}\frac{\partial A_3}{\partial x}\right)_{ref}}{\sqrt{G}\Big|_{ref}}$$

$$+\left[\frac{e}{c}\frac{\partial A_2}{\partial x}\right]_{ref} + \kappa\left(p_y\right)_{ref} + \kappa\left(\frac{e}{c}\frac{\partial A_1}{\partial x} y - \frac{e}{c}\frac{\partial A_3}{\partial x} x\right)_{ref} = 0$$

$$E \equiv \left(H - e\varphi\right); \quad G = \frac{E^2}{c^2} - m^2 c^2 - p_x^2 - p_y^2; \sqrt{G}\Big|_{ref} = p_o \ldots$$

Note: The term(s) that do not vanish at the limit are identified by the square brackets [...]

and using the above expansions, we derive the well-know equation for the curvature of the trajectory:

$$K(s) \equiv \frac{1}{\rho(s)} = -\frac{e}{p_o c}\left(B_y\Big|_{ref} + \frac{E_o}{p_o c} E_x\Big|_{ref} \right). \tag{123}$$

Differentiation on ***y*** is similar

$$-\frac{\partial h^*}{\partial y}\bigg|_{ref} = -(1+Kx)_{ref}\frac{\left(\left[\frac{eE}{c^2}\frac{\partial\varphi}{\partial y}\right]+p_x\frac{e}{c}\frac{\partial A_1}{\partial y}+p_y\frac{e}{c}\frac{\partial A_3}{\partial y}\right)_{ref}}{\sqrt{G}\big|_{ref}}$$

$$-\left[\frac{e}{c}\frac{\partial A_2}{\partial y}\right]_{ref}-(p_x)_{ref}+\kappa\left(\frac{e}{c}\frac{\partial A_1}{\partial y}y-\frac{e}{c}\frac{\partial A_3}{\partial y}x\right)_{ref}=0$$

and yields

$$B_x\big|_{ref} = \frac{E_o}{p_o c}E_y\big|_{ref}, \tag{124}$$

That represents only the absence of "vertical curvature". The difference between (123) and (124) arises from the choice of coordinates in Frenet-Serret system: ***x*** (i.e., $\boldsymbol{q_1}$) corresponds to the plane where trajectory bends.

The conditions in (120) for the arrival time of the reference particle and values of its Hamiltonian are also informative, but not surprising. First, the condition on the arrival time

$$\frac{dt_o(s)}{ds} = -\frac{\partial h^*}{dH}\bigg|_{ref} = \frac{H-e\varphi}{c^2\sqrt{G}}\bigg|_{ref} = \frac{H_o-e\varphi_o}{p_o c^2} \equiv \frac{E_o}{p_o c^2} = \frac{1}{\mathrm{v}_o(s)} \tag{125}$$

gives an understandable definition of velocity along trajectory: ***v=ds/dt,*** and the velocity of the reference particle $\mathrm{v}_o = p_o c^2/E_o$.

The condition on energy (3D Hamiltonian) gives

$$\left.\frac{\partial h^*}{\partial t}\right|_{ref} = (1+Kx)_{ref} \frac{\left(\left[\frac{eE}{c^2}\frac{\partial\varphi}{\partial t}\right] + p_x \frac{e}{c}\frac{\partial A_1}{\partial t} + p_y \frac{e}{c}\frac{\partial A_3}{\partial t}\right)_{ref}}{\left.\sqrt{G}\right|_{ref}}$$

$$-\left(\frac{e}{c}\frac{\partial A_2}{\partial t}\right)_{ref} + \kappa\left(\frac{e}{c}\frac{\partial A_1}{\partial x} y - \frac{e}{c}\frac{\partial A_3}{\partial x} x\right)_{ref} = \left.\frac{eE_o}{p_o c^2}\frac{\partial\varphi}{\partial t}\right|_{ref} \qquad (126)s$$

$$\frac{dH_o(s)}{ds} = \left.\frac{\partial h^*}{d\,t}\right|_{ref} = \left.\frac{eE_o}{p_o c^2}\frac{\partial\varphi}{\partial t}\right|_{ref};$$

which can be transferred using $H = E + e\varphi$ and $d\varphi_o(s,t_o(s)) = \frac{\partial\varphi_o}{\partial s} ds + \frac{\partial\varphi_o}{\partial t}\frac{ds}{\mathrm{v}_o(s)}$ into the energy gain of the reference particle along is trajectory:

$$\frac{dE_o(s)}{ds} = \frac{d\left(H_o(s) - \varphi_o(s,t_o(s))\right)}{ds} = -e\left.\frac{\partial\varphi}{\partial s}\right|_{ref} \equiv eE_2(s,t_o(s)). \qquad (127)$$

Time/Hamiltonian Canonical Pair

As discussed before, accelerator designers face the problem of ensuring that the reference particle faithfully follows the reference trajectory. Our goal is to use the above conditions to the maximum, and, as we see below, to eliminate zero- order terms from the equations of motion. By selecting the reference trajectory as basis for our coordinate system, we set the transverse coordinates and momenta at zero at the reference orbit. Hence, two canonical pairs have a good and solid origin.

The third pair ***(-t,H)*** is odd; it is not zero for the reference particle. Furthermore, it has different units. Hence, we can move step forward with a more natural Canonical pair $\{q_\tau = -ct, p_\tau = H/c\}$ - whose generating function is obvious: $\Phi\left(q = -t, \tilde{P} = p_\tau\right) = -ct \cdot p_\tau$. In this case, the analogy is complete: $q_\tau = -ct$ has the dimension of distance and is just ***–x*** in 4D space, while $p_\tau = H/c$ has the dimension of momentum and is just ***P*** in 4D space.

* It worth mentioning that q_τ is positive for particles in front of the reference particle i.e. those arriving earlier (head of the bunch) and negative for those arriving later (tail of the bunch)

From Lecture 3: Most General Form of the Accelerator Hamiltonian

$$h^* = -\left(1+Kx\right)\sqrt{\frac{\left(H-e\varphi\right)^2}{c^2} - m^2c^2 - \left(P_1 - \frac{e}{c}A_1\right)^2 - \left(P_3 - \frac{e}{c}A_3\right)^2}$$

$$-\frac{e}{c}A_2 + \kappa x\left(P_3 - \frac{e}{c}A_3\right) - \kappa y\left(P_1 - \frac{e}{c}A_1\right)$$

$$x' = \frac{dx}{ds} = \frac{\partial h^*}{\partial P_1}; \quad \frac{dP_1}{ds} = -\frac{\partial h^*}{\partial x}; \qquad y' = \frac{dy}{ds} = \frac{\partial h^*}{\partial P_3}; \quad \frac{dP_3}{ds} = -\frac{\partial h^*}{\partial y}$$

$$t' = \frac{dt}{ds} = \frac{\partial h^*}{\partial P_t} \square -\frac{\partial h^*}{\partial H}; \quad \frac{dP_t}{ds} = -\frac{\partial h^*}{\partial t} \square \quad \frac{dH}{ds} = \frac{\partial h^*}{\partial t}$$

We also should select variables that are zero at the reference orbit. The following pair is one of better choices:

$$\left\{ \tau = -c(t - t_o(s)),\ \ \delta = \left(H - E_o(s) - e\varphi_o(s,t)\right)/c \right\}, \tag{128}$$

which are zero for the reference particle. Generation function is easily to come with:

$$\Phi(q,\tilde{P},s) = \tilde{P}_1 x + \tilde{P}_3 y - \left(E_o(s) + c\delta\right)\left(t - t_o(s)\right) - e\int^{t} \varphi_o(s,t_1)\,dt_1, \tag{129}$$

and it produces what is desired:

$$P_1 = \frac{\partial\Phi}{\partial x} = \tilde{P}_1;\ \ P_3 = \frac{\partial\Phi}{\partial y} = \tilde{P}_3; H = \frac{\partial\Phi}{\partial(-t)} = E_o + c\delta + e\varphi_o(s,t);$$

$$\tilde{q}_1 = \frac{\partial\Phi}{\partial\tilde{P}_1} = x;\ \ \tilde{q}_3 = \frac{\partial\Phi}{\partial\tilde{P}_3} = y;\ \ \tilde{q}_\delta = \frac{\partial\Phi}{\partial\delta} = -c\left(t - t_o(s)\right) = \tau$$

$$\tilde{h} = h + \frac{\partial\Phi}{\partial s} = h + \frac{E_o(s) + c\delta}{\mathrm{v_o}(s)} + E_o'(s)\tau/c - e\int^{t} \varphi_o'(s,t_1)\,dt_1$$

The change to the Hamiltonian comprised only of meaningful terms as well as just a trivial function of ***s***, ***g(s):***

$$\frac{\partial \Phi}{\partial s} = \frac{c}{\mathrm{v_o}(s)}\delta - e\varphi_{//}(s,\tau) + g(s);$$

$$g(s) = E_o(s)/\mathrm{v_o}(s) - e\int^{t_o(s)} \varphi_o'(s,t_1)\,dt_1$$

$$\varphi_{//}(s,\tau) =_{def} \frac{\partial}{\partial s}\int_0^{-\tau/c}\Big(\varphi_o(s,t_o(s)+\zeta) - \varphi_o(s,t_o(s))\Big)d\zeta\ \square \tag{131}$$

$$-\int_0^{-\tau/c}\Big(E_2(s,t_o(s)+\zeta) - E_2\big|_{ref}\Big)d\zeta$$

where we used eq. (127) as $E_o'(s) = -\left.\frac{\partial \varphi}{\partial s}\right|_{ref}$. Additive ***g(s)*** simply can be dropped from the Hamiltonian - it does not change equations of motion.

Now the only remaining task is to express the new Hamiltonian function with an updated canonical pair (130) and (115):

$$\tilde{h} = -(1+Kx)\sqrt{p_o^2 + \frac{2E_o}{c}\left(\delta - \frac{e}{c}\varphi_\perp\right) + \left(\delta - \frac{e}{c}\varphi_\perp\right)^2 - \left(P_1 - \frac{e}{c}A_1\right)^2 - \left(P_3 - \frac{e}{c}A_3\right)^2} + \tag{132}$$

$$-\frac{e}{c}A_2 + \kappa x\left(P_3 - \frac{e}{c}A_3\right) - \kappa y\left(P_1 - \frac{e}{c}A_1\right) + \frac{c}{\mathrm{v_o}}\delta - \frac{e}{c}\varphi_{//}(s,\tau)$$

where we used following trivial expansion and definition:

$$\frac{\left(E_o + c\delta + e\varphi_o(s,t) - e\varphi\right)^2}{c^2} - m^2c^2 = p_o{}^2 + \frac{2E_o}{c}\left(\delta - \frac{e}{c}\varphi_\perp\right) + \left(\delta - \frac{e}{c}\varphi_\perp\right)^2; \tag{133}$$

$$\varphi_{\perp def} = \varphi(s,x,y,t) - \varphi_o(s,t) \equiv \varphi(s,x,y,t) - \varphi(s,0,0,t)$$

Scaling variables.

Frequently, it is useful to scale one of canonical variables. Typical scaling in accelerator physics involves dividing the canonical momenta P_1, P_3, δ by the momentum of the reference particle:

$$\pi_1 = \frac{P_1}{p_o}; \ \ \pi_3 = \frac{P_3}{p_o}; \ \ \pi_o = \frac{\delta}{p_o}. \qquad (134)$$

These variables are dimensionless and also are close to $x', y', \ \delta E / p_o c$ for small deviations. Such scaling only is allowed in Hamiltonian mechanics when the scaling parameter is constant, i.e., is not function of *s*. Thus, scaling by the particle's momentum remains within the framework of Hamiltonian mechanics only if the reference particle's momentum is constant, that is, when the longitudinal electric field is zero along the reference particle's trajectory (i.e. at moment $t = t_o(s)$). One similarly can scale the coordinates by a constant.

$$\xi_1 = \frac{x}{L}; \ \ \xi_3 = \frac{y}{L}; \ \ \xi_o = \frac{\tau}{L}.$$

Scaling by a constant is easy; divide the Hamiltonian by the constant and rename the variables. Hence, transforming (134) with constant, called $p_{o,}$ will make Hamiltonian (132) into

$$h = -(1+Kx)\sqrt{1 + \frac{2E_o}{p_o c}\left(\delta - \frac{e}{p_o c}\varphi_\perp\right) + \left(\delta - \frac{e}{p_o c}\varphi_\perp\right)^2 - \left(\pi_1 - \frac{e}{p_o c}A_1\right)^2 - \left(\pi_3 - \frac{e}{p_o c}A_3\right)^2} +$$

$$- \frac{e}{p_o c}A_2 + \kappa x\left(\pi_3 - \frac{e}{p_o c}A_3\right) - \kappa y\left(\pi_1 - \frac{e}{p_o c}A_1\right) + \frac{c}{\mathrm{v}_o}\delta - \frac{e}{p_o c}\varphi_{//}(s,\tau)$$

(132 @ **constant energy**)

Usage of this Hamiltonian is very popular for storage rings or transport channels, wherein the energy of the particles remains constant in time. It should not be employed for particles undergoing an acceleration.

Normalizing momenta.

Dimensionless momenta are useful and this is scaling them by constant mc is frequently used:

$$\pi_{1,3} = \frac{P_{1.3}}{mc}; \; \pi_0 = \frac{\delta}{mc}. \quad . \qquad (134^*)$$

These variables are dimensionless and also are close to $x', y', \; \delta E / p_o c$ for small deviations.

Scaling by a constant is easy; divide the Hamiltonian by the constant and rename the variables. Hence, transforming (134) with constant, called p_o, will make Hamiltonian (132) into

$$\tilde{h} = -(1+Kx)\sqrt{\frac{E_o^2 - mc^2}{mc^2} + \frac{2E_o}{mc^2}\left(\pi_0 - \frac{e}{mc^2}\varphi_\perp\right) + \left(\pi_0 - \frac{e}{mc^2}\varphi_\perp\right)^2 - \left(\pi_1 - \frac{e}{mc^2}A_1\right)^2 - \left(\pi_3 - \frac{e}{mc^2}A_3\right)^2} +$$

$$-\frac{e}{mc^2}A_2 + \kappa x\left(\pi_3 - \frac{e}{mc^2}A_3\right) - \kappa y\left(\pi_1 - \frac{e}{mc^2}A_1\right) + \frac{c}{v_o}\pi_0 - \frac{e}{mc^2}\varphi_{//}(s,\tau)$$

It can further "beatified" using dimensionless variables

$$\gamma_o(s) = \frac{E_o(s)}{mc^2}; \; \beta_o(s) = \frac{v_o(s)}{c}; \; a_k = \frac{eA_k}{mc^2}; \; a_\parallel = \frac{e\varphi_\parallel}{mc^2}; \; a_\perp = \frac{e\varphi_\perp}{mc^2},$$

With the normalized accelerator Hamiltonian of

$$\tilde{h} = -(1+Kx)\sqrt{\gamma_o^2 - 1 + 2\gamma_o^2(\pi_0 - a_\perp) + (\pi_0 - a_\perp)^2 - (\pi_1 - a_1)^2 - (\pi_3 - a_3)^2} + \qquad (132n)$$

$$-a_2 + \kappa x(\pi_3 - a_3) - \kappa y(\pi_1 - a_1) + \frac{\pi_0}{\beta_o} - a_{//}(s,\tau).$$

Expanding the Hamiltonian.

Expanding the Hamiltonian (132) is a nominal tool in accelerator physics that allows the separation of the effects of various orders and sometimes the use of perturbation-theory approaches. Having completed the process of creating canonical variables, which are zero for the reference particle, the next step is to assume (which is true for operational accelerators) that the relative deviations of momenta are small

$$\left|\frac{P_1}{p_o}\right| << 1; \quad \left|\frac{P_3}{p_o}\right| << 1; \quad \left|\frac{\delta}{p_o}\right| << 1;$$

and that the EM fields are sufficiently smooth around the reference trajectory to allow expansion in terms of $x;\ y;\ \tau$. We will consider* all six variables to be of the same order (of infinitesimally, ε). We call the order of expansion to be the maximum total power in a product that is any combination of x,y,τ,P_1,P_2,δ. Unless there is a good reason not to do so, we truncate the series using this rule.

*Sometimes, one can keep explicit the time dependence of fields and expand only the rest of the variables. One such case is an approximate, and useful, description of synchrotron oscillation.

The general expansion of Hamiltonian (132) can be accomplished via the already derived expansion for 4-potential and the well-known expansion of the square root:

$$\tilde{h} = -(1+Kx)\,p_o\sqrt{\begin{aligned}&1+\frac{2E_o}{p_oc}\left(\frac{\delta}{p_o}-\frac{e}{p_oc}\varphi_\perp\right)+\left(\frac{\delta}{p_o}-\frac{e}{p_oc}\varphi_\perp\right)^2 +\\ &-\left(\frac{P_1}{p_o}-\frac{e}{p_oc}A_1\right)^2-\left(\frac{P_3}{p_o}-\frac{e}{p_oc}A_3\right)^2\end{aligned}}\;; \tag{135}$$

$$+\frac{e}{c}A_2+\kappa x\left(P_3-\frac{e}{c}A_3\right)-\kappa y\left(P_1-\frac{e}{c}A_1\right)+\frac{c}{\mathrm{v}_o}\delta-\frac{e}{c}\varphi_{//}(s,\tau)$$

$$\sqrt{1+g} = 1+\sum_{n=1}^{\infty}(-1)^{n-1}\frac{g^n}{2^n}\frac{(2n-3)!!)}{n!} = 1+\frac{g}{2}+O\left(g^2\right). \tag{136}$$

First order terms. Using expansion (136), we would dispose of p_o (s) as valueless, and look initially at the first-order terms in the expansion about the reference particle and its trajectory:

$$\tilde{h} = C_x x + C_y y + C_\tau \tau + C_{P_1} \frac{P_1}{p_o} + C_{P_3} \frac{P_3}{p_o} + + C_\delta \frac{\delta}{p_o} + O(\varepsilon^2) \tag{137}$$

with

$$\begin{gathered} x' = \frac{\partial \tilde{h}}{\partial P_1} = \frac{C_{P_1}}{p_o} + O(\varepsilon); y' = \frac{\partial \tilde{h}}{\partial P_3} = \frac{C_{P_3}}{p_o} + O(\varepsilon); \tau' = \frac{C_\delta}{p_o} + O(\varepsilon); \\ P_1' = -\frac{\partial \tilde{h}}{\partial x} = -C_x + O(\varepsilon); P_3' = -\frac{\partial \tilde{h}}{\partial y} = -C_y + O(\varepsilon); \delta' = -C_\tau + O(\varepsilon). \end{gathered} \tag{138}$$

Since we defined the reference trajectory and reference particle in such a way that

$$x = 0, y = 0, \tau = 0, P_1 = 0; P_2 = 0, \delta = 0 \tag{139}$$

is exact solution of the equation of motion. Hence, we shall conclude that all first order term in the accelerator Hamiltonian expansion vanish – in other terms, conditions for the reference particle makes them to be zero.

In future lectures we will consider EM field errors, e.g. violations of the condition for the reference particle, and first order terms would appear as perturbation of the reference orbit and conditions for reference particle.

Second order (oscillator) expansion. We continue with ideal condition and expand the Hamiltonian to the next – the most important– order

$$\tilde{h} = \frac{P_1^2 + P_3^2}{2p_o} + F\frac{x^2}{2} + Nxy + G\frac{y^2}{2} + L(xP_3 - yP_1) + \frac{\delta^2}{2p_o}\cdot\frac{m^2c^2}{p_o^2} + U\frac{\tau^2}{2} + g_x x\delta + g_y y\delta + F_x x\tau + F_y y\tau; \qquad (140)$$

$$\frac{F}{p_o} = \left[-K\cdot\frac{e}{p_o c}B_y - \frac{e}{p_o c}\frac{\partial B_y}{\partial x} + \left(\frac{eB_s}{2p_o c}\right)^2\right] - \frac{e}{p_o \mathrm{v}_o}\frac{\partial E_x}{\partial x} - 2K\frac{eE_x}{p_o \mathrm{v}_o} + \left(\frac{meE_x}{p_o^2}\right)^2;$$

$$\frac{G}{p_o} = \left[\frac{e}{p_o c}\frac{\partial B_x}{\partial y} + \left(\frac{eB_s}{2p_o c}\right)^2\right] - \frac{e}{p_o \mathrm{v}_o}\frac{\partial E_y}{\partial y} + \left(\frac{meE_z}{p_o^2}\right)^2;$$

$$\frac{2N}{p_o} = \left[\frac{e}{p_o c}\frac{\partial B_x}{\partial x} - \frac{e}{p_o c}\frac{\partial B_y}{\partial y}\right] - K\cdot\frac{e}{p_o c}B_x - \frac{e}{p_o \mathrm{v}_o}\left(\frac{\partial E_x}{\partial y} + \frac{\partial E_y}{\partial x}\right) - 2K\frac{eE_y}{p_o \mathrm{v}_o} + \left(\frac{meE_z}{p_o^2}\right)\left(\frac{meE_x}{p_o^2}\right)$$

$$L = \kappa + \frac{e}{2p_o c}B_s; \qquad \frac{U}{p_o} = \frac{e}{pc^2}\frac{\partial E_s}{\partial t}; \quad g_x = \frac{(mc)^2\cdot eE_x}{p_o^3} - K\frac{c}{\mathrm{v}_o}; \; g_y = \frac{(mc)^2\cdot eE_y}{p_o^3}; \qquad (141)$$

$$F_x = \frac{e}{c}\frac{\partial B_y}{\partial ct} + \frac{e}{\mathrm{v}_o}\frac{\partial E_x}{\partial ct}; F_y = -\frac{e}{c}\frac{\partial B_x}{\partial ct} + \frac{e}{\mathrm{v}_o}\frac{\partial E_y}{\partial ct}.$$

If momentum p_o is constant, we can use (134) and rewrite Hamiltonian of the linearized motion (140) as

$$\tilde{h}_n = \frac{\pi_1{}^2 + \pi_3{}^2}{2} + f\,\frac{x^2}{2} + n\cdot xy + g\,\frac{y^2}{2} + L(x\pi_3 - y\pi_1) + \frac{\pi_o{}^2}{2}\cdot\frac{m^2c^2}{p_o{}^2} + u\frac{\tau^2}{2} + g_x x\pi_o + g_y y\pi_o + f_x x\tau + f_y y\tau \quad ; \tag{142}$$

with

$$f = \frac{F}{p_o};\ n = \frac{N}{p_o};\ g = \frac{G}{p_o};\ u = \frac{U}{p_o};\ f_x = \frac{F_x}{p_o};\ f_y = \frac{F_y}{p_o}; \tag{144}$$

Note that

$$x' = \frac{\partial h_n}{\partial \pi_1} = \pi_1 - Ly; \quad y' = \frac{\partial h_n}{\partial \pi_3} = \pi_3 + Lx; \quad ; \tag{145}$$

i.e. as soon as L=0, we can use traditional x' and y' as reduced momenta.

Normalizing momenta.

Dimensionless momenta are useful and this is scaling them by constant mc is frequently used:

$$\pi_{1,3} = \frac{P_{1,3}}{mc};\ \pi_0 = \frac{\delta}{mc}. \quad . \qquad (134^*)$$

These variables are dimensionless and also are close to $x', y', \delta E / p_o c$ for small deviations.

Scaling by a constant is easy; divide the Hamiltonian by the constant and rename the variables. Hence, transforming (134) with constant, called p_o, will make Hamiltonian (132) into

$$\tilde{h} = -(1+Kx)\sqrt{\frac{E_o^2 - mc^2}{mc^2} + \frac{2E_o}{mc^2}\left(\pi_0 - \frac{e}{mc^2}\varphi_\perp\right) + \left(\pi_0 - \frac{e}{mc^2}\varphi_\perp\right)^2 - \left(\pi_1 - \frac{e}{mc^2}A_1\right)^2 - \left(\pi_3 - \frac{e}{mc^2}A_3\right)^2} +$$
$$-\frac{e}{mc^2}A_2 + \kappa x\left(\pi_3 - \frac{e}{mc^2}A_3\right) - \kappa y\left(\pi_1 - \frac{e}{mc^2}A_1\right) + \frac{c}{v_o}\pi_0 - \frac{e}{mc^2}\varphi_{//}(s,\tau)$$

It can further "beatified" using dimensionless variables

$$\gamma_o(s) = \frac{E_o(s)}{mc^2};\ \beta_o(s) = \frac{v_o(s)}{c};\ a_k = \frac{eA_k}{mc^2};\ a_\parallel = \frac{e\varphi_\parallel}{mc^2};\ a_\perp = \frac{e\varphi_\perp}{mc^2},$$

With the normalized accelerator Hamiltonian of

$$\tilde{h} = -(1+Kx)\sqrt{\gamma_o^2 - 1 + 2\gamma_o^2\left(\pi_0 - a_\perp\right) + \left(\pi_0 - a_\perp\right)^2 - \left(\pi_1 - a_1\right)^2 - \left(\pi_3 - a_3\right)^2} + \qquad (132n)$$
$$-a_2 + \kappa x\left(\pi_3 - a_3\right) - \kappa y\left(\pi_1 - a_1\right) + \frac{\pi_0}{\beta_o} - a_{//}(s,\tau).$$

For a flat reference orbit - $\kappa = 0$, in the absence of transverse coupling (L=0, N=0) and transverse electric fields, the accelerator Hamiltonian has the form which is used in most of the text books and papers:

$$\tilde{h} = \frac{P_1^2 + P_3^2}{2p_o} + F\frac{x^2}{2} + G\frac{y^2}{2} + \frac{\delta^2}{2p_o}\cdot\frac{m^2c^2}{p_o^2} + U\frac{\tau^2}{2} + g_x x\delta$$

$$or \qquad ; \qquad (146)$$

$$\tilde{h}_n = \frac{x'^2 + y'^2}{2p_o} + f\frac{x^2}{2} + g\frac{y^2}{2} + \frac{\pi_o^2}{2}\cdot\frac{m^2c^2}{p_o^2} + u\frac{\tau^2}{2} + g_x x\delta$$

with

$$f = \left[-K\cdot\frac{e}{p_oc}B_y - \frac{e}{p_oc}\frac{\partial B_y}{\partial x}\right]; \; g = \frac{e}{p_oc}\frac{\partial B_x}{\partial y}; \;\; u = \frac{e}{pc^2}\frac{\partial E_s}{\partial t}; \;\; g_x = -K\frac{c}{\mathrm{v}_o}. \qquad (147)$$

Finally, see an Additional Materials, where Mathematica used for expanding the Accelerator Hamiltonian to an arbitrary order

What we learned today?

- That distance along the reference particle trajectory, *s*, is a natural choice of independent coordinate in accelerator physics
  - Remember, magnets and vacuum chambers are bolted to the floor and are not floating in time
- Accelerator coordinate system than determined by *curvilinear* Frenet-Serret coordinates, e.g. we can not use just simple Cartesian coordinates in most of the cases
- With *s* as independent variable, $-P_2$ becomes the accelerator Hamiltonian with (x,P_1), (y,P_3) and $(t,-H)$ being Canonical pairs
- The Hamiltonian can be expanded to any arbitrary order "about" reference particle's trajectory, momentum/energy and "arriving time" to azimuth *s*
- The condition for reference particle remove first order terms in the Hamiltonian expansion
- Second order is the lowest remaining term in the Hamiltonian.
- It plays fundamental role in accelerator physics since leads to a set of linear s-dependent ordinary differential equation – subject for next class

About differential operators in curvilinear coordinates read:
https://www.jfoadi.me.uk/documents/lecture_mathphys2_05.pdf