Phase Stability, Closed Form

To get a closed form solution, we will convert the difference equations derived in the last lecture to differential equations. We can do so, because phase and energy are continuous enough parameters as suggested by following the trajectory around the phase space. In practical terms, it means that the phase and energy change gradually and in small enough amounts from one turn to another.

start with

$$\begin{cases} \phi_{n+1} = \phi_n + \dfrac{(\omega_{rf}\tau)\,\eta}{\beta^2 E_s}\,\Delta E_{n+1} \\ \Delta E_{n+1} = \Delta E_n + eV\left(\sin(\phi_n) - \sin(\phi_s)\right) \end{cases}$$

treat n as the independant variable: $\dfrac{\phi_{n+1} - \phi_n}{1} \to \dfrac{\Delta\phi}{\Delta n} \to \dfrac{d\phi}{dn}$

$$\frac{d}{dn}\left[\frac{d\phi}{dn} = \frac{\eta\,\omega_{rf}\,\tau\,c^2}{v^2 E_s}\,\Delta E\right] \quad (2.39)$$

$$\frac{d\Delta E}{dn} = eV\,(\sin\phi - \sin\phi_s) \quad 2.40$$

$$\frac{d^2\phi}{dn^2} = \frac{\eta\,\omega_{rf}\,\tau\,c^2}{v^2 E_s}\,\frac{d\Delta E}{dn}$$

$$\Rightarrow \boxed{\frac{d^2\phi}{dn^2} = \frac{\eta\,\omega_{rf}\,\tau\,c^2}{v^2 E_s}\,eV\,(\sin\phi - \sin\phi_s)} \quad (2.41)$$

what about the dependence of v, E_s on n?

→ dropped based on assumption of small dE_s/dn (will be treated in next section)

Multiply both sides by $\dfrac{d\phi}{dn}$ & integrate

$$\int \frac{d^2\phi}{dn^2}\frac{d\phi}{dn}\,dn = \frac{\eta\,\omega_{rf}\,\tau\,c^2 eV}{V^2 E_s}\int (\sin\phi - \sin\phi_s)\frac{d\phi}{dn}\,dn$$

$$\Rightarrow \frac{1}{2}\left(\frac{d\phi}{dn}\right)^2 = -\frac{\eta\,\omega_{rf}\,\tau\,eVc^2}{V^2 E_s}(\cos\phi + \phi\sin\phi_s) + C_1$$

or

$$\frac{1}{2}\left(\frac{d\phi}{dn}\right)^2 + \frac{\eta\,\omega_{rf}\,\tau\,eVc^2}{V^2 E_s}[\cos\phi + \phi\sin\phi_s] = \text{constant} \quad (2.44)$$

"Kinetic energy" + "potential Energy" = "Total Energy"

$$T + V = U$$

By this, we mean that 2.44 has the same form as

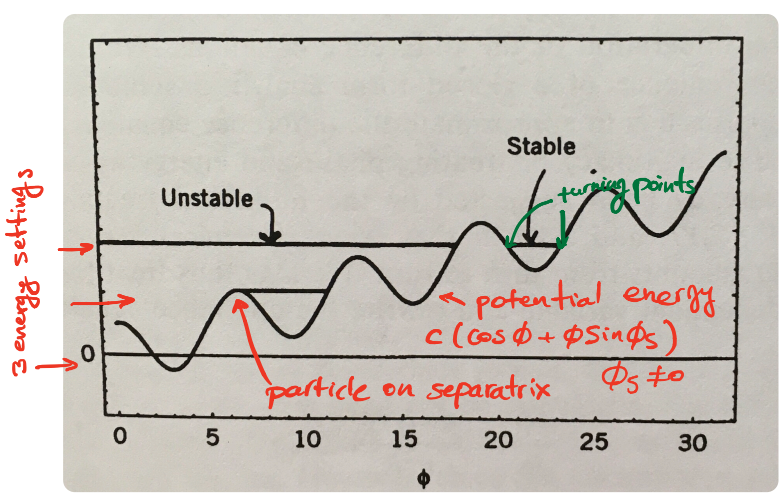
$$\frac{1}{2}\dot{r}^2 + V(r) = \text{Total energy}$$

Not that the terms themselves represent energies of specific particles. This allows us to use our intuition from dynamic motion to determine regions of stability.

Consider three cases w/ three "energies"

→ Potential energy here is for $\phi_s \neq 0$

→ Model the behavior of the system as a ball rolling down a hill



→ stable case is where "kinetic energy" is bounded on either side by "potential energy", where the "ball" would encounter a turning point.

→ The unstable fixed point, which is at the limit between converging sequence of turning points (stable) and unstable motion occurs at $\phi = \pi - \phi_s$

Particle motion in ϕ-ΔE space

Combine 2.39 & 2.44 to replace $\frac{d\phi}{dn}$ & get ΔE vs. ϕ

$$\frac{1}{2}\left[\frac{\eta\, \omega_{rf}\, \tau\, c^2}{v^2 E_s}\Delta E\right]^2 + \frac{\eta\, \omega_{rf}\, \tau\, eV c^2}{v^2 E_s}(\cos\phi + \phi \sin\phi_s) = \text{constant}$$

↑ multiply both sides by the inverse of this factor

$$\Delta E^2 + \frac{2eV v^2 E_s}{\eta\, \omega_{rf}\, \tau\, c^2}\left(\cos\phi + \phi \sin(\phi_s)\right) = \text{constant}$$

Now that we have demonstrated the equivalence of the difference and differential equations (sort of!), we can use these results to treat phase oscillations, by which we mean to find how the phase difference changes from turn to turn.

To do so, we linearize the equations of motion, that is to say, rewrite the equation of phase variation in terms of

$$\Delta\phi = \phi - \phi_s$$

And assume that $\Delta\phi$ is small, allowing us to only keep the terms with the linear terms of this factor and ignore the seconds and third powers of $\Delta\phi$

$$(2.41):\quad \frac{d^2\phi}{dn^2} = \frac{\eta(\omega_{rf}\tau)eV}{\beta^2 E_s}(\sin\phi - \sin\phi_s)$$

$\Delta\phi = \phi - \phi_s$ is small

$$\therefore \sin(\phi_s + \Delta\phi) - \sin\phi_s$$

$$= \cos\phi_s \sin\Delta\phi + \underbrace{\cos\Delta\phi}_{=1} \sin\phi_s - \sin\phi_s$$

$$\simeq \cos\phi_s \, \Delta\phi \quad + \sin\phi_s - \sin\phi_s$$

$$= \cos\phi_s \, \Delta\phi$$

$$\frac{d^2\phi}{dn^2} = \frac{d^2}{dn^2}(\phi_s + \Delta\phi) = \frac{d^2\Delta\phi}{dn^2}$$

(ϕ_s does not change from one turn to next)

$$2.41 \Rightarrow \frac{d^2\Delta\phi}{dn^2} + \underbrace{\frac{(-)\eta\, \omega_{rf} \tau\, eV c^2}{v^2 E_s 4\pi^2} \cos\phi_s}_{\text{call this } \nu_s^2} (4\pi^2) = 0$$

$$\nu_s = \sqrt{-\frac{\eta(\omega_{rf}\tau)\, eV \cos\phi_s}{4\pi^2 \beta^2 E_s}}$$: synchrotron oscillation tune

$$\frac{d^2\Delta\phi}{dn^2} + \underbrace{(2\pi\nu_s)^2}_{\text{equivalent to an } \omega_s^2} \Delta\phi = 0$$

The solution of this equation is

- an exponential (unstable) for $(2\pi\nu_s)^2 < 0$, i.e. if ν_s is imaginary
- a sinusoide (stable) for $(2\pi\nu_s)^2 > 0$, i.e. if ν_s is real

for ν_s to be real, $\eta \cos\phi_s$ must be less than zero,

i.e. if $\eta < 0$, $\gamma < \gamma_t$, motion is stable for $\cos\phi_s > 0$

if $\eta > 0$, $\gamma > \gamma_t$, motion is stable for $\cos\phi_s < 0$

Thus, circular accelerators that cross the transition energy during the acceleration cycle must perform a phase jump in the radio frequency system at the appropriate time in order to maintain phase stability.

ν represents the number of synchrotron oscillations per accelerating station. To get the more familiar units of $(\text{time})^{-1}$, we will use the angular frequency of synchrotron oscillations, which is

$$\frac{2\pi\nu_s}{t} \equiv \Omega_s = \left[-\frac{\eta\,\omega_{rf}\,c^2 eV\cos\phi_s}{\tau v^2 E_s}\right]^{1/2}$$

At the transition energy, synchrotron oscillation period becomes infinite and there is no phase focusing

$$\Omega_s = 0 \text{ at } \eta = 0 \Rightarrow \frac{1}{\Omega_s} = \text{period} \Rightarrow \infty$$

$$\Delta\phi = C_1 n^2 + C_2 n + C_3 \quad (\text{no focusing})$$

In case of a linac,

$\eta \to \frac{1}{\gamma^2}$ (no difference in path lengths between accelerating sections)

as $E_s \to$ high, ν_s & $\Omega_s \to 0$, $\frac{d^2(\Delta\phi)}{dn^2} = 0$

e.g. SLAC linac; $\gamma \gg 1$,

$\Delta\phi \approx$ constant $\to$ e^- starting at some phase will stay at that phase.

To reiterate the constraints used in this section, we assumed that the synchronous energy and particle velocity do not change very much at each accelerating station, and also that the phase oscillation amplitudes are small; these assumptions allowed us to derive the differential equations in this section (equation 2.41) and linearizre and solve it. Next, we will relax these constraints.

A diabetic damping and longitudinal emittance

In the previous section , we assumed that the change in synchronous energy and other parameters is sufficiently small over one period of interest so that their derivatives could be ignored. Here, we relax this restriction. We will find that if these changes are slow over time, the corresponding changes in the oscillation amplitudes can be given by a simple expression.

To be able to use our knowledge and intuition from the simple harmonic oscillators, we will derive equations in terms of time, t, rather than in terms of acceleration stage passage number, n.

start with energy gain expressions at each stage:

$$(2.33) \quad (E_s)_{n+1} = (E_s)_n + eV\sin\phi_s$$
$$(2.34) \quad E_{n+1} = E_n + eV\sin\phi_n$$

diff. form $\Longrightarrow$

$$\frac{dE_s}{dn} = eV\sin\phi_s$$
$$\frac{dE}{dn} = eV\sin\phi$$

note: $\frac{d}{dn} = \frac{dt}{dn}\frac{d}{dt} = \tau\frac{d}{dt}$

$$\therefore \tau(E)\frac{dE}{dt} = eV\sin\phi$$

→ τ, or transition time through an accelerator section depends on the energy of particle, which we said changes in non-negligable ways

$$- \quad \tau(E_s)\frac{dE_s}{dt} = eV\sin\phi_s$$

$$\tau(E)\frac{dE}{dt} - \tau(E_s)\frac{dE_s}{dt} = eV(\sin\phi - \sin\phi_s)$$

Taylor expansion: $\tau(E) = \tau(E_s) + \left(\frac{d\tau}{dE}\right)_{E_s}(E - E_s)$

$$\therefore \; \tau(E_s)\frac{dE}{dt} + (E-E_s)\left.\frac{d\tau}{dE}\right|_{E=E_s}\frac{dE}{dt} - \tau(E_s)\frac{dE_s}{dt} = eV(\sin\phi - \sin\phi_s)$$

$= \frac{d\tau}{dt}$ with $d\tau$ measured in the neighborhood of E_s

$$\tau(E_s)\frac{d}{dt}(E-E_s) + (E-E_s)\frac{d\tau}{dt} = eV(\sin\phi - \sin\phi_s)$$

$$\boxed{\frac{d}{dt}(\tau\Delta E) = eV(\sin(\phi) - \sin\phi_s)} \qquad 2.58$$

With the equation for the evolution of energy difference in hand, we will now look for an equation for phase oscillation

To get the time equation for ϕ start from (2.39):

$$\frac{d\phi}{dn} = \frac{\eta\,\omega_{rf}\,\tau\,c^2}{v^2 E_s}\Delta E$$

$$\cancel{\tau}\,\frac{d\phi}{dt} = \frac{\eta\,\omega_{rf}\,\cancel{\tau}\,c^2}{v^2 E_s}\Delta E$$

$$\Delta\phi = \phi - \phi_s$$

$$\frac{d\Delta\phi}{dt} = \frac{\eta\,\omega_{rf}}{\beta^2 E_s}\Delta E - \frac{d\phi_s}{dt}$$

note: the synchronous phase is maintained in synchrotron acceleration so just like $\frac{d\phi_s}{dn}$, $\frac{d\phi_s}{dt} = 0$ also

$$(2.59) \qquad \boxed{\frac{d\Delta\phi}{dt} = \lambda\Delta E}$$

want to express ΔE as a function of $\Delta\phi$

$$2.58 \Rightarrow \frac{d}{dt}(\tau \Delta E) = eV \cos\phi_s \Delta\phi$$

(Same trick as pg 4 above)

$$= \tau \left(\frac{eV \cos\phi_s}{\tau} \right) \Delta\phi$$

(2.60) $$\boxed{\frac{d}{dt}(\tau \Delta E) = \tau \mu \Delta\phi}$$

Use 2.59 and 2.60 to get an expression in terms of $\Delta\phi$ only

$$\frac{d}{dt}[2.59] \Rightarrow \frac{d^2\Delta\phi}{dt^2} = \frac{d}{dt}(\lambda \Delta E)$$

$$= \frac{d\lambda}{dt}\Delta E + \lambda \frac{d\Delta E}{dt}$$

$$2.60 \Rightarrow \frac{d}{dt}(\tau \Delta E) = \tau\mu\Delta\phi \Rightarrow \tau \frac{d\Delta E}{dt} = \tau\mu\Delta\phi - \Delta E \frac{d\tau}{dt}$$

$$\therefore \frac{d^2\Delta\phi}{dt^2} = \frac{d\lambda}{dt}\Delta E + \frac{\lambda}{\tau}\left[\tau\mu\Delta\phi - \Delta E \frac{d\tau}{dt}\right]$$

⇓ Collect terms with ΔE

$$\Rightarrow \frac{d^2\Delta\phi}{dt^2} = \left[\frac{d\lambda}{dt} - \frac{\lambda}{\tau}\frac{d\tau}{dt}\right]\Delta E + \lambda\mu\Delta\phi$$

⇓ Replace ΔE from (2.59)

$$\frac{d^2\Delta\phi}{dt^2} = \tau\left[\frac{1}{\tau}\frac{d\lambda}{dt} - \frac{\lambda}{\tau^2}\frac{d\tau}{dt}\right]\frac{1}{\lambda}\frac{d\Delta\phi}{dt} + \lambda\mu\Delta\phi$$

$$\Rightarrow \frac{d^2\Delta\phi}{dt^2} - \frac{1}{\lambda/\tau}\frac{d}{dt}\left(\frac{\lambda}{\tau}\right)\frac{d\Delta\phi}{dt} + \Omega_s^2\Delta\phi = 0 \quad (2.61)$$

Ω_s is the same as the angular freq. of synchrotron oscillations given previously. Also note that λ contains η while μ contains $\cos\phi_s$, so for stable phase oscillations, either $\lambda>0$ & $\mu<0$ or $\lambda<0$ & $\mu>0$, similar to the condition previously derived

So now, we have a condition for phase oscillations for the case where the synchronous energy and velocity are changing with time. We are still looking for solutions for particles with phase and energies near the synchronous values, but now this synchronous value changes with time.

Let's solve equation 2.61:

A standard approach for solving a second order differential equation of this type is to choose a trial solution of the form

$$\Delta\phi = uv$$

And pick v such that the first derivative term is zero in the equation for u.

Collecting the u terms, we get

i.e. $$v\frac{d^2u}{dt^2} + \underbrace{\left[2\frac{dv}{dt} - \frac{1}{\lambda/\tau}\frac{d}{dt}\left(\frac{\lambda}{\tau}\right)v\right]}_{\text{set this to zero}}\frac{du}{dt}$$

$$+\left[\frac{d^2v}{dt^2} - \frac{1}{\lambda/\tau}\frac{d}{dt}\left(\frac{\lambda}{\tau}\right)\frac{dv}{dt} + \Omega_s^2 v\right]u = 0 \quad (2.62)$$

i.e. $$2\frac{dv}{dt} = \frac{1}{\lambda/\tau}\frac{d}{dt}\left(\frac{\lambda}{\tau}\right)v$$

↳ solve: $$\int\frac{dv}{v} = \int\frac{1}{2}\cdot\frac{1}{\lambda/\tau}\frac{d}{dt}\left(\frac{\lambda}{\tau}\right)dt$$

$$\log v = \frac{1}{2}\log(\lambda/\tau) \rightarrow v^2 = \frac{\lambda}{\tau} \Rightarrow v = \sqrt{\pm\frac{\lambda}{\tau}}$$

We choose the sign that makes v real

Insert v into 2.62 to get an expression for u. The 2nd term drops out (as it was used to solve for v). We get

$$\frac{d^2u}{dt^2} + \left\{ \frac{1}{2(\lambda/\tau)} \frac{d^2}{dt^2}\left(\frac{\lambda}{\tau}\right) - \frac{3}{4}\frac{1}{(\lambda/\tau)^2}\left[\frac{d}{dt}\left(\frac{\lambda}{\tau}\right)\right]^2 + \Omega_s^2 \right\} u = 0$$

$$u = f(t)\, e^{i\int \omega dt}$$ → ω^2 is the quanity in brackets

↑ slowly varing function

← called method of integrated phase

set ω in initial trial equal to Ω_s (other terms are small)

$$\therefore \ddot{f} + i(2\dot{f}\Omega_s + f\dot{\Omega}_s) = 0 \qquad \dot{\square} \equiv \frac{d\square}{dt}$$

⇑ negligable since f varies slowly in time

$$\therefore f = \frac{1}{\sqrt{\Omega_s}}$$

$$\therefore \Delta\phi = A\sqrt{\frac{\pm\lambda}{\tau\Omega_s}}\cos\left(\int \Omega_s dt + \delta_1\right) \qquad (2.68)$$

(cos: real function; A, δ_1: from initial conditions)

$$\sqrt{\frac{\lambda}{\tau\Omega_s}} = \left[\pm \frac{\eta\omega_{rf}c^2}{v^2\tau E_s eV\cos\phi_s}\right]^{1/4} \qquad (2.69)$$

In the high energy limit of a circular accelerator, $\gamma \gg \gamma_t$

Phase oscillation amplitude ∝ fourth root of the synchronous energy (E_s)

This is an example of adiabatic damping.
meaning the phase oscillation amplitude reduces as the synchronous energy gets higher and higher

If we look at the energy oscillation equation, we get a similar equation, which results in

$$\Delta E = B\sqrt{\frac{\mp \mu}{\tau \Omega_s}} \cos\left(\int \Omega_s \, dt + \delta_2\right)$$

$$\sqrt{\frac{\mu}{\tau \Omega_s}} = \left[\mp \frac{v^2 E_s eV \cos\phi_s}{\eta \omega_{rf} \tau^3 c^2}\right]$$

again, for $\gamma >> \gamma_t$, ΔE increases with E_s, but the fractional energy difference, $\Delta E / E_s$ will decrease

Classical mechanics tells us that a simplification in the description of motion is gained by the use of canonically conjugated pairs.

From classical mechanics, we know that for a particle undergoing periodic motion, the area of its trajectory in the appropriate phase space is an adiabatic invariant. We have a similar case here with our variables:

product of amplitudes of $\Delta\phi$ & $\Delta E \propto \frac{1}{\tau}$

Area of phase space ellipse of $\Delta\phi - \Delta E$ is

$$\pi \, \Delta\hat{\phi} \, \Delta\hat{E} = \frac{\pi A B}{\tau}$$

if we use time of passage Δt instead of phase of passage, $\Delta\phi$, $\Delta\hat{\phi} = \omega_{rf} \Delta\hat{t}$ & can use $\Delta E - \Delta t$ space:

$$\boxed{\pi\, \Delta\hat{E}\, \Delta\hat{E} = \frac{\pi A B}{\omega_{rf}\tau} = \text{constant}}$$ ← because τ changes such that $\omega_{rf}\tau$ corresponding to ϕ_s remain constant

The area in phase space which contains the particles in a bunch is termed the longitudinal emittance. It is hoped (!) that this emittance is smaller than the area of the entire stable region.

In ΔE-Δt coordinates, for a bucket with $\phi_s = 0$ or $180°$, the area of the bucket is

(stationary bucket (ideal particle is not accelerated in this case))

$$A = \frac{16\,(V/c)}{\omega_{rf}} \sqrt{\frac{eV \cdot E_s}{2\pi h |\eta|}}$$

In this case, the beam will have a longitudinal emittance

$$S = \frac{\pi (V/c)(\Delta\hat{\phi})^2}{\omega_{rf}} \sqrt{\frac{eV E_s}{2\pi h |\eta|}}, \quad \text{where } \Delta\phi = \Delta\hat{\phi}\cos(\quad)$$

At transition, our requirement that the parameters of the system change slowly does not hold. So, the equations derived here do not apply. Let's understand what happens when we cross a transition.