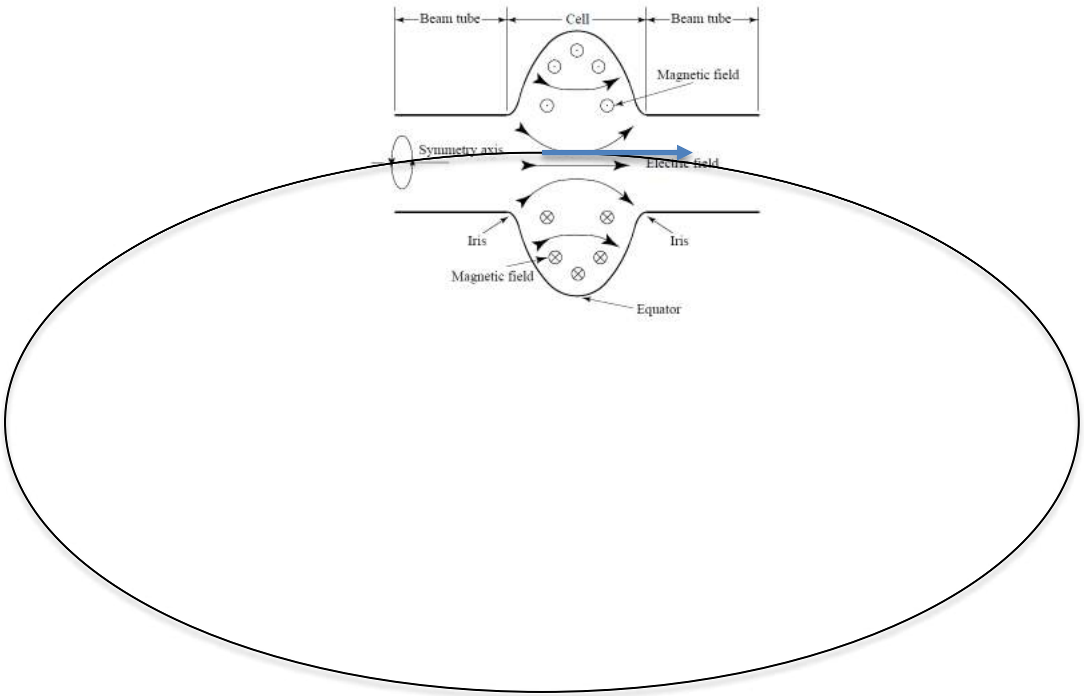
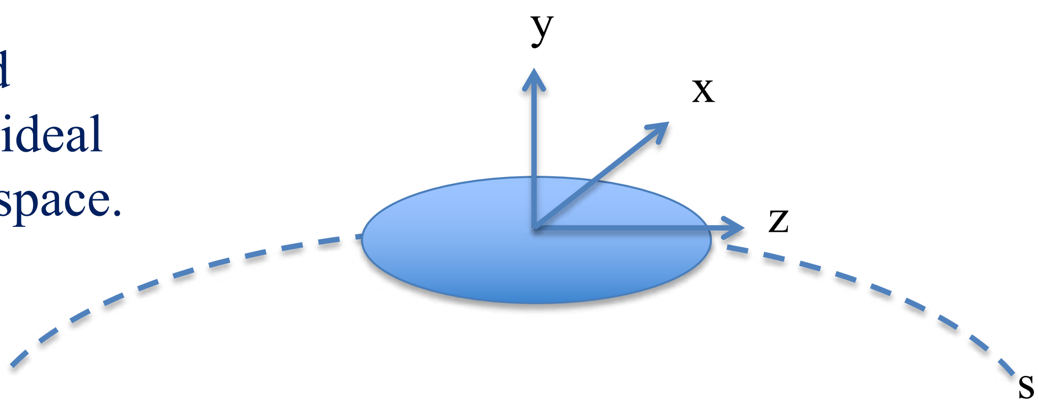
PHY 554
Fundamentals of Accelerator Physics
Lecture 13: Longitudinal Dynamics

Vladimir N. Litvinenko



6-D Phase Space

A complete description of a charged particle motion with respect to the 'ideal particle' must be done in 6D phase space.



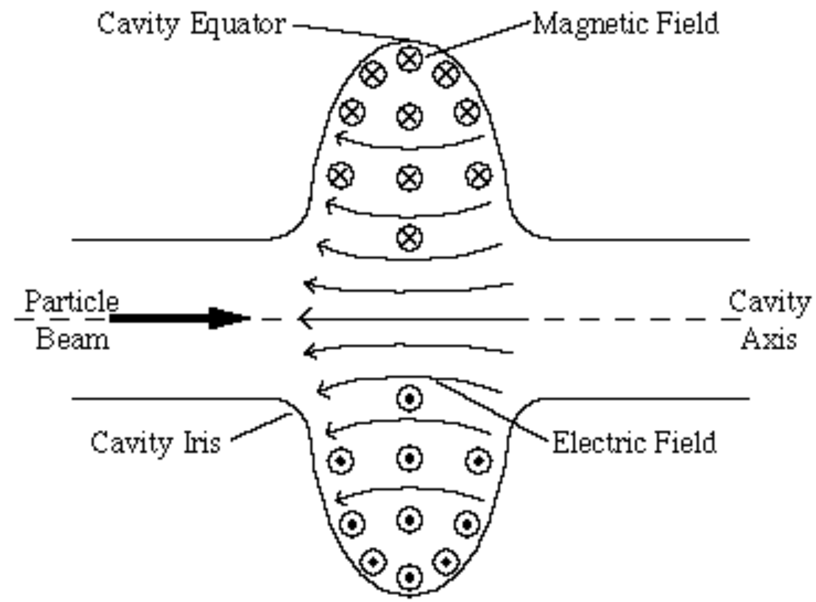
$$X^T = \{x, x', y, y', z, \delta\}$$

$$z = -ct \qquad \delta = \frac{p - p_o}{p_o}$$

- Longitudinal dynamics is important in
  - Storage rings
  - Beam transport in linacs
  - Applications, such as Free Electron Lasers

Charged Particle passing an RF Cavity

From previous lectures: Let us consider a ultra-relativistic particle passing a RF cavity, with the field E and voltage V.



$$E_Z(s,t) = E_o(s)\sin(\omega t + \phi_s);$$

The energy gain of one charged particle with position z in a bunch:

$$\Delta\mathbf{E} = e\int E_o(s)\sin(\omega t + \phi_s)\,ds;$$

$$t = t_o + \frac{s}{\mathrm{V}};\Delta\mathbf{E} = eV_{RF}\sin(\phi + \phi_s); \quad \phi = \omega t_o$$

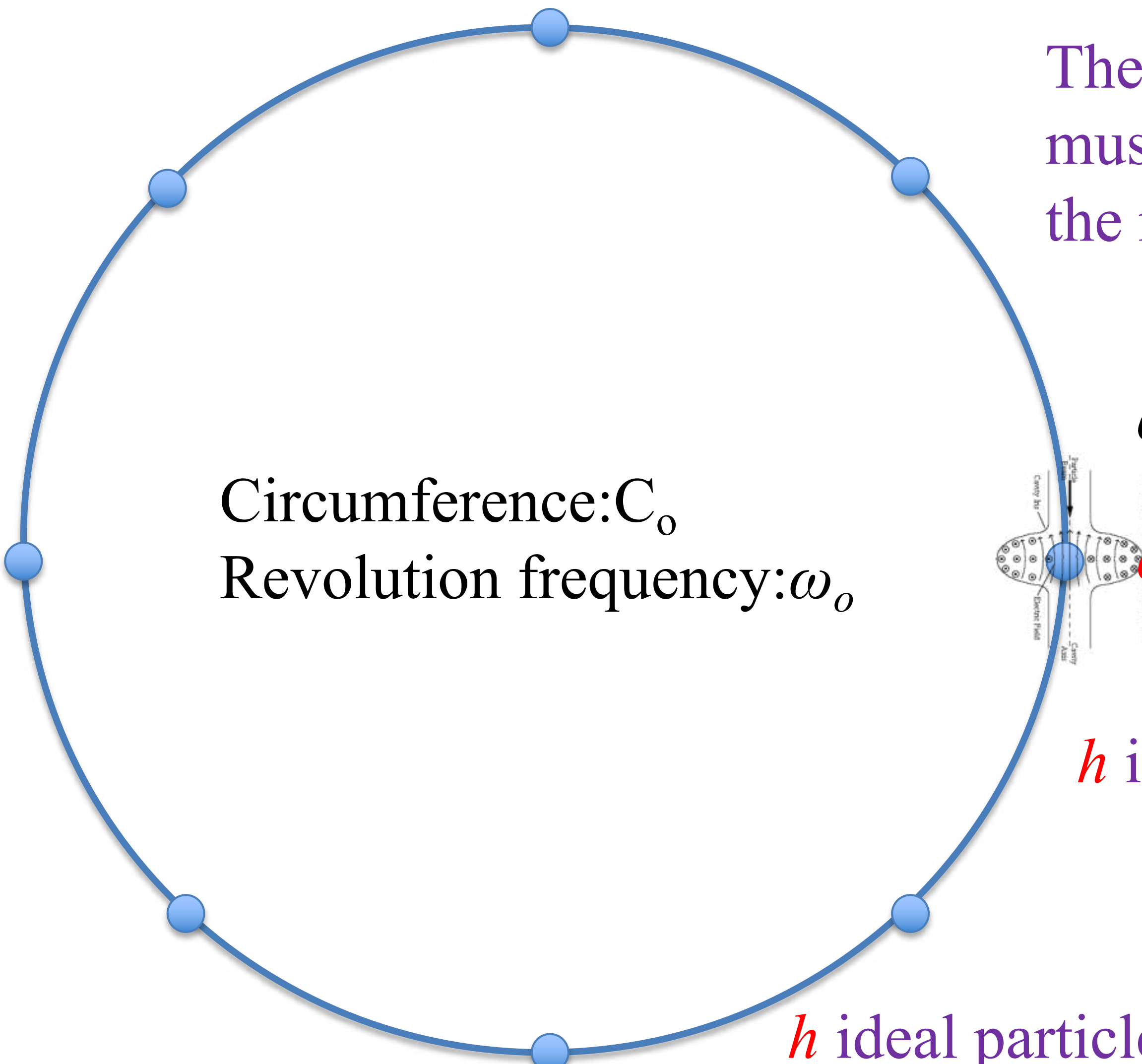
$$V_{RF} = \int E_o(s)\cos\frac{\omega s}{\mathrm{V}}\,ds = T\int E_o(s)\,ds$$

T is transit time factor

Synchrotron Motion in a Storage Ring

- Longitudinal motion in circular accelerators is called synchrotron oscillation
- The origin of this term originates from "synchrotron" where particles are "synchronized" with oscillating electric field in RF cavity(ies)
  - Like "betatron" motion name for transverse degrees of freedom originated from betatron, it is purely historical slang
- Hence, the terminology of transverse oscillations is extended to the synchrotron oscillations and synchrotron tune, Q_s, for stable oscillations
- In contrast with transverse motion, where particles typically can execute multiple oscillations per tur, synchrotron oscillations are usually very slow with $Q_s << 1$
- The later is used for a simplified description of slow synchrotron oscillations by separating them from "fast" betatron oscillations

RF Synchronization in a ring



The frequency of the cavity must be integer harmonic of the revolution frequency:

$$\omega_o \equiv \omega_{rev} \equiv 2\pi f_o; f_o = \frac{1}{T_o} = \frac{\mathrm{v}_o}{C_o}$$

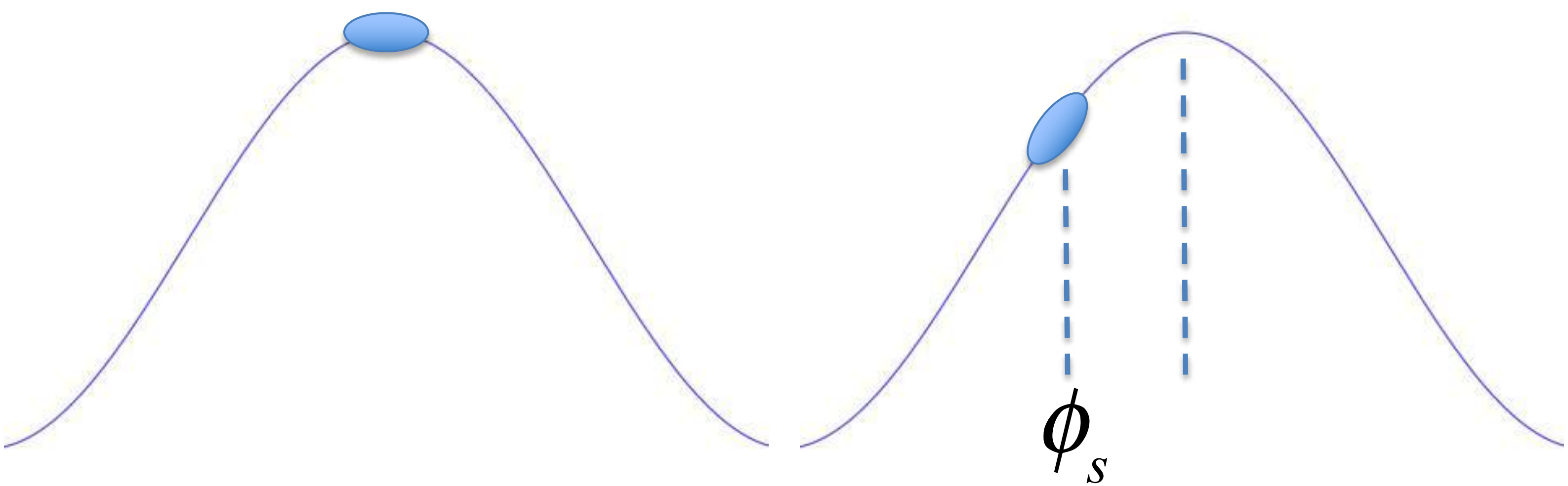
$$\omega_{rf} = h\omega_o; h - \text{integer}$$

h is called harmonic number.

h ideal particles can circulate in the ring. They are called <u>*synchronous particles*</u>

Charged Particle in RF Cavity II

- We name the synchronous particle's phase ϕ_s
- For number of very good reasons, we don't want the particle to experience the highest accelerating voltage (on crest).



$$\Delta\mathbf{E} = eV_{RF}\sin\left(\phi + \phi_s\right)$$

Energy change by the RF cavity

Synchronous particle energy change in the cavity is given by its synchronous phase

$$\Delta \mathbf{E} = eV_{RF}\sin\left(\phi + \phi_s\right)$$

Non-zero value of energy change can be related to acceleration/deceleration of the beam or compensation for energy losses in the ring (such as radiation losses)

The energy of a particle displaced by distance $z=-ct$ from synchronous particle changes as

$$\mathbf{E}^2 = p^2c^2 + m^2c^4$$

$$\Delta \mathbf{E} << \mathbf{E} \Rightarrow \mathbf{E}\Delta\mathbf{E} = c^2 p\Delta p$$

$$\Delta\delta = \frac{\Delta p}{p_o} = \frac{\Delta \mathbf{E}}{\mathbf{E}_o}\left(\frac{\mathbf{E}_o}{cp_o}\right)^2 = \frac{\Delta \mathbf{E}}{\mathbf{E}_o\beta_o^2}; \beta_o = \frac{cp_o}{\mathbf{E}_o} \equiv \frac{\mathrm{v}_o}{c}$$

$$\Delta\delta = \frac{eV_{rf}}{\mathbf{E}_o\beta_o^2}\sin\left(\phi_s - kz\right)$$

One turn map

Synchronous particles arrive to RF at t=0 and the relative change of their momentum is

$$z_s = 0; \;\; \Delta\delta_s = \frac{eV_{rf}}{\mathbf{E}_o \beta_o^2} \sin\phi_s$$

Using relative values, we can re-write the above equations in dimensionless form

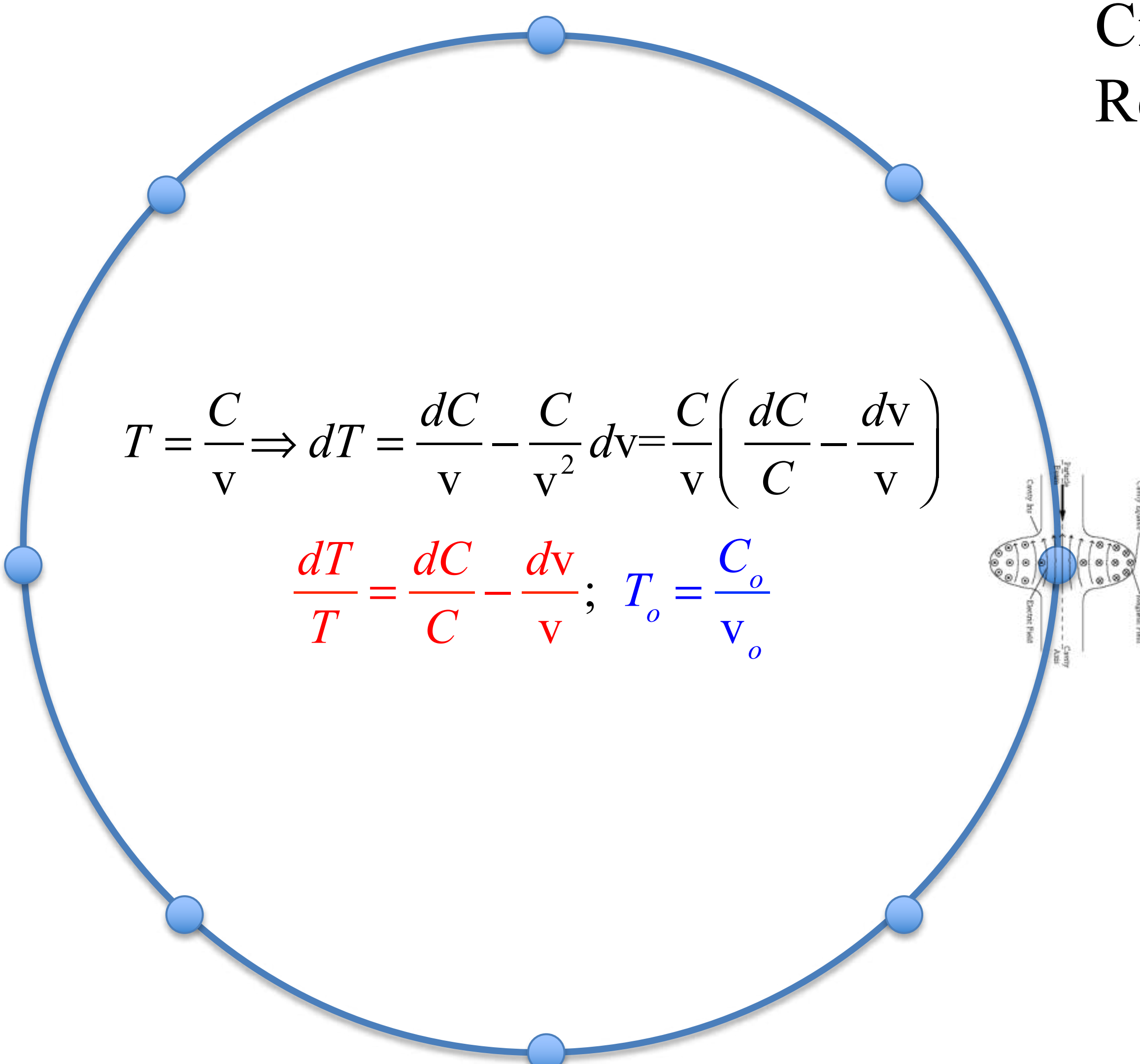
δ changes between n^{th} to $(n+1)^{th}$ turn in the ring as:

$$\delta_{n+1} = \delta_n + \frac{eV_{rf}}{\mathbf{E}_o \beta_o^2} \left(\sin\phi_n - \sin\phi_s \right)$$

$$\phi_n = \phi_s + \omega t_n \equiv \phi_s - k z_n$$

Now we know, how the momentum deviation evolves, and we need to determine evolution of the arrival time

Revolution time of a particle



$$T=\frac{C}{\mathrm{v}}\Rightarrow dT=\frac{dC}{\mathrm{v}}-\frac{C}{\mathrm{v}^2}d\mathrm{v}=\frac{C}{\mathrm{v}}\left(\frac{dC}{C}-\frac{d\mathrm{v}}{\mathrm{v}}\right)$$

$$\frac{dT}{T}=\frac{dC}{C}-\frac{d\mathrm{v}}{\mathrm{v}};\quad T_o=\frac{C_o}{\mathrm{v}_o}$$

Circumference: C_o

Revolution frequency: ω_o

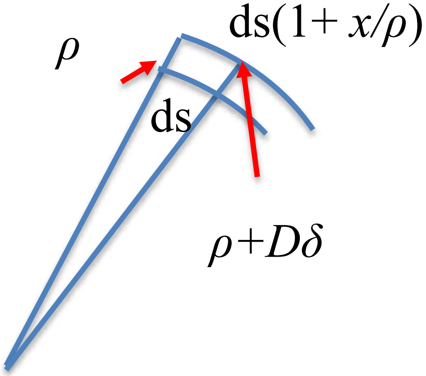
$$\beta=\frac{\mathrm{v}}{c}=\frac{pc}{\mathbf{E}}\rightarrow\Delta\mathrm{v}=c\left(\frac{\Delta p}{\mathbf{E}}-\frac{p\Delta\mathbf{E}}{\mathbf{E}^2}\right);$$

$$\Delta\mathbf{E}=c^2\frac{p\Delta p}{\mathbf{E}}$$

Revolution time of a particle

- Let's consider the consequence of the energy deviation
  - Its velocity changes: $\dfrac{\Delta \mathrm{v}}{\mathrm{v}_o} = \dfrac{\delta}{\gamma_o^{\;2}}$
  - And the pass length change

$x = x_\beta + D \cdot \delta; \ \langle x_\beta \rangle = 0$



$$\frac{\Delta C}{C_o} = \frac{1}{C_o}\oint \frac{x(s)}{\rho(s)} ds \Rightarrow \frac{\Delta C}{C_o} = \frac{\delta}{C_o}\oint \frac{D(s)}{\rho(s)} ds = \alpha_c \delta$$

- The arrival time difference

$$\frac{\Delta T}{T_o} = \frac{\Delta C}{C_o} - \frac{\Delta \mathrm{v}}{\mathrm{v}} = \left(\alpha_c - \gamma_o^{-2}\right)\delta \equiv \eta\delta; \quad \eta = \alpha_c - \frac{1}{\gamma_o^2}$$

Change in RF phase

Then we can translate the arriving time to the *rf* phase variable: $\Delta T = T_o \eta \delta$

$$\Delta\phi = \omega_{rf}\Delta T = \omega_{rf} T_o \eta\delta; \;\; \omega_{rf} T_o = 2\pi h$$

Change to turn by turn mapping format:

$$\phi_{n+1} = \phi_n + 2\pi h\eta \cdot \delta_{n+1}$$

Combined with the earlier change in the energy change, we have the longitudinal one-turn map:

$$\delta_{n+1} = \delta_n + \frac{eV_{rf}}{\mathbf{E}_o \beta_o^2}\left(\sin\phi_n - \sin\phi_s\right)$$

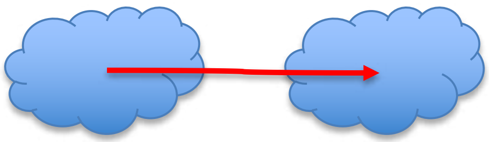
$$\phi_{n+1} = \phi_n + 2\pi h\eta \cdot \delta_{n+1}$$

Fixed Points

- For any nonlinear map, the first step is attempt to find fixed point(s)

$$\delta_{n+1} = \delta_n + \frac{eV_{rf}}{\mathbf{E}_o \beta_o^2}\left(\sin\phi_n - \sin\phi_s\right)$$

$$\phi_{n+1} = \phi_n + 2\pi h\eta \cdot \delta_{n+1}$$



- The fixed point is defined as:

$$\delta_{n+1} = \delta_n \qquad \phi_{n+1} = \phi_n$$

- And located at

$$\phi_{n+1} = \phi_n \rightarrow \delta = 0$$

$$\delta_{n+1} = \delta_n \rightarrow \sin\phi = \sin\phi_s$$

$$\phi = \phi_s \ \ \& \ \ \phi = \pi - \phi_s$$

- Next step is to find if fixed points are stable are not?

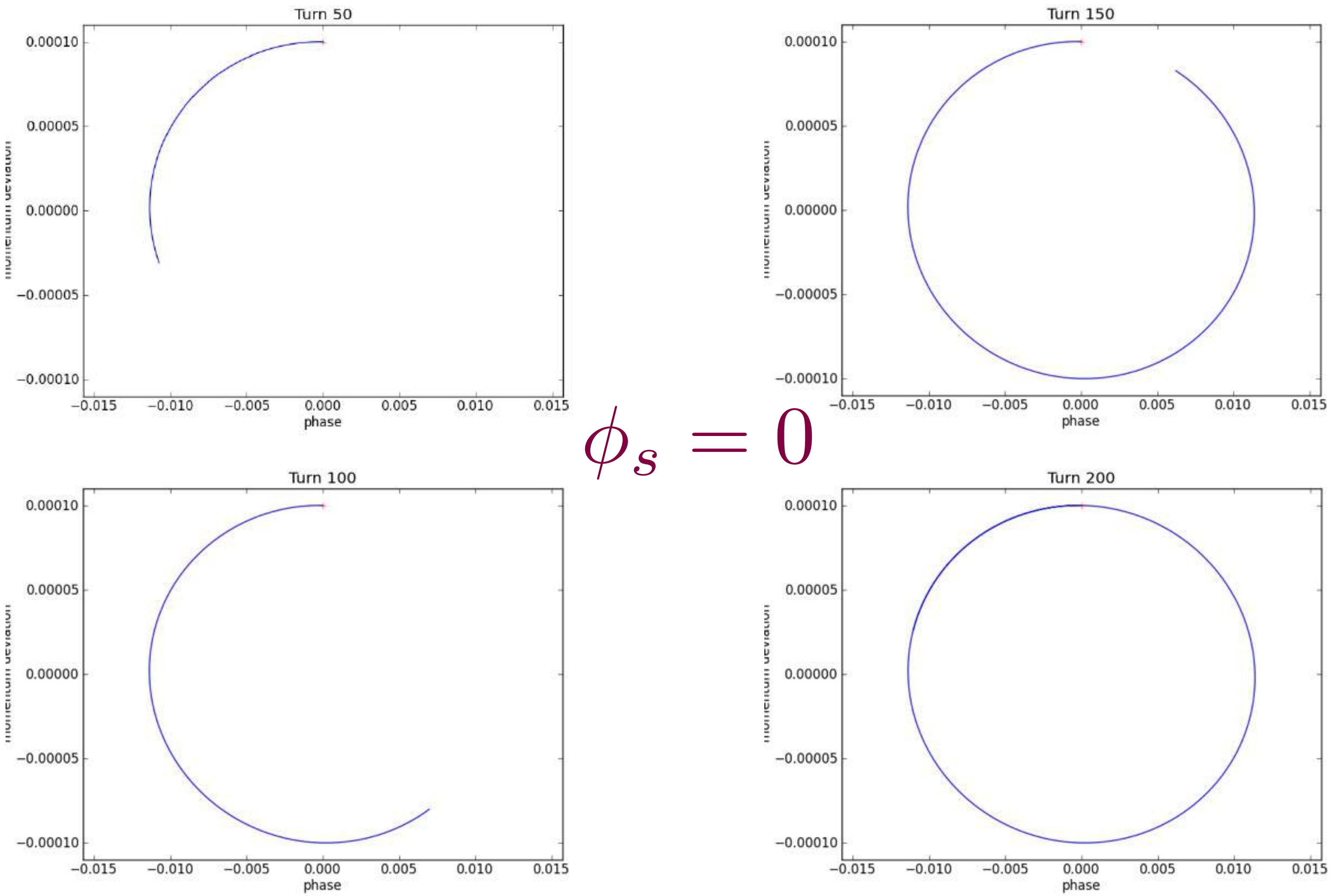
An Example

- Consider the example with following parameters:
  - Proton beam with 100 GeV or 15 GeV
  - Cavity voltage 5 MV, 360 harmonic
  - Compaction factor 0.002
  - No net acceleration.
  - Initial condition:

$$\phi = \phi_s$$

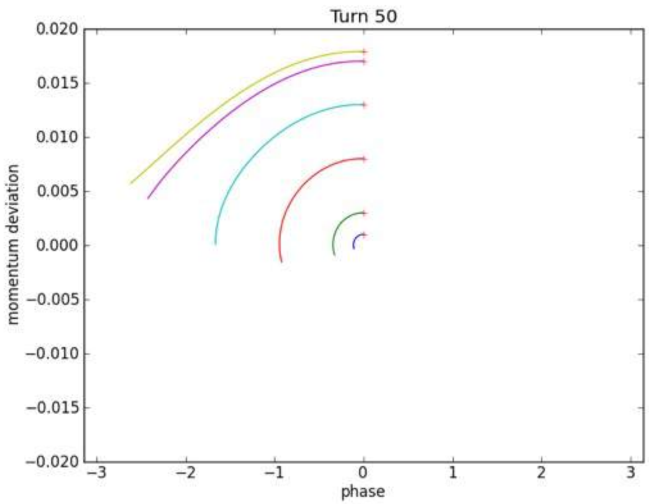
$$\delta_o = \varepsilon << 1$$

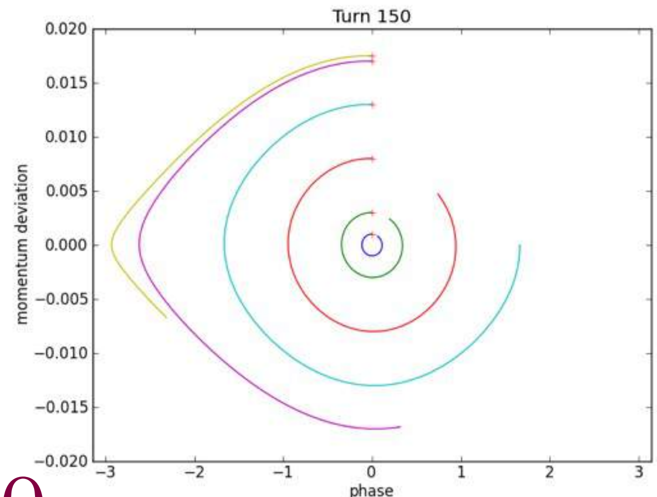
Phase stability for 15GeV



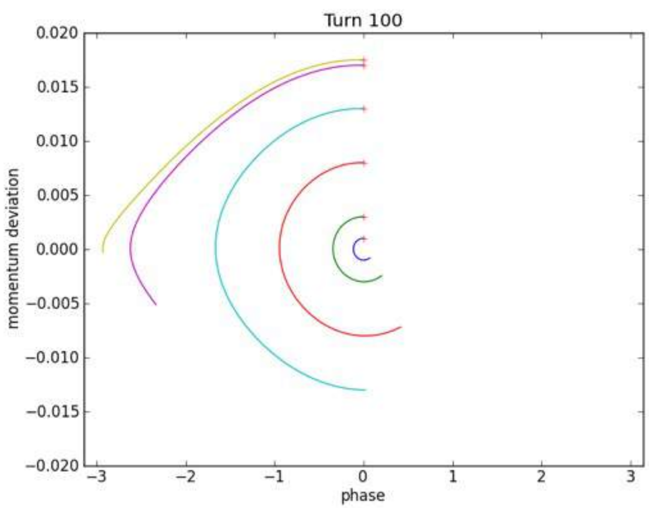
$$\phi_s = 0$$

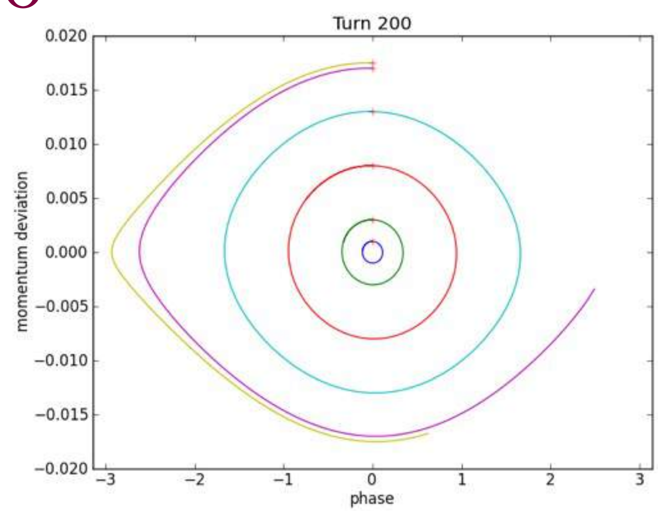
Phase stability, cont'd



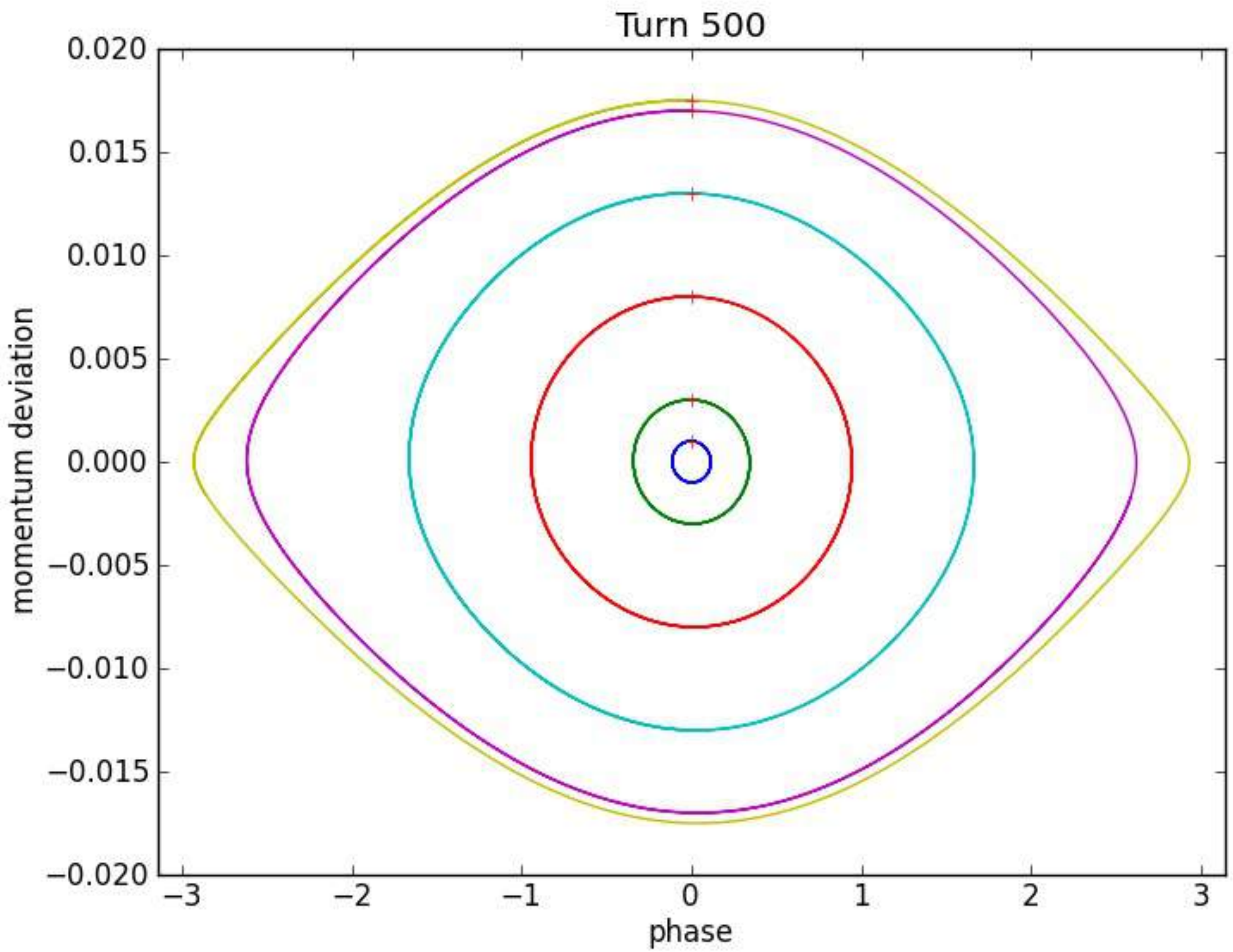


$$\phi_s = 0$$

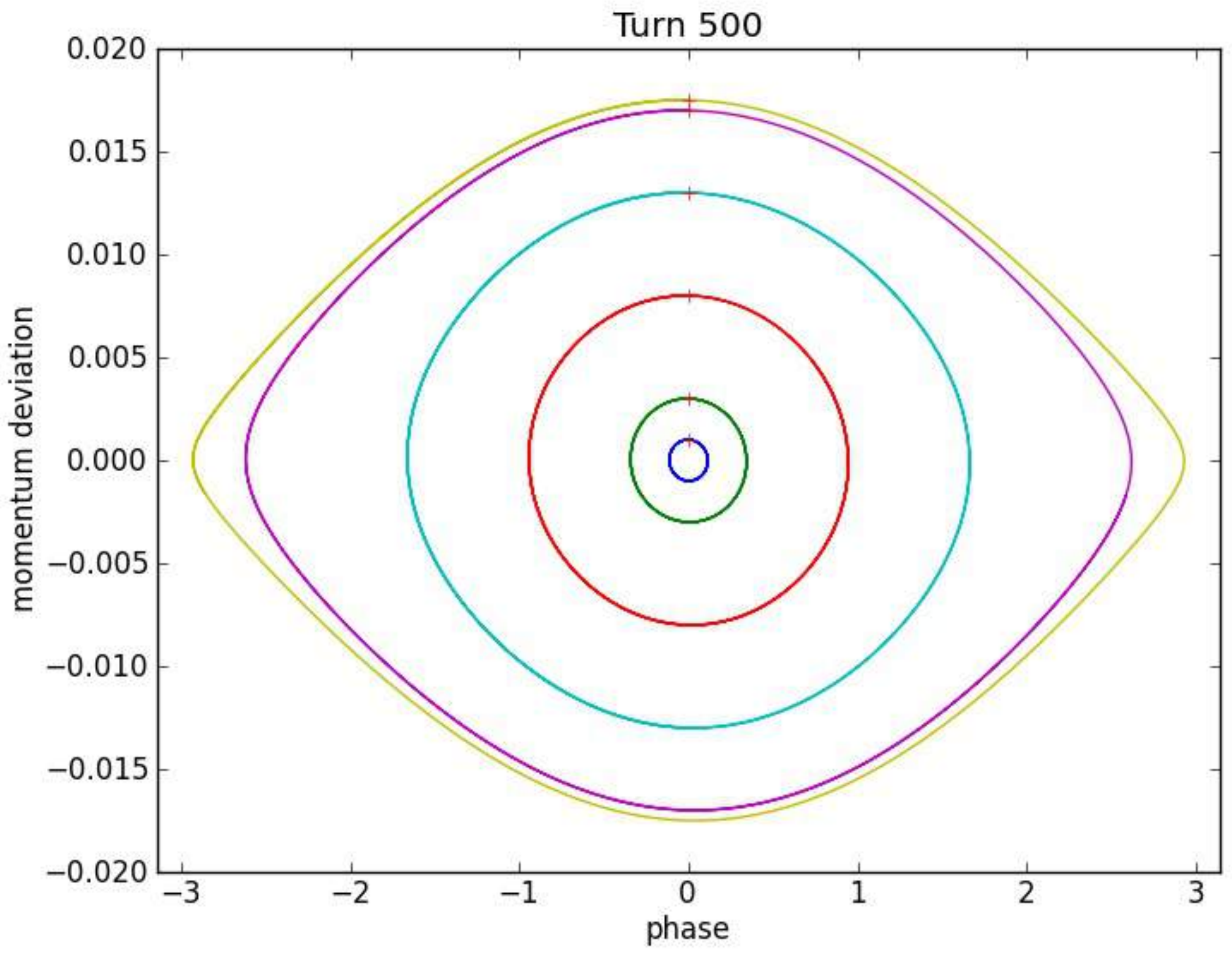




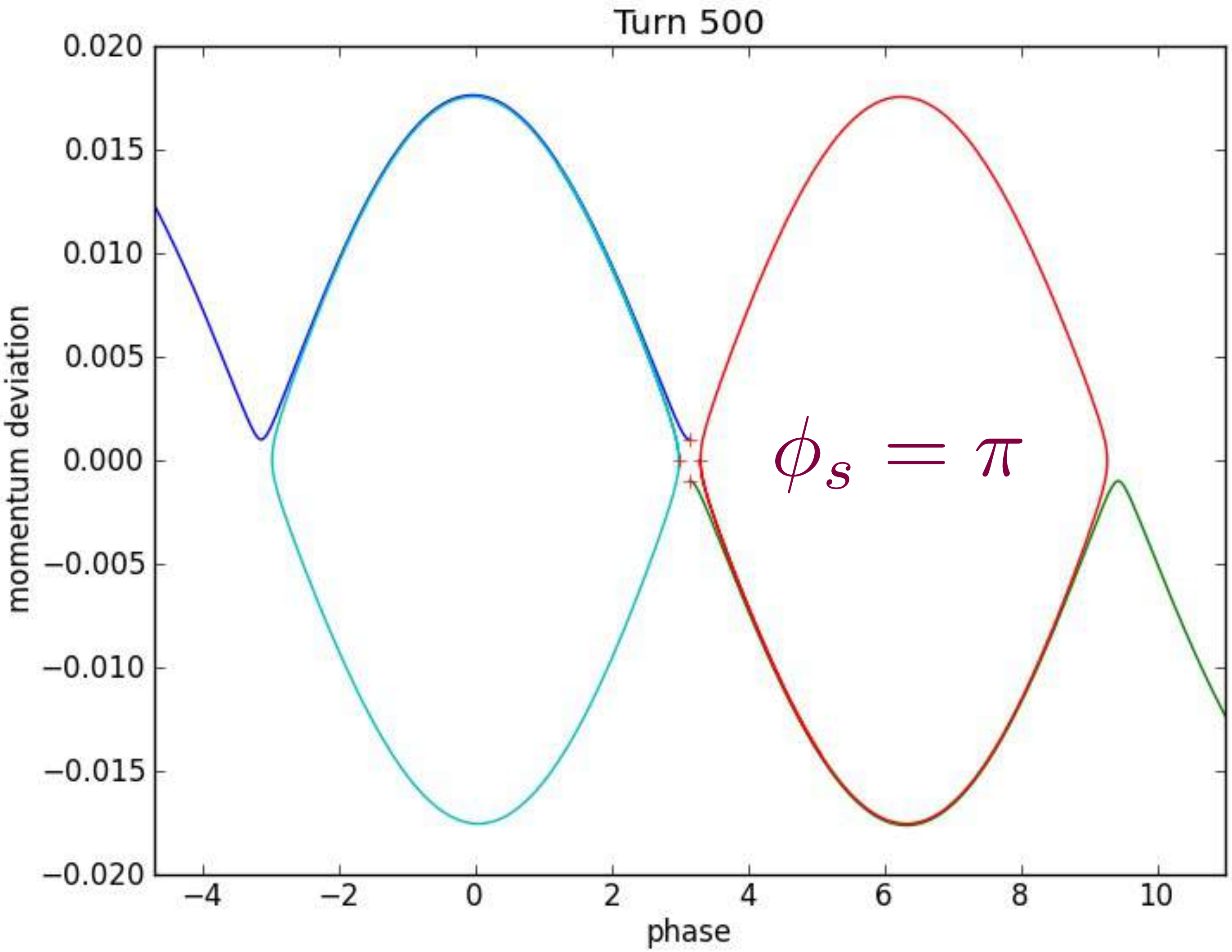
Phase stability cont'd



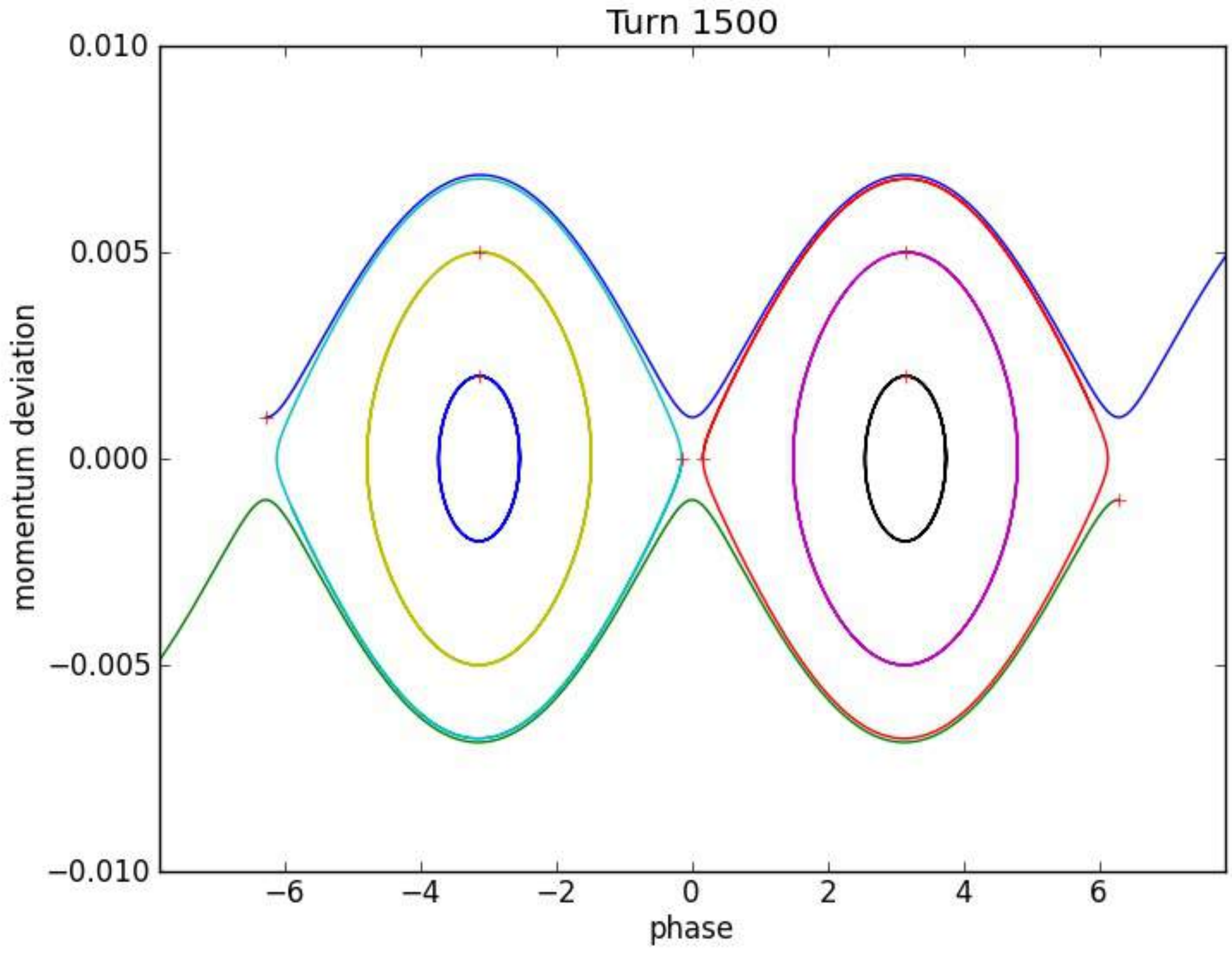
Phase stability cont'd



Phase stability for 100 GeV



Phase stability for 100 GeV



From Map to Hamiltonian

Finite differential map:

$$\delta_{n+1} - \delta_n = \frac{eV_{rf}}{\mathbf{E}_o \beta_o^2}\left(\sin\phi_n - \sin\phi_s\right)$$

$$\phi_{n+1} - \phi_n = 2\pi h\eta \cdot \delta_{n+1}$$

Can be approximately described by differential equations when variations are small

$$\dot{\delta} = \frac{d\delta}{dn} \cong \frac{eV_{rf}}{\mathbf{E}_o \beta_o^2}\left(\sin\phi - \sin\phi_s\right); \; \dot{\phi} = \frac{d\phi}{dn}_{n+1} \cong 2\pi h\eta \cdot \delta$$

$$\dot{f} \equiv \frac{df}{dn}; \ddot{f} \equiv \frac{d^2 f}{dn^2}$$

$$\ddot{\phi} = 2\pi h\eta \frac{eV_{rf}}{\mathbf{E}_o \beta_o^2}\left(\sin\phi - \sin\phi_s\right)$$

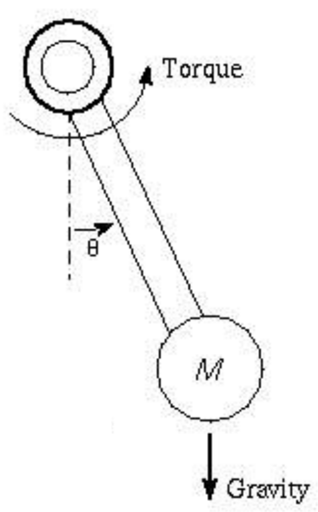
with the turn number as independent variable and an effective Hamiltonian of a pendulum!

$$H = 2\pi h\eta \cdot \frac{\delta^2}{2} + \frac{eV_{rf}}{\mathbf{E}_o \beta_o^2}\left\{\cos\phi - \cos\phi_s + \sin\phi_s\left(\phi - \phi_s\right)\right\}$$

Hamiltonian equations of motion

$$H = 2\pi h\eta \cdot \frac{\delta^2}{2} + \frac{eV_{rf}}{\mathbf{E}_o\beta_o^2}\left\{\cos\phi - \cos\phi_s + \sin\phi_s\left(\phi - \phi_s\right)\right\}$$

$$\dot{\phi} = \frac{\partial H}{\partial \delta} = 2\pi h\eta \cdot \delta; \quad \dot{\delta} = -\frac{\partial H}{\partial \phi} = \frac{eV_{rf}}{\mathbf{E}_o\beta_o^2}\left(\sin\phi - \sin\phi_s\right)$$



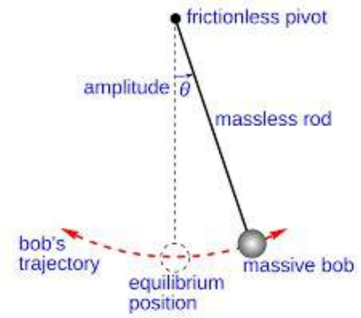
Forced pendulum

$$H = \frac{p^2}{2} - \frac{g}{R}\left(\cos\theta - \cos\theta_s\right) + \frac{T}{MR^2}\left(\theta - \theta_s\right)$$

$$\dot{\theta} = \frac{\partial H}{\partial \dot{\theta}} = p; \quad \dot{p} = -\frac{\partial H}{\partial \theta} = \frac{1}{MR^2}\left(T - MRg\sin\theta\right)$$

$$\sin\theta_s = \frac{T}{MgR}$$

At zero accelerating phase - simple pendulum



$$\phi_s = 0;\ H = 2\pi h\eta \cdot \frac{\delta^2}{2} + \frac{eV_{rf}}{\mathbf{E}_o \beta_o^2}(\cos\phi - 1)$$

$$\dot{\phi} = 2\pi h\eta \cdot \delta;\ \dot{\delta} = -\frac{\partial H}{\partial \phi} = \frac{eV_{rf}}{\mathbf{E}_o \beta_o^2}\sin\phi$$

$$H = \frac{p^2}{2m} + mgR(\cos\phi - 1)$$

$2\pi h\eta$	'MASS' (not unique)	$m > 0$
$\eta < 0, \phi_s = 0;\ \eta > 0, \phi_s = \pi$	Stable phase	$\theta = 0$
$\delta_{\max/\min} = \pm\sqrt{\frac{2eV_{rf}}{\pi h\lvert\eta\rvert \cdot \mathbf{E}_o \beta_o^2}}$	Bucket height For stable motion	$2\sqrt{gR}$
$\Omega_s = \sqrt{-\eta\cos\phi_s \cdot \frac{2\pi heV_{rf}}{\mathbf{E}_o \beta_o^2}}$	Angular frequency for small oscillation	$\sqrt{g / R}$

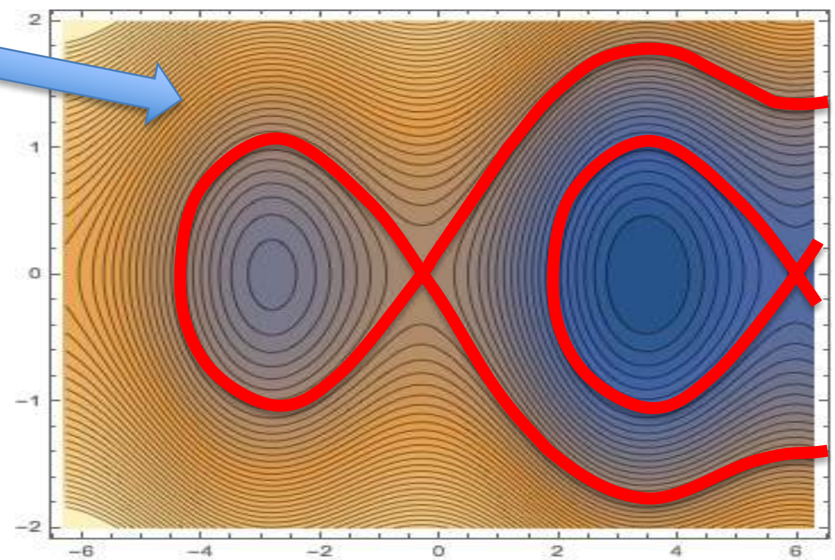
Energy conservation and trajectories

$$H(\delta,\phi) = 2\pi h\eta \cdot \frac{\delta^2}{2} + \frac{eV_{rf}}{\mathbf{E}_o\beta_o^2}\left\{\cos\phi - \cos\phi_s + \sin\phi_s(\phi - \phi_s)\right\}$$

$$\frac{\partial H}{\partial t} = 0 \rightarrow H(\delta,\phi) = const \rightarrow trajectories \;\; H(\delta,\phi) = H_o$$

$\eta > 0, \sin\phi_s = 0.3$

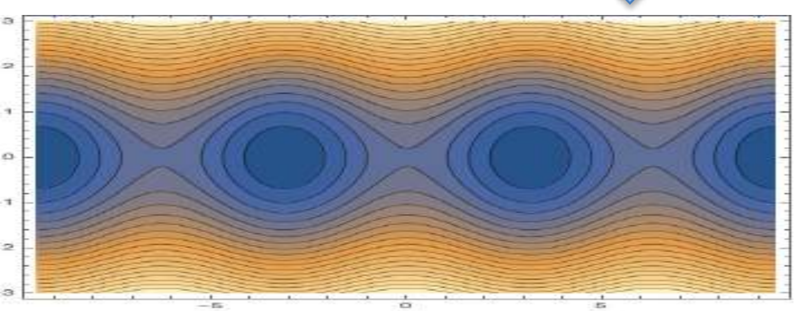
$\delta, a.u.$



ϕ

$\eta > 0, \phi_s = \pi$

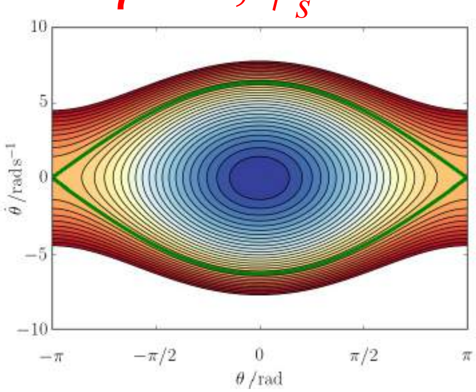
$\delta, a.u.$



ϕ

$\eta < 0, \phi_s = 0$

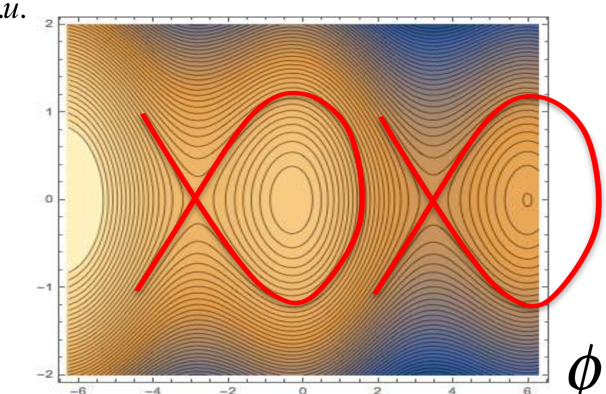
$\delta, a.u.$



ϕ

$\eta < 0, \sin\phi_s = 0.3$

$\delta, a.u.$



ϕ

Small amplitude approximation
Stability criterion

Back to the 2nd order differential equation

$$\ddot{\phi} = 2\pi h\eta \frac{eV_{rf}}{\mathbf{E}_o\beta_o^2}\left(\sin\phi - \sin\phi_s\right)$$

For small phase deviations we can linearize it

$$\Delta\phi = \phi - \phi_s; \left|\Delta\phi\right| << 1 \rightarrow \sin\left(\phi_s + \Delta\phi\right) = \sin\phi_s + \Delta\phi\cos\phi_s + O\left(\Delta\phi^2\right)$$

$$\Delta\ddot{\phi} = 2\pi h\eta \frac{eV_{rf}\cos\phi_s}{\mathbf{E}_o\beta_o^2}\cdot\Delta\phi \rightarrow \phi = ae^{i\Omega_s n}; \boldsymbol{\Omega}_s = \sqrt{-\frac{2\pi heV_{rf}}{\mathbf{E}_o\beta_o^2}\eta\cos\phi_s}$$

And find stability condition:

$$2\eta\cos\phi_s < 0$$

Synchrotron tune

The ‘tune’ is defined as

$$Q_s = \frac{\Omega_s}{2\pi} = \sqrt{-\frac{heV_{rf}}{2\pi \mathbf{E}_o \beta_o^2}\eta \cos\phi_s} = \nu_s \sqrt{|\cos\phi_s|}$$

Synchrotron tune for zero crossing

$$\nu_s = \sqrt{\frac{heV_{rf}}{2\pi \mathbf{E}_o \beta_o^2}|\eta|}$$

- Typical Numbers
  - Hadron rings: $Q_s \sim 10^{-3}$
  - Electron rings: $Q_s \sim 10^{-2}$

Small Amplitude Approximation Hamiltonian

$$H = 2\pi h\eta \cdot \frac{\delta^2}{2} + \frac{eV_{rf}}{\mathbf{E}_o\beta_o^2}\left\{\cos\phi - \cos\phi_s + \sin\phi_s\left(\phi - \phi_s\right)\right\}$$

When the phase is close to the synchronous phase:

$$H = 2\pi h\eta \cdot \frac{\delta^2}{2} - \frac{eV_{rf}\cos\phi_s}{\mathbf{E}_o\beta_o^2} \cdot \frac{\Delta\phi^2}{2}$$

$$\cos\left(\phi_s + \Delta\phi\right) - \cos\phi_s + \Delta\phi\sin\phi_s = -\cos\phi_s\frac{\Delta\phi^2}{2} + O\left(\Delta\phi^3\right)$$

The phase space trajectory will be upright ellipse for fixed ‘energy’

$$\left(\frac{\delta}{a_\delta}\right)^2 + \left(\frac{\Delta\phi}{a_\phi}\right)^2 = 1$$

$$\frac{a_\delta}{a_\phi} = \sqrt{\frac{eV_{rf}\left|\cos\phi_s\right|}{2\pi h\left|\eta\right|\beta^2\mathbf{E}_o}} = \frac{Q_s}{h\left|\eta\right|}$$

Transition energy

Transition happens when:

$$\eta = \alpha_c - \frac{1}{\gamma_o^2} \rightarrow 0$$

$$\gamma_T = 1/\sqrt{\alpha_c} \quad when \quad \alpha_c > 0$$

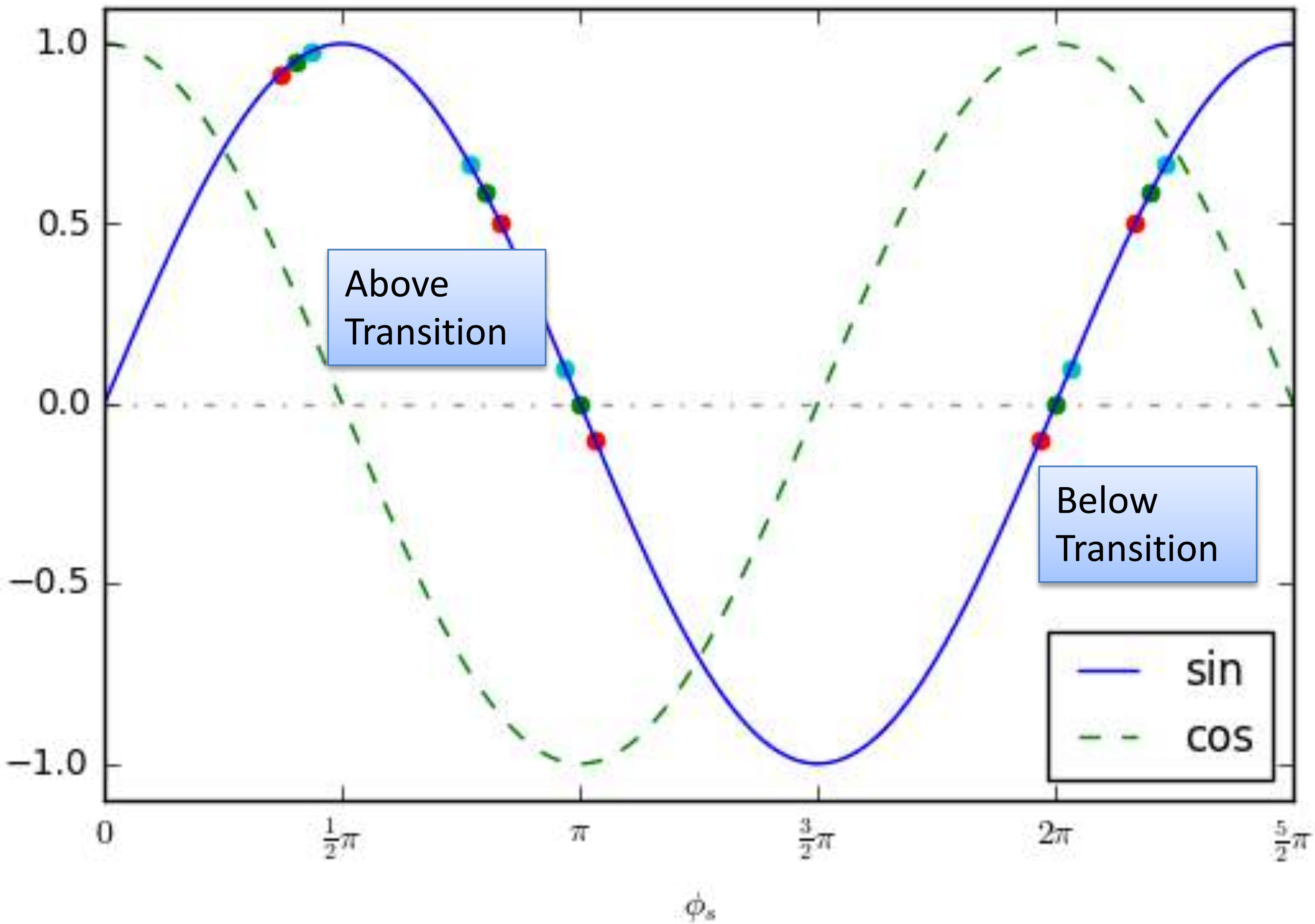
Below transition: High energy particles arrives earlier

$$\gamma < \gamma_T, \quad \eta < 0$$

Above transition: High energy particles arrives later

$$\gamma > \gamma_T, \quad \eta > 0$$

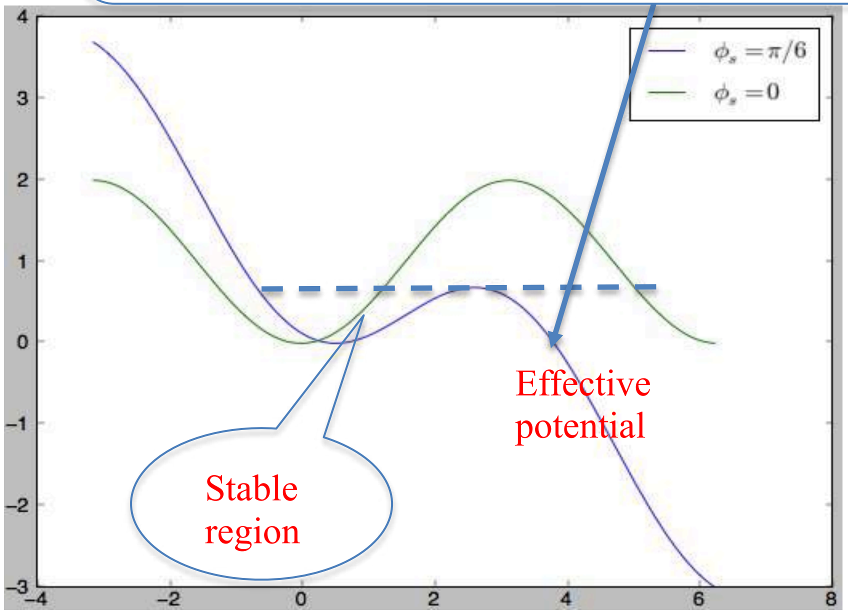
Physics Picture



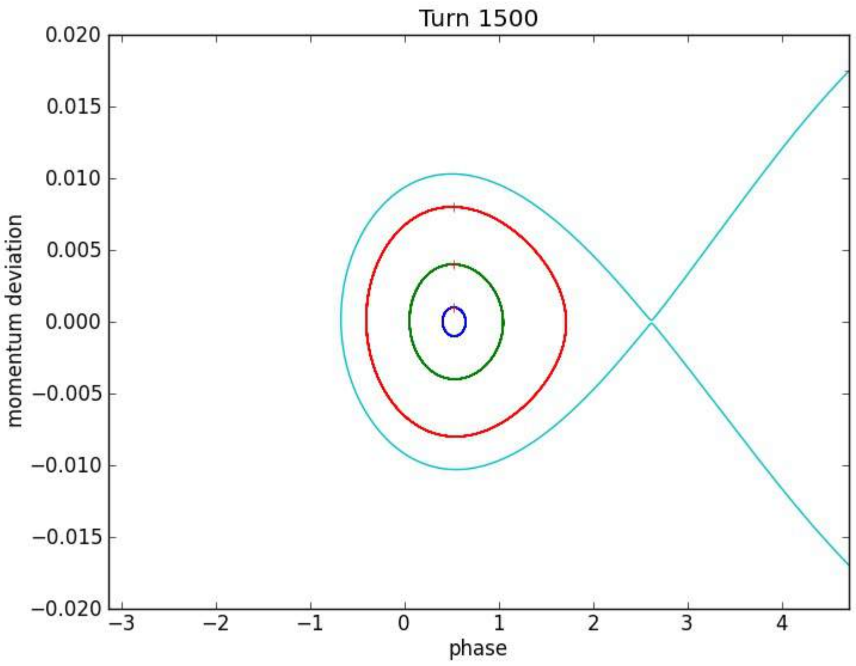
Non-zero acceleration phase

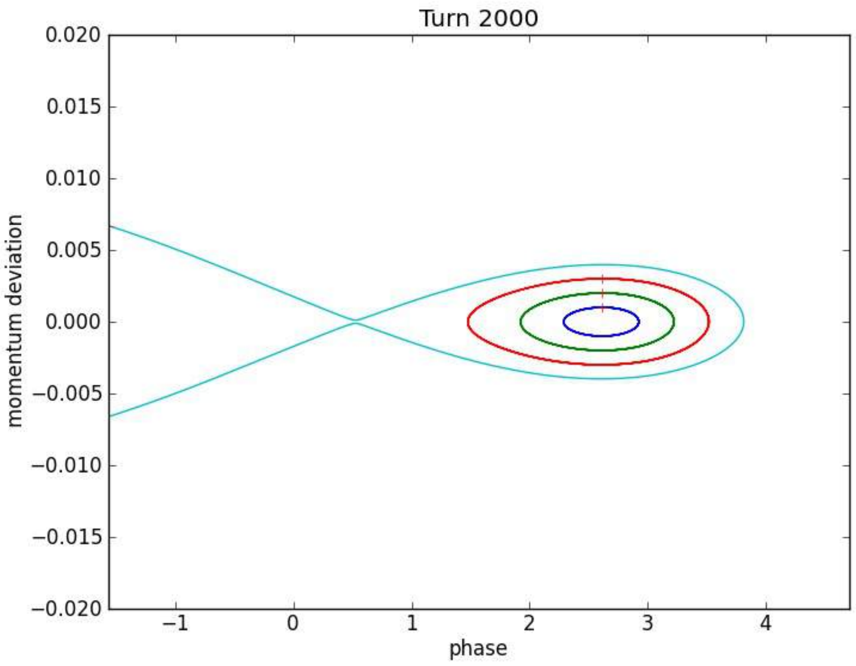
- In lepton (electron & positron) storage rings, as well in future high (TeVs) energy hadron rings , we need acceleration for synchronous particle to compensate energy loss.
- For now, we assume that the energy loss per turn is energy independent, and not net acceleration for synchronous particle.

$$H = 2\pi h\eta \cdot \frac{\delta^2}{2} + \frac{eV_{rf}}{\mathbf{E}_o \beta_o^2}\left\{\cos\phi - \cos\phi_s + \sin\phi_s\left(\phi - \phi_s\right)\right\}$$



Phase space





$\phi_s = \pi/6$

15 GeV

$\phi_s = 5\pi/6$

100 GeV

Longitudinal Phase Space

- We can define longitudinal phase space area from the conjugate variables

$$\left\{ t = \phi / \omega_{rf}, \Delta \mathbf{E} = \beta_o^2 \mathbf{E}_o \cdot \delta \right\}$$

- The phase space area remain constant even in acceleration
- If we stay with, (ϕ, δ), the phase space area is constant only without net acceleration.

Longitudinal Phase Space II

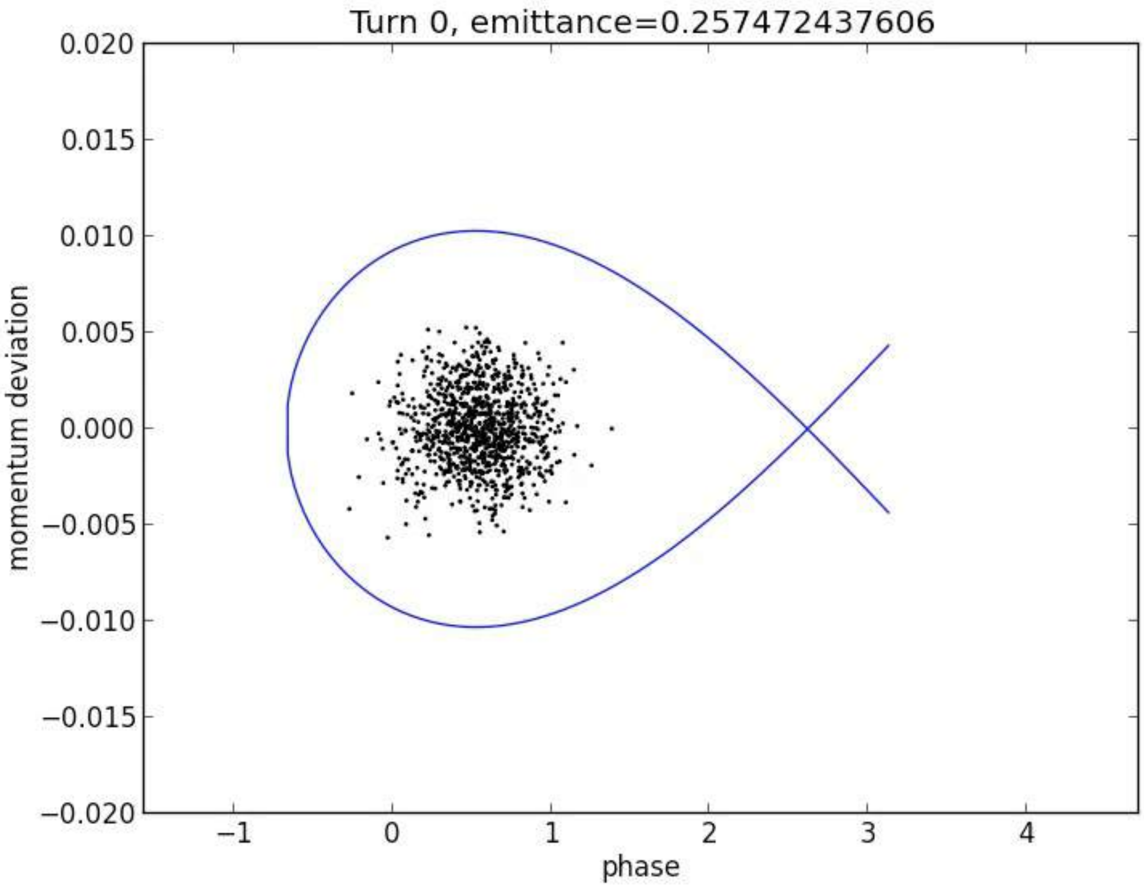
- We may take a Gaussian beam distribution then the rms phase space area is simply:

$$A_{rms} = \pi \sigma_t \sigma_E$$

$$A_{95\%} = 6 A_{rms}$$

- The shape in phase space is conserved only
  - When the beam distribution matches the bucket
  - When the beam oscillation is very small (linear).

Phase Space Area
Examples and Evolution I

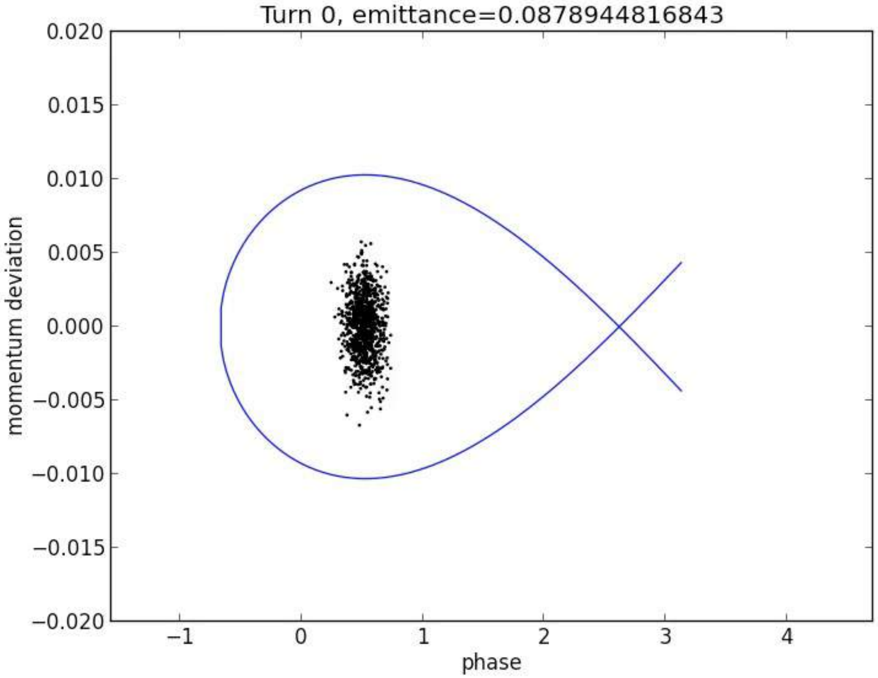


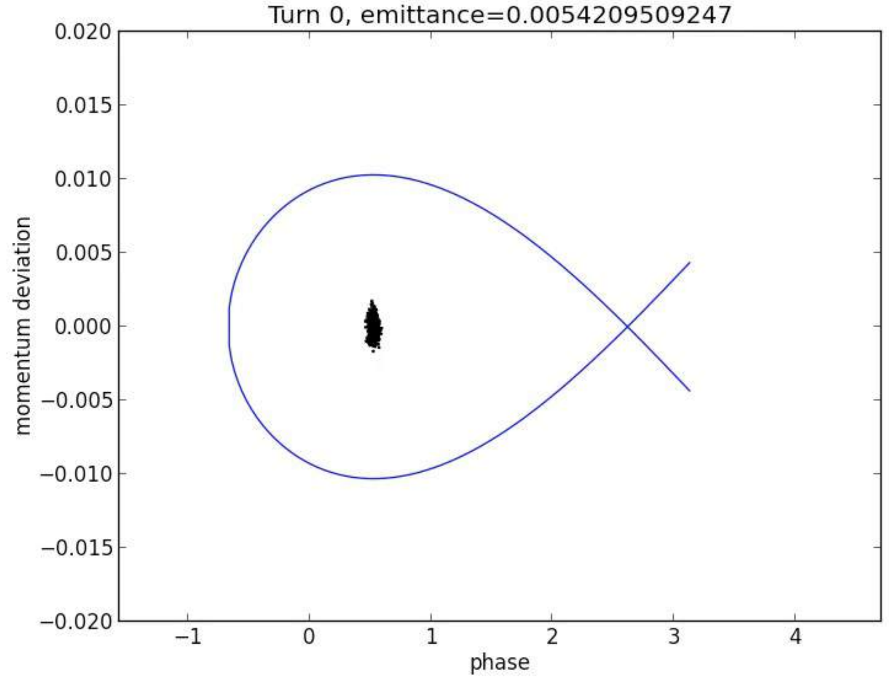
A Matched case
(Perfect injection):

Initial conditions match:

$$\frac{a_\delta}{a_\phi} = \frac{Q_s}{h|\eta|}$$

Phase Space Area
Examples and Evolution II

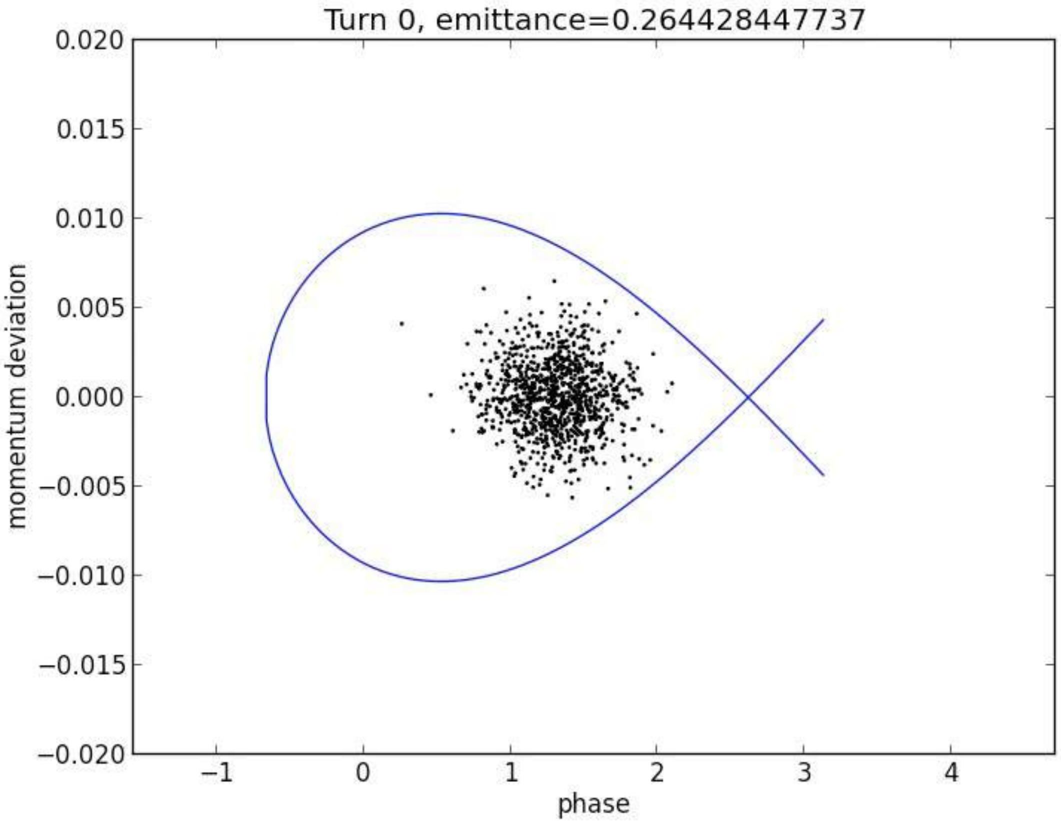




An unmatched case

$$\frac{a_\delta}{a_\phi} = \frac{3Q_s}{h|\eta|}$$

Phase Space Area
Examples and Evolution III



Time jitter at injection, other wise same as the matched case:

The phase error is:

$$\phi_{err} = \pi / 4$$

What have we learned today?

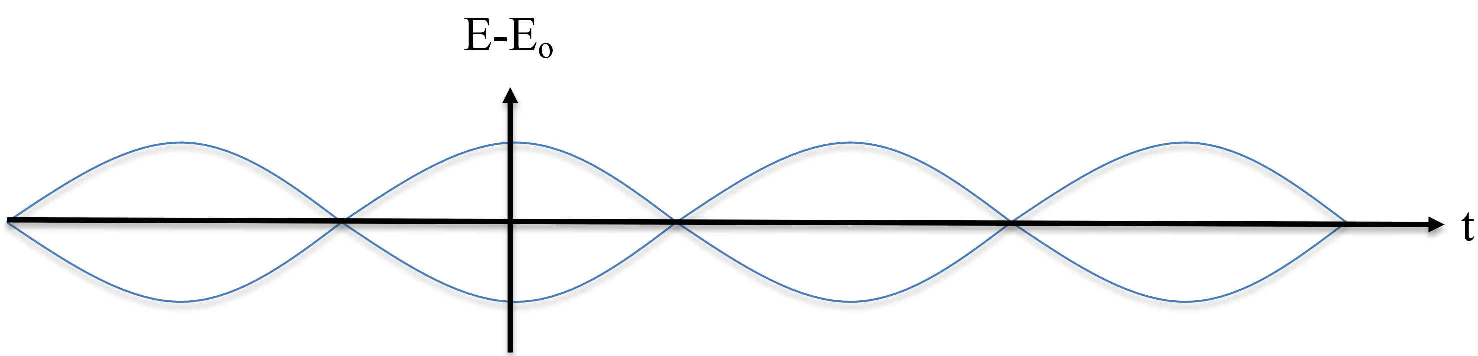
- Stable longitudinal (e.g. energy – arrival time) motion of particles in circular accelerators is called synchrotron oscillations
- Synchrotron motion is described with respect to a synchronous (ideal, or moving as designed) particle, which may experience acceleration, deceleration or energy loss by various processes such as radiation
- RF frequency is a integer harmonic (h) of the synchronous particle revolution frequency and has to be adjusted if velocity of the beam changes
- h bunches can be operated (accelerated) simultaneously in such storage ring/synchrotron. Area in energy-phase space for each bunch is called "RF bucket"

What have we learned today?

- Stable longitudinal (e.g. energy – arrival time) motion of particles in circular accelerators is called synchrotron oscillations
- Synchrotron motion is described with respect to a synchronous (ideal, or moving as designed) particle, which may experience acceleration, deceleration or energy loss by various processes such as radiation
- RF frequency is a integer harmonic (h) of the synchronous particle revolution frequency and has to be adjusted if velocity of the beam changes
- h bunches can be operated (accelerated) simultaneously in such storage ring/synchrotron. Area in energy-phase space for each bunch is called “RF bucket”

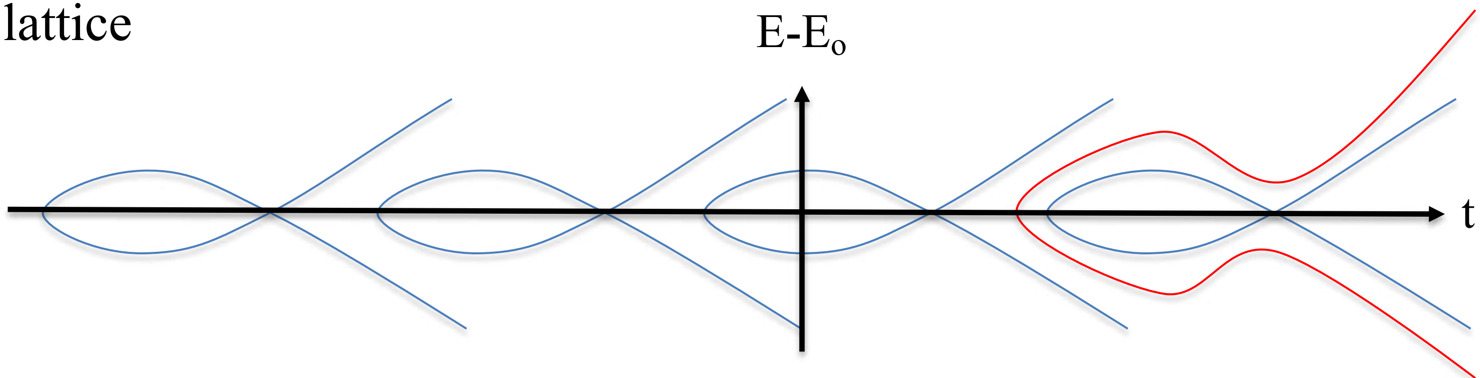
What have we learned today?

- Accurate description of synchrotron motion is described by a map, e.g. change of the energy and the phase in finite differentials. Synchrotron oscillation have stability areas separated by trajectory (called separatrix) from area of unstable motion.
- Synchrotron oscillations are typically very slow, $Q_s<<1$, which allows to describe them by differential equation identical that that of a pendulum. Such description has time independent Hamiltonian (e.g. it is a constant!) and use the Hamiltonian contour plot as particles trajectories in the phase space
- When synchronous particles have zero energy change in RF cavity (no acceleration, no energy loss), separatrices a symmetric and particles outsize the separatrix acceptance are drifting in phase without being "lost": particles with higher energy above the separatrix never cross to the lower part and vice versa.



What have we learned today?

- When synchronous particles gaining or losing energy, topology of the phase space changes: particles can cross from the upper part to the bottom part (typical for qn acceleration or a energy loss case) or from the bottom to top (for a deceleration case) – these particles will be lost by hitting energy acceptance of the lattice



- Synchrotron oscillations are intrinsically nonlinear and asynchronous: oscillations with larger amplitudes are slower than at small amplitudes. Period turns into infinity at the separatrix: particle never reaches a saddle point.
- Pair (-t, E) is a canonical pair and Liouville theorem guaranties preservation of the phase space occupied by particles
- During injection of particles, phase and energy errors as well as mismatch of the particles distribution can lead to effective “emittance” growth by particles entrapping “empty space”. Nonlinearity is the cause of this mixing.

Credits

- Credit to Prof. Yue Hao are for the animations of the particle's motions in slides 33-35