PHY 554. Homework 1.

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HW 1 (5 point): Future Circular Collider (FCC,) is under consideration by world physics community as a potentially next high energy collider.

(a) 1 point: The tunnel circumference would be 100 km https://en.wikipedia.org/wiki/Future_Circular_Collider. What average magnetic field is required to circulate 50 TeV proton beam?

(b) 1 point: It is also considered for electron-positron collider with beam energy up to 175 GeV. What average magnetic field is required to circulate 175 GeV electron or positron beam?

(c) 2 points: Show that the same ring (set of magnets) can be used to circulate electrons and positrons with the same energy but moving in opposite (colliding) directions. Specifically, write equation of motion for an electron and a positron and show that they can travel by the same trajectory but in opposite directions

(d) 1 point: Can we use the same trick to circulate and collide two proton beams?

Solution: (a &b) Bending angle of the beam trajectory is given by a simple formula

$$R = \frac{pc}{eB}; d\theta = \frac{ds}{R} = \frac{eB}{pc} ds;$$

where s is the length of the particle's trajectory. Since the circumference of the ring is nothing else that length of trajectory after which particle returns to the same point in space and direction, e.g. its trajectory bends by 360 degrees

$$2\pi \equiv \int_0^{2\pi} d\theta = \int_0^C \frac{eB}{pc} ds \equiv \frac{e\langle B \rangle}{pc} C;$$

$$\langle B \rangle = 2\pi \frac{pc}{eC}.$$

where we use definition of average value for B. While pc is slightly smaller that the energy, it is practically indistinguishable from the energy

$$pc = \sqrt{E^2 - m_p^2 c^4} = 49.99999999\ TeV;$$

$$m_p c^2 = 0.938272\ GeV.$$

Relative difference is only 1.7 10^{-10} ,which is beyond accuracy of any measurements. Hence, $pc=50\ TeV$ and

$$\frac{pc}{e}\left[T \cdot km\right] = \frac{50\left[TeV\right]}{0.2997925} = 166.782\left[T \cdot km\right]$$

#a

$$\langle B \rangle = 2\pi \frac{pc}{eC} = 10.479225T \sim 10.5T$$

This level of magnetic field requires superconducting technology and is currently beyond state of the art – can be generated in a special devices but moss-produced.

Similarly, *pc* of 175 GeV electrons and positrons is indistinguishable from their energy (difference ~10^{-12}!) The same exercise

$$\frac{pc}{e}[T\cdot km] = \frac{0.15[TeV]}{0.2997925} = 0.583737167[T\cdot km]$$

$$\langle B\rangle = 2\pi\frac{pc}{eC} = 0.036677T \sim 367\,Gs$$ #b

e.g. very low field easily achievable by many means.

(c) Let's consider a positively charge particle with charge +e moving in time independent magnetic field along an orbit $\vec{r}_+(t)$ and the equation of its motion is satisfied:

$$\frac{d\vec{p}_+}{dt} = \frac{+e}{c}\left[\vec{\mathrm{v}}_+ \times \vec{B}\left(\vec{r}_+(t)\right)\right] = \frac{+e}{\gamma mc}\left[\vec{p}_+ \times \vec{B}\left(\vec{r}_+(t)\right)\right];$$

Let's now reverse the direction of motion $\vec{r}_-(t) = \vec{r}_+(-t)$ and the sign of particle's charge $+e \to -e$:

$$is\,\frac{d\vec{p}_-}{dt} \stackrel{?}{=} \frac{-e}{\gamma mc}\left[\vec{p}_- \times \vec{B}\left(\vec{r}_-(t)\right)\right] \quad ???? \quad \vec{p}_-(t) = \gamma m\frac{d\vec{r}_-(t)}{dt} = \gamma m\frac{d\vec{r}_+(-t)}{dt} = -\vec{p}_+(-t)$$

$$left\ side: \frac{d\vec{p}_-(t)}{dt} = -\frac{d\vec{p}_+(-t)}{dt} = \left.\frac{d\vec{p}_+}{dt}\right|_{-t}$$

$$rightside: \frac{-e}{\gamma mc}\left[\vec{p}_- \times \vec{B}\left(\vec{r}_+(-t)\right)\right] = \frac{e}{\gamma mc}\left[\vec{p}_+(-t) \times \vec{B}\left(\vec{r}_+(-t)\right)\right]$$

$$\Rightarrow \frac{d\vec{p}_-}{dt} \equiv \frac{-e}{\gamma mc}\left[\vec{p}_- \times \vec{B}\left(\vec{r}_+(-t)\right)\right];\#$$

e.g. the equation of motion for negatively charge particle is satisfied. While the above exercise shows details of the necessary transformation, using second order deferential equation on particles trajectory also reveals the nature:

$$\frac{d\vec{p}}{dt} = \gamma m\frac{d^2\vec{r}}{dt^2} \Rightarrow \frac{d^2\vec{r}}{dt^2} = \frac{+e}{\gamma mc}\left[\frac{d\vec{r}}{dt} \times \vec{B}\right];$$

$$\vec{r}(t) = \vec{\rho}(-t); \dot{\vec{r}} = -\dot{\vec{\rho}}; \ddot{\vec{r}} = \ddot{\vec{\rho}}; \to \frac{d^2\vec{\rho}}{dt^2} \equiv \frac{-e}{\gamma mc}\left[\frac{d\vec{\rho}}{dt} \times \vec{B}\right].$$

Important note: this statement is correct for motion of particles in time-independent magnetic field. Presence of electric field will break this symmetry.

(d) The answer is no – as indicated in the above derivation, reversal of the direction in magnetic field of motion requires reversal of the sign of the charge. Otherwise, the sign of the force changes for proton propagating in the opposite direction. It means that its trajectory will bend in opposite direction: it is called dipole separator.

But, if constant radial electric field could be used to bend particles trajectory, it would support collision of the beams with the same charge and energy:

$$\frac{d\vec{p}}{dt} = \gamma m \frac{d^2\vec{r}}{dt^2} \Rightarrow \frac{d^2\vec{r}}{dt^2} = \frac{+eE}{\gamma mc};$$

$$\vec{r}(t) = \vec{\rho}(-t); \ddot{\vec{r}} = \ddot{\vec{\rho}}; \rightarrow \frac{d^2\vec{\rho}}{dt^2} \equiv \frac{+eE}{\gamma mc}.$$

Unfortunately electric fields available in the lab cannot support GeV and TeV colliders. Typical magnetic field of 1 T generates force equivalent to that generated 300 MV/m electric field. Arcs happening in vacuum at much lover electric fileds…

HW 2 (2 points): For a classical microtron having energy gain per pass of 1.022 MeV and operational RF frequency 3 GHz (3 x 10^9 Hz) find required magnetic field (Hint: use k=1). What will be radius of first orbit in this microtron?

Solution: Again, lets start from the radius of curvature

$$\rho = \frac{pc}{eB_y}$$

and calculate time of flight for a given energy

$$T = \frac{2\pi\rho}{\mathrm{v}} = \frac{2\pi}{eB_y} \cdot \frac{pc}{\mathrm{v}} = \frac{2\pi}{eB_y} \cdot \frac{E}{\mathrm{c}}$$

The energy at n-turn is equal to the rest energy electron energy 0.511 MeV plus n-fold energy gain:

$$E_n = mc^2 + n \cdot \Delta E; \quad \Delta E = 2mc^2;$$

$$E_n = (2n+1)mc^2 = (2n+1) \cdot 0.511\,MeV$$

Now we should use synchronization condition, e.g. that each turn should take an integer number of RF cycles

$$T_n = N(n) \cdot T_o;\ T_o = \frac{1}{f_{RF}};\ T_n = \frac{2\pi}{eB_y} \cdot \frac{E_n}{\mathrm{c}} = (2n+1)\frac{2\pi mc}{eB_y};$$

$$N(n) = k \cdot (2n+1);\ \frac{2\pi mc}{eB_y} = kT_o = \frac{k}{f_{RF}} \rightarrow B_y = \frac{1}{k}\frac{2\pi mc^2}{e(cT_o)}$$

where *k* is a positive integer. Putting number together for *k=1,* we get for first pass

$$cT_o = \frac{c}{f_o} = 9.993\,cm \cong 10\,cm;$$

$$\frac{2\pi m_e c^2}{e} = 2\pi\frac{0.511...}{0.29979...} \cong 2\pi \cdot 1.705\ kGs\ cm = 10.71\ kGs\ cm$$

$$B_y \cong 1.071\ kGs$$

$$\gamma = 2n+1 \Rightarrow \quad \gamma_1 = 3; \quad \beta_1 = \sqrt{1-\gamma_1^{-2}} \cong 0.943$$

$$p_1 c = \gamma_1 \beta_1\, mc^2 \cong 1.445\,MeV$$

$$\rho_1[cm] \cong \frac{p_1 c\left[MeV\right]}{0.3 \cdot B_y} = 4.5\,cm$$

The other way to find radius of first orbit: it takes 3 RF periods and orbit circumference is

$$T_1 = 3 \cdot T_o;\ \ C_1 = 2\pi\rho_1 = \mathrm{v}_1 T_1 = \beta_1 c T_1 = 26.26\,cm$$

Naturally, dividing the circumference by 2π we get the same 4.5 cm radius of the first orbit.

HW 3 (3 points): Find available energy (so called C.M. energy) for a head-on collision of electrons and protons in two proposed electron-hadron colliders eRHIC and LHeC:

(a) eRHIC plans to collide 18 GeV electrons with 275 GeV protons;
(b) LHeC plans to collide 60 GeV electrons with 7 TeV protons

(c) eRHIC plans to collide 20 GeV electrons with 250 GeV protons;
(d) LHeC plans to collide 60 GeV electrons with 7 TeV protons

$$p_p^\mu = \left\{E_p / c, p_p, 0, 0\right\} \qquad\qquad p_e^\mu = \left\{E_e / c, -p_e, 0, 0\right\}$$

First let's find the c.m. energy using 4-momenta of both particles:

$$p_e^\mu = \left\{E_e / c, -p_e, 0, 0\right\}; p_p^\mu = \left\{E_p / c, p_p, 0, 0\right\};$$

$$p_c^\mu = \left\{\frac{E_p + E_e}{c}, p_p - p_e, 0, 0\right\}; \frac{E_{cm}{}^2}{c^2} = p_c^\mu p_c^\mu = \left(\frac{E_p + E_e}{c}\right)^2 - \left(p_p - p_e\right)^2;$$

$$E_{cm}{}^2 = \left(E_p + E_e\right)^2 - \left(p_p c - p_e c\right)^2 = E_p{}^2 - \left(p_p c\right)^2 + E_e{}^2 - \left(p_e c\right)^2 + 2\left(E_p E_e + p_p p_e c^2\right)$$

$$E_p{}^2 - \left(p_p c\right)^2 = \left(m_p c^2\right)^2; \;\; E_e{}^2 - \left(p_e c\right)^2 = \left(m_e c^2\right)^2;$$

$$E_{cm}{}^2 = m_p{}^2 c^4 + m_e{}^2 c^4 + 2 E_p E_e \left(1 + \beta_p \beta_e\right);$$

$$E_{cm} = \sqrt{m_p{}^2 c^4 + m_e{}^2 c^4 + 2 E_p E_e \left(1 + \beta_p \beta_e\right)}$$

For ultra-relativistic case ($\gamma_p >> 1; \gamma_e >> 1$, 1-$\beta_p \beta_e$ <<1) we can approximately write

$$E_{cm} \simeq 2\sqrt{E_p E_e}$$

(a) eRHIC with 20 GeV electrons and 250 GeV protons:
Exact E_{cm} = 141.4242197 GeV, approximate 141.4213562 is accurate in 5 digits.

(b) LHeC with 60 GeV electrons with 7 TeV protons: E_{cm} = 1.296 TeV.
Difference between exact and proximate formulae 2.6E-7 is negligible.