PHY 564
Advanced Accelerator Physics
Lecture 15
Applications of parameterization and the phase-action variables to standard accelerator problems

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Applications of parameterization to standard accelerator problems

Complete parameterization developed in previous lecture can be used to solve most (if not all) of standard problems in accelerator. Incomplete list is given below:

1. Dispersion
2. Orbit distortions
3. AC dipole (periodic excitation)
4. Tune change with quadrupole (magnets) changes
5. Chromaticity
6. Beta-beat
7. Weak coupling
8. Synchro-betatron coupling
9. Beyond Hamiltonian system - weak (slow) damping
10. ….and diffusion
11. ……

We do not plan to go through all these examples while focusing on general methodology and use selected examples to demonstrate power of the symplectic linear parameterization. We will use complex form of parameterization since it gives more transparent frequency content of the oscillations, but one can do similar exercise using real notations – after all results in real life are in real notations.

Sample I. Let's start from simplest problems such as dispersion and closed orbit. We found a general form of parameterization of linear motion in Hamiltonian system, which is solution of homogeneous linear equations, where ***B*** is constant vector:

$$\frac{dX}{ds} = \mathbf{D}(s) \cdot X; \; X = \tilde{\mathbf{U}}(s) \cdot B \tag{1}$$

A standards problems is a solution of inhomogeneous equations:

$$\frac{dX}{ds} = \mathbf{D}(s) \cdot X + F(s); \tag{2}$$

It can be done analytically by varying the constant ***B:***

$$X = \tilde{\mathbf{U}}(s) B(s) \Rightarrow \tilde{\mathbf{U}} \cdot B' = F(s) \Rightarrow B' = \tilde{\mathbf{U}}^{-1}(s) F(s) \Rightarrow B(s) = B_o + \int_{s_o}^{s} \tilde{\mathbf{U}}^{-1}(\xi) F(\xi) d\xi$$

A general solution is a specific solution of inhomogeneous equation plus arbitrary solution of the homogeneous – result you expect in linear ordinary differential equations (in this case with s-depended coefficients):

$$X(s) = \tilde{\mathbf{U}}(s) A_o + \tilde{\mathbf{U}}(s) \int_{s_o}^{s} \tilde{\mathbf{U}}^{-1}(\xi) F(\xi) d\xi; \; \tilde{\mathbf{U}}^{-1} = \frac{i}{2} \mathbf{S} \cdot \tilde{\mathbf{U}}^{T} \cdot \mathbf{S} \tag{3}$$

For a periodic force (orbit distortions, dispersion function) $F(s+C)=F(s)$ one can fine periodic solution $X(s+C)=X(s)$:

$$X(s)=\tilde{\mathbf{U}}(s)B(s);\; B(s)=A_o+\tilde{\mathbf{U}}(s)\int_{s_o}^{s}\tilde{\mathbf{U}}^{-1}(\xi)F(\xi)d\xi;$$

$$X(s)=X(s+C)\Rightarrow\tilde{\mathbf{U}}(s)A_o+\tilde{\mathbf{U}}(s)\int_{s_o}^{s}\tilde{\mathbf{U}}^{-1}(\xi)F(\xi)d\xi=\tilde{\mathbf{U}}(s+C)A_o+\tilde{\mathbf{U}}(s+C)\int_{s_o}^{s+C}\tilde{\mathbf{U}}^{-1}(\xi)F(\xi)d\xi$$

$$\tilde{\mathbf{U}}(s+C)=\mathbf{T}(s)\tilde{\mathbf{U}}(s)=\tilde{\mathbf{U}}(s)\Lambda;\; \int_{s_o}^{s+C}\tilde{\mathbf{U}}^{-1}(\xi)F(\xi)d\xi=\int_{s_o}^{s}\tilde{\mathbf{U}}^{-1}(\xi)F(\xi)d\xi+\int_{s}^{s+C}\tilde{\mathbf{U}}^{-1}(\xi)F(\xi)d\xi;$$

$$\tilde{\mathbf{U}}(s)B(s)=\tilde{\mathbf{U}}(s)\Lambda B(s)+\tilde{\mathbf{U}}(s)\Lambda\int_{s}^{s+C}\tilde{\mathbf{U}}^{-1}(\xi)F(\xi)d\xi \tag{4}$$

$$\tilde{\mathbf{U}}^{-1}(s)\times\left\{\tilde{\mathbf{U}}(s)B(s)-\tilde{\mathbf{U}}(s)\Lambda B(s)=\tilde{\mathbf{U}}(s)\Lambda\int_{s}^{s+C}\tilde{\mathbf{U}}^{-1}(\xi)F(\xi)d\xi\right\}\Rightarrow$$

$$(\mathbf{I}-\Lambda)B(s)=\Lambda\int_{s}^{s+C}\tilde{\mathbf{U}}^{-1}(\xi)F(\xi)d\xi\equiv\int_{s-C}^{s}\tilde{\mathbf{U}}^{-1}(\xi)F(\xi)d\xi\Rightarrow B(s)=(\mathbf{I}-\Lambda)^{-1}\int_{s-C}^{s}\tilde{\mathbf{U}}^{-1}(\xi)F(\xi)d\xi$$

$$X(s)=\tilde{\mathbf{U}}(s)B(s)=\tilde{\mathbf{U}}(s)(\mathbf{I}-\Lambda)^{-1}\int_{s-C}^{s}\tilde{\mathbf{U}}^{-1}(\xi)F(\xi)d\xi$$

It is easy to see that $X(s+C)=X(s)$ exists if none of the eigen values is not equal 1 – otherwise matrix $(\mathbf{I}-\Lambda)$ would have zero determinant and can not be inverted!

Specific examples: Orbit distortions caused by the field errors, transverse dispersion.

When the conditions for the equilibrium particle and the reference trajectory are slightly violated:

$$X^T = \{x, P_1, y, P_3, \tau, \delta\}; F^T = \left\{0, \frac{e}{c}\left(\delta B_y + \frac{E_o}{p_o c}\delta E_x\right), 0, \frac{e}{c}\left(\delta B_x - \frac{E_o}{p_o c}\delta E_y\right), 0, 0\right\}$$

$$K(s) \equiv \frac{1}{\rho(s)} - \frac{e}{p_o c}\left(B_y\big|_{ref} + \frac{E_o}{p_o c}E_x\big|_{ref}\right) - f_x; \quad f_x = \frac{e}{p_o c}\left(\delta B_y + \frac{E_o}{p_o c}\delta E_x\right). \quad (5)$$

$$\frac{e}{p_o c}\left(B_x\big|_{ref} - \frac{E_o}{p_o c}E_y\big|_{ref}\right) = -f_y = \frac{e}{p_o c}\left(\delta B_x - \frac{E_o}{p_o c}\delta E_y\right)$$

Plugging (5) into (4) will give one the periodic closed orbit for such a case. For transverse dispersion the finding reduces to

$$\tilde{h} = \frac{P_1^2 + P_3^2}{2p_o} + F\frac{x^2}{2} + Nxy + G\frac{y^2}{2} + L(xP_3 - yP_1) + \frac{\delta^2}{2p_o}\cdot\frac{m^2c^2}{p_o^2} + g_x x\delta + g_y y\delta$$

with

$$F = S\frac{\partial H}{\partial X} = \left\{0, -g_x, 0, -g_y, 0, -\frac{m^2c^2}{p_o^3}\right\}^T. \quad (6)$$

1D ACCELERATOR

$$\tilde{U} = \begin{bmatrix} \mathrm{w} & \mathrm{w} \\ \mathrm{w}' + \frac{i}{\mathrm{w}} & \mathrm{w}' - \frac{i}{\mathrm{w}} \end{bmatrix} \begin{bmatrix} e^{i\psi} & 0 \\ 0 & e^{-i\psi} \end{bmatrix}; \tilde{U}^{-1} = \frac{i}{2} \begin{bmatrix} e^{-i\psi} & 0 \\ 0 & e^{i\psi} \end{bmatrix} \begin{bmatrix} \mathrm{w}' - \frac{i}{\mathrm{w}} & -\mathrm{w} \\ -\mathrm{w}' \text{-} \frac{i}{\mathrm{w}} & \mathrm{w} \end{bmatrix};$$

$$X(s) = \frac{i}{2} \int_{s-C}^{s} \begin{bmatrix} \mathrm{w(s)} & \mathrm{w(s)} \\ \mathrm{w}'(\mathrm{s}) + \frac{i}{\mathrm{w(s)}} & \mathrm{w}'(\mathrm{s}) - \frac{i}{\mathrm{w(s)}} \end{bmatrix} \cdot \begin{bmatrix} \left(1 - e^{i\mu}\right)^{-1} e^{i\psi(\mathrm{s}) - i\psi(\xi)} & 0 \\ 0 & \left(1 - e^{-i\mu}\right)^{-1} e^{-i(\psi(\mathrm{s}) - i\psi(\xi))} \end{bmatrix} \cdot \begin{bmatrix} \mathrm{w}'(\xi) - \frac{i}{\mathrm{w}(\xi)} & \text{-}\mathrm{w}(\xi) \\ \mathrm{w}'(\xi) - \frac{i}{\mathrm{w}(\xi)} & \mathrm{w}(\xi) \end{bmatrix} F(\xi) d\xi$$

$X^T = \{x, x'\}; F^T = \frac{e}{p_o c} \delta B_y \{0,1\}$ - orbit; $F^T = K(s)\{0,1\}$ for dispersion, i.e. $F^T = f(s)\{0,1\}$

$$X(s) = \int_{s-C}^{s} \begin{bmatrix} \dfrac{\mathrm{Re}\left(\mathrm{w(s)w}(\xi) e^{i(\psi(\mathrm{s}) - \psi(\xi) - \mu/2)} \left(\dfrac{e^{-i\mu/2} - e^{i\mu/2}}{-i} \right)^{-1} \right)}{\ldots\ldots} \end{bmatrix} f(\xi) d\xi$$

i.e. $$x(s) = \frac{\mathrm{w(s)}}{2\sin\mu/2} \oint_C f(\xi) \mathrm{w}(\xi) \cos(\psi(\mathrm{s}) - \psi(\xi) - \mu/2) d\xi \qquad (8)$$

First example: orbit distortion

$$f_x(s) = -\frac{e\delta B_y(s)}{p_o c}; \quad f_y(s) = \frac{e\delta B_x(s)}{p_o c}$$

$$\delta x(s) = -\frac{\mathrm{w(s)}}{2\sin\mu/2}\oint_C \frac{e\delta \mathrm{B_y}(\xi)}{p_o c}\mathrm{w}(\xi)\cos(\psi(\mathrm{s}) - \psi(\xi) - \mu/2)d\xi$$

$$\delta y(s) = \frac{\mathrm{w(s)}}{2\sin\mu/2}\oint_C \frac{e\delta \mathrm{B_x}(\xi)}{p_o c}\mathrm{w}(\xi)\cos(\psi(\mathrm{s}) - \psi(\xi) - \mu/2)d\xi \tag{9}$$

but this is not the end of the story for horizontal motion! What about change of the orbiting time?

Second example: Dispersion

$$f_x(s) = K_o(s)\pi_l = K_o(s)\pi_\tau/\beta_o;$$

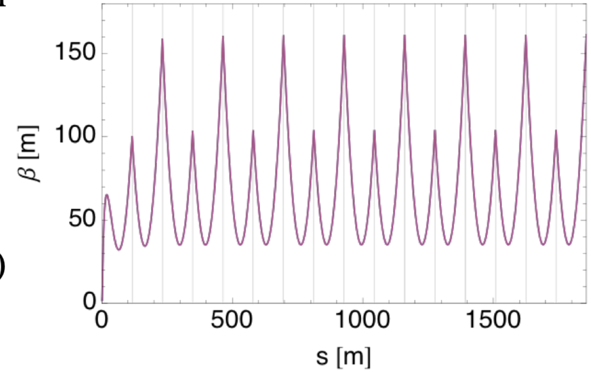
$$x(s) = \eta_x(s)\cdot\pi_l = \eta_x(s)\cdot\pi_\tau/\beta_o;$$

$$\eta_x(s) = -\frac{\mathrm{w(s)}}{2\sin\mu/2}\oint_C K_o(\xi)\mathrm{w}(\xi)\cos(\psi(\mathrm{s}) - \psi(\xi) - \mu/2)d\xi \tag{10}$$

Sample II: Beta-beat – 1D case

Let's consider a case when we are designing a circular accelerator comprised of various parts and want parameterization parameters (in this case envelope function w) to have a specific s-dependence. For example, we want it to fit a long periodic arc of an accelerator or generate a very specific w(s) – consistent with equations of motion - to satisfy a specific function needed from accelerator: minimize emittance, achromatic lattice….

It is simple fact that any solution can be expanded upon the eigen vectors of periodic system (FODO cell repeated and again is an example). Let 's consider that at azimuth $s=s_o$ initial value of "injected" eigen vector **V** being different from the periodic solution **Y**. We expand it as



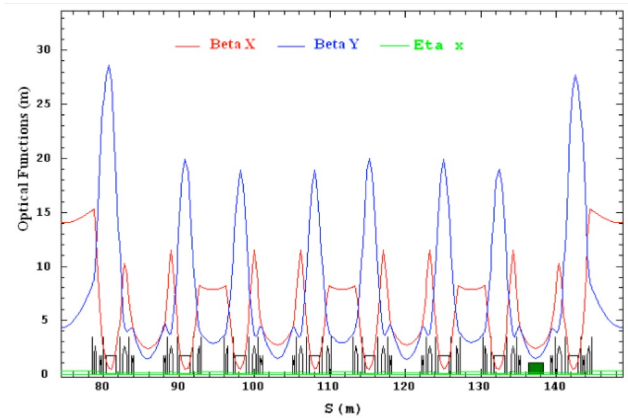
$$V(s_o) = aY_k(s_o) + bY_k^*(s_o) = \begin{bmatrix} v_o \\ v_o' + \dfrac{i}{v_o} \end{bmatrix}; Y_k = \begin{bmatrix} w_o \\ w_o' + \dfrac{i}{w_o} \end{bmatrix}$$

$$a = \frac{1}{2i}\ Y_k^{*T}(s_o) S V(s_o)\ ;\ b = \frac{1}{-2i}\ Y_k^T(s_o) S V(s_o) \qquad (11)$$

$$a = \frac{1}{2i}\left\{v_o w_o' - w_o v_o' + i\left(\frac{v_o}{w_o} + \frac{w_o}{v_o}\right)\right\}; b = -\frac{1}{2i}\left\{v_o w_o' - w_o v_o' + i\left(\frac{v_o}{w_o} - \frac{w_o}{v_o}\right)\right\};$$

$$\frac{d}{ds}\tilde{Y}(s) = \mathbf{D}(s)\cdot\tilde{Y}(s);\quad \tilde{Y}(s) = Y(s)e^{i\psi(s)}; Y(s+C) = Y(s)$$

It is self-evident that



$$\tilde{V}' = D\tilde{V};\quad \tilde{V}(s) = a\tilde{Y}_k(s) + b\tilde{Y}_k^*(s) = \begin{bmatrix} v \\ v' + \dfrac{i}{v} \end{bmatrix} e^{i\varphi} = Y_k = a\begin{bmatrix} w \\ w' + \dfrac{i}{w} \end{bmatrix} e^{i\psi} + b\begin{bmatrix} w \\ w' - \dfrac{i}{w} \end{bmatrix} e^{-i\psi}$$

$$|v|^2 = \frac{|w|^2}{4}\left|ae^{i\psi} + be^{-i\psi}\right|^2 = \frac{|w|^2}{4}\left(|a|^2 + |b|^2 - 2\mathrm{Re}\left(ab^* e^{2i\psi}\right)\right)$$

i.e. beta-function will beat with double of the betatron phase.

Sample III: Perturbation theory (ala quantum mechanics)

Small variation of the linear Hamiltonian terms (including coupling)

$$\frac{dX}{ds} = \left(\mathbf{D}(s) + \varepsilon \mathbf{D}_1(s)\right) \cdot X = \left(\mathbf{SH}(s) + \varepsilon \mathbf{SH}_1(s)\right) \cdot X$$
$$\frac{d\tilde{Y}_k(s)}{ds} = \mathbf{D}(s)\tilde{Y}_k(s); k = 1,..,n. \tag{13}$$

Assuming that changes are very small we can express the changes in the eigen vectors basis:

$$\tilde{Y}_{1k} = \tilde{Y}_k e^{i\delta\phi_k} + \varepsilon c_k \tilde{Y}_k^{\ *} + \varepsilon \sum_{j\neq k} \left(a_{kj}\tilde{Y}_j + b_{kj}\tilde{Y}_j^*\right) + O(\varepsilon^2);\ k = 1,...,n$$
$$\tilde{Y}_{1k}^{\ *} = \tilde{Y}_k^{\ *} e^{-i\delta\phi_k} + \varepsilon c^*_{\ k}\tilde{Y}_k + \varepsilon \sum_{j\neq k} \left(a^*_{\ kj}\tilde{Y}_j^* + b^*_{\ kj}\tilde{Y}_j\right) + O(\varepsilon^2); \tag{14}$$
$$\frac{d\tilde{Y}_{1k}}{ds} = \left(\mathbf{D}(s) + \varepsilon \mathbf{D}_1(s)\right) \cdot \tilde{Y}_{1k} + o(\varepsilon^2);$$

We need substitute the expansion of the new eigen vectors into the differential equation and to keep first order term of ε

$$\tilde{Y}_{1k} = \tilde{Y}_k e^{i\delta\phi_k} + \varepsilon c_k \tilde{Y}_k^* + \varepsilon \sum_{j\neq k} \left(a_{kj}\tilde{Y}_j + b_{kj}\tilde{Y}_j^* \right) + O(\varepsilon^2);\ k = 1,...,n$$

$$\tilde{Y}_k' e^{i\delta\phi_k} + \delta\phi_k' \tilde{Y}_k e^{i\delta\phi_k} + \varepsilon c_k' \tilde{Y}_k^* + \varepsilon c_k \tilde{Y}_k'^* + \varepsilon \sum_{j\neq k} \left(a_{kj}'\tilde{Y}_j + b_{kj}'\tilde{Y}_j^* \right) + \varepsilon \sum_{j\neq k} \left(a_{kj}\tilde{Y}_j' + b_{kj}\tilde{Y}_j'^* \right) =$$

$$\mathbf{D}\left(\tilde{Y}_k e^{i\delta\phi_k} + \varepsilon c_k \tilde{Y}_k^* + \varepsilon \sum_{j\neq k} \left(a_{kj}\tilde{Y}_j + b_{kj}\tilde{Y}_j^* \right) \right) + \varepsilon \mathbf{D}_1(s)\tilde{Y}_k e^{i\delta\phi_k} + O(\varepsilon^2)$$

$$\tilde{Y}_j' = \mathbf{D}\tilde{Y}_k; \tilde{Y}_j'^* = \mathbf{D}\tilde{Y}_j^*.$$

and all terms in red cancel each other leaving us with

$$\delta\phi_k' \tilde{Y}_k e^{i\delta\phi_k} + \varepsilon c_k' \tilde{Y}_k^* + \varepsilon \sum_{j\neq k} \left(a_{kj}'\tilde{Y}_j + b_{kj}'\tilde{Y}_j^* \right) = \varepsilon \mathbf{D}_1(s)\tilde{Y}_k e^{i\delta\phi_k}$$

which we can split into individual equations for each component using symplectic orthogonality of the eigen vectors

$$\tilde{Y}_k^{T*} S \tilde{Y}_j = -\tilde{Y}_k^T S \tilde{Y}_j^* = 2i\delta_{ik};\ \tilde{Y}_k^T S \tilde{Y}_j = \tilde{Y}_k^{T*} S \tilde{Y}_j^* = 0$$

Multiplying by $\tilde{Y}_m^{\ *}S$ or $\tilde{Y}_m S$ from the left yields:

$$-2\delta\phi_k' = \varepsilon \tilde{Y}_k^* \mathbf{SD}_1(s)\tilde{Y}_k \to \delta\phi' = \frac{\varepsilon}{2} Y_k^{\ *T}\mathbf{H}_1(s)Y_k; \quad \mathbf{SD}_1 = -\mathbf{H}_1;$$

$$-2ic' = \tilde{Y}_k^T \mathbf{SD}_1(s)\tilde{Y}_k e^{i\delta\phi_k} \to c' = \frac{1}{2i} Y_k^{\ T}\mathbf{H}_1(s)Y_k e^{i(2\psi_k+\delta\phi_k)} \cong \frac{1}{2i} Y^T{}_k \mathbf{H}_1 Y_k e^{2i\psi_k}$$

$$2ia_{kj}' = \tilde{Y}_j^{T*}\mathbf{SD}_1(s)\tilde{Y}_k e^{i\delta\phi_k} \to a_{kj}' = \frac{-1}{2i} Y_j^{\ *T}\mathbf{H}_1(s)Y_k e^{i(\psi_k-\psi_j+\delta\phi_k)} \cong \frac{-1}{2i} Y_j^{\ *T}\mathbf{H}_1(s)Y_k e^{i(\psi_k-\psi_j)}; j \neq k$$

$$-2ib_{kj}' = \tilde{Y}_j^T\, \mathbf{SD}_1(s)\tilde{Y}_k e^{i\delta\phi_k} \to b_{kj}' = \frac{1}{2i} Y_j^{\ T}\mathbf{H}_1(s)Y_k e^{i(\psi_k+\psi_j+\delta\phi_k)} \cong \frac{1}{2i} Y_j^{\ T}\mathbf{H}_1(s)Y_k e^{i(\psi_k+\psi_j)}; j \neq k.$$

with solutions in form of integrals:

$$\delta\phi(s) = \phi_o + \frac{\varepsilon}{2}\int_0^s Y_k^{\ *T}\mathbf{H}_1 Y_k\, d\xi; \; c(s) = c_o + \frac{1}{2i}\int_0^s d\xi Y_k^{\ T}\mathbf{H}_1 Y_k e^{i(2\psi_k+\delta\phi_k)};$$

$$a_{kj} = a_{kjo} - \frac{1}{2i}\int_o^s d\xi Y_j^{\ *T}\mathbf{H}_1 Y_k e^{i(\psi_k-\psi_j+\delta\phi_k)}; b_{kj} = b_{kjo} + \frac{1}{2i}\int_o^s d\xi Y_j^{\ T}\mathbf{H}_1 Y_k e^{i(\psi_k+\psi_j+\delta\phi_k)};$$

$$Y_{1k}(s) = \tilde{Y}_{1k} e^{-i(\psi_k+\delta\phi_k)} = Y_k + \varepsilon c_k Y_k^{\ *} e^{-i(2\psi_k+\delta\phi_k)}\left(c_o + \frac{1}{2i}\int_0^s d\xi Y_k^{\ T}\mathbf{H}_1 Y_k e^{i(2\psi_k+\delta\phi_k)} \right) +$$

$$\varepsilon \sum_{j\neq k} \begin{pmatrix} Y_j e^{-i(\psi_k-\psi_j+\delta\phi_k)}\left(a_{kjo} - \frac{1}{2i}\int_o^s d\xi Y_j^{\ *T}\mathbf{H}_1 Y_k e^{i(\psi_k-\psi_j+\delta\phi_k)} \right) + \\ Y^*{}_j e^{-i(\psi_k+\psi_j+\delta\phi_k)}\left(b_{kjo} + \frac{1}{2i}\int_o^s d\xi Y_j^{\ T}\mathbf{H}_1 Y_k e^{i(\psi_k+\psi_j+\delta\phi_k)} \right) \end{pmatrix} + O(\varepsilon^2)$$

Now we want to have periodic eigen vectors, e.g.

$$\tilde{Y}_{1k}(s+C) = \tilde{Y}_{1k}(s)e^{i\mu_{1k}}; \mu_{1k} = \mu_k + \frac{\varepsilon}{2}\int_0^C Y_k^{*T}\mathbf{H}_1 Y_k \, d\xi;$$

into periodic functions, we need to choose the initial conditions

$$d(s) = e^{-i\theta(s)}\left(d_o - \frac{1}{2i}\int_o^s d\xi f(\xi)e^{i\theta(\xi)}\right); \; f(\xi+C) = f(\xi).$$

to make a coefficient looking

$$e^{-i\theta(s+C)}\left(d_o + \int_o^{s+C} d\xi f(\xi)e^{i\theta(\xi)}\right) = e^{-i\theta(s)}\left(d_o + \int_o^s d\xi f(\xi)e^{i\theta(\xi)}\right);$$

$$\int_o^{s+C} d\xi f(\xi)e^{i\theta(\xi)} = \left(e^{i\Delta\theta(C)} - 1\right)\left(d_o + \int_o^s d\xi f(\xi)e^{i\theta(\xi)}\right);$$

$$\left(d_o + \int_o^s d\xi f(\xi)e^{i\theta(\xi)}\right) = \frac{1}{e^{i\Delta\theta(C)} - 1}\int_o^{s+C} d\xi f(\xi)e^{i\theta(\xi)}.$$

Final expression is:

$$\tilde{Y}_{1k} e^{-i(\psi_k+\delta\phi_k)} = Y_{1k}(s) = Y_k + \varepsilon \frac{Y_k^{\ *} e^{-i(2\psi_k+\delta\phi_k)}}{2i\left(1-e^{i(2\mu_k+\delta\mu_k)}\right)} \int_s^{s+C} d\xi Y_k^{\ T} \mathbf{H}_1 Y_k e^{i(2\psi_k+\delta\phi_k)} +$$

$$\varepsilon \sum_{j\neq k} \left(\begin{array}{l} -\dfrac{Y_j e^{i(\psi_j-\psi_k-\delta\phi_k)}}{2i\left(1-e^{i(\mu_k-\mu_j+\delta\mu_k)}\right)} \displaystyle\int_s^{s+C} d\xi Y_j^{\ *T} \mathbf{H}_1 Y_k e^{i(\psi_k-\psi_j+\delta\phi_k)} + \\ \dfrac{Y_j^{\ *} e^{-i(\psi_j+\psi_k+\delta\phi_k)}}{2i\left(1-e^{i(\mu_j+\mu_k+\delta\mu_k)}\right)} \displaystyle\int_s^{s+C} d\xi Y_j^{\ T} \mathbf{H}_1 Y_k e^{i(\psi_k+\psi_j+\delta\phi_k)} \end{array} \right) + O(\varepsilon^2) \qquad (15)$$

We should note, that while it was easy to keep $\delta\mu_k, \delta\phi_k$ in the final expression (15), it belongs to the next order correction and generally speaking should be dropped.

One should be aware of the resonant case $e^{i(\mu_k-\mu_i)} = 1$, including parametric resonance $e^{2i\mu_k} = 1$, when one should solve self-consistently the set of (14-14). It is well known case well described in weak coupling resonance case or in the case of parametric resonance.

Sample IV: small variation of the gradient. It can come from errors in quadrupoles or from a deviation of the energy from the reference value. In 1D case (reduced) it is simple addition to the Hamiltonian: (including sextupole term!)

$$H_1 = \delta K_1 \frac{z^2}{2};\ z = \{x,y\};$$
$$\pi_l = p/p_o - 1 \tag{16}$$
$$\delta K_{1\ x,y} = \mp\delta\left(\frac{e}{pc}\frac{\partial B_y}{\partial x}\right) = \mp\left(\frac{e}{pc}\delta\frac{\partial B_y}{\partial x}\right) - K_1\pi_l \mp \left(\frac{e}{pc}\frac{\partial^2 B_y}{\partial x^2}D_x\right)\pi_l + o\left(\pi_l^2\right)$$

Plugging our parameterization into the residual Hamiltonian we get:

$$z = w(s)\sqrt{2I}\cos\left(\psi(s)+\varphi\right)$$
$$H_1 = \delta K_1(s)\cdot w^2(s)\cdot I\cdot\cos^2\left(\psi(s)+\varphi\right) \tag{17}$$

The easiest way is to average the Hamiltonian (on the phase of fast betatron oscillation – our change is small! And does not affect them strongly) to have a well-known fact that the beta-function is also a Green function (modulo 4π) of the tune response on the variation of the focusing strength.

$$\langle H_1\rangle = \frac{\langle \delta K_1(s)\cdot w^2(s)\rangle}{2}\cdot I \equiv \frac{\langle \delta K_1(s)\cdot \beta(s)\rangle}{2}\cdot I$$
$$\langle \varphi'\rangle = \frac{\partial\langle H_1\rangle}{\partial I} = \frac{\langle \delta K_1(s)\cdot \beta(s)\rangle}{2}; \tag{18}$$
$$\Delta\varphi = \frac{1}{2}\oint \delta K_1(s)\cdot\beta(s)ds;\ \Delta Q = \frac{\Delta\varphi}{2\pi} = \frac{1}{4\pi}\oint \delta K_1(s)\cdot\beta(s)ds;$$

Direct way will be to put it into the equations of motion and to find just the same, that <I'>=0 and the above result. Finally, putting a weak thin lens as a perturbation gives a classical relation:

$$\delta K_1(s) = \frac{1}{f}\delta\left(s - s_o\right)$$
$$\Delta Q = \frac{\Delta\varphi}{2\pi} = \frac{1}{4\pi}\frac{\beta_o(s)}{f} \tag{19}$$

In general case of change in Hamiltonian of linear motion

$$H = \frac{1}{2} X^T \left(\mathbf{H}_o + \delta \mathbf{H}_1\right) X;\ X \to \{\varphi_k, I_k\} \to \delta H_1\left(\varphi_k, I_k, s\right);$$

$$\Delta\mu_k = \frac{\partial}{\partial I_k} \int_o^C \left\langle \delta H_1\left(\varphi_k, I_k, s\right)\right\rangle_{\varphi_k} ds. \tag{20}$$

$$H_1\left(\varphi_k, I_k, s\right) = \frac{1}{2} A^T \tilde{U}^T \delta \mathbf{H}_1 \tilde{U} A =$$

$$\frac{1}{8}\left\{\sum_{k=1}^{n} \sqrt{2I_k}\left(Y_k e^{i(\psi_k+\varphi_k)} + Y^*{}_k e^{-i(\psi_k+\varphi_k)}\right)\right\}^T \delta \mathbf{H}_1 \left\{\sum_{k=1}^{n} \sqrt{2I_k}\left(Y_k e^{i(\psi_k+\varphi_k)} + Y^*{}_k e^{-i(\psi_k+\varphi_k)}\right)\right\}$$

$$\left\langle H_1\left(\varphi_k, I_k, s\right)\right\rangle_\varphi = \frac{1}{2}\sum_{k=1}^{n} I_k \operatorname{Re}\left(Y^{*T}{}_k \delta \mathbf{H}_1(s) \tilde{Y}_k\right); \frac{d\varphi_k}{ds} = \frac{\partial \langle H_1\rangle}{\partial I_k} = \frac{1}{2}\operatorname{Re}\left(Y^{*T}{}_k\ \delta \mathbf{H}_1(s) \tilde{Y}_k\right); \tag{22}$$

or

$$\frac{d\varphi_k}{ds} = \frac{\partial H_1}{\partial I_k} = \frac{1}{4}\left(Y_k e^{i(\psi_k+\varphi_k)} + Y_k^* e^{-i(\psi_k+\varphi_k)}\right)^T \delta \mathbf{H}_1 \left(Y_k e^{i(\psi_k+\varphi_k)} + Y_k^* e^{-i(\psi_k+\varphi_k)}\right)$$

$$\left\langle \frac{d\varphi_k}{ds}\right\rangle = \left\langle \frac{\partial H_1}{\partial I_k}\right\rangle = \frac{1}{2} Y_k^{*T} \delta \mathbf{H}_1(s) \tilde{Y}_k.$$

with $\operatorname{Im}(Y_k^{*T} \delta \mathbf{H}_1 \tilde{Y}_k) = 0$ since

$$\left(Y_k^{*T} \delta \mathbf{H}_1 \tilde{Y}\right)^* = \left(Y_k^T\ \delta \mathbf{H}_1 \tilde{Y}^*\right) = \left(Y_k^{*T} \delta \mathbf{H}_1 \tilde{Y}\right)^T = \left(Y_k^{*T} \delta \mathbf{H}_1 \tilde{Y}\right)$$

Finally, the tune change is just an integral:

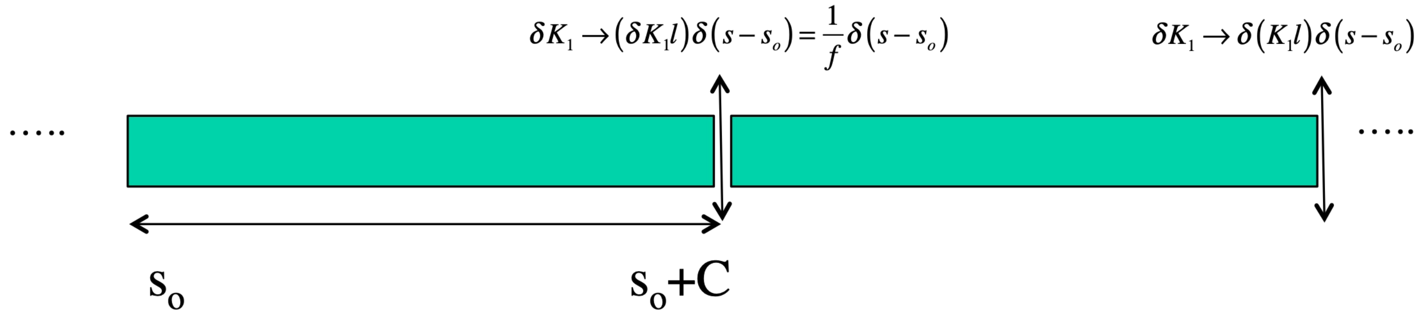
$$\Delta Q_k = \frac{\Delta\mu_k}{2\pi} = \frac{1}{4\pi}\int_0^c Y_k^{*T}(s) \delta \mathbf{H}_1(s) \tilde{Y}_k(s) ds \tag{23}$$

Just to drive it home: 1D case

$$\Delta Q_k = \frac{\Delta\mu_k}{2\pi} = \frac{1}{4\pi}\int_0^c \mathrm{Re}\left(\begin{bmatrix} w & w' + \frac{i}{w}\end{bmatrix}\begin{bmatrix}\delta K_1 & 0 \\ 0 & 0\end{bmatrix}\begin{bmatrix} w \\ w' + \frac{i}{w}\end{bmatrix}\right) ds = \qquad (24)$$

$$\frac{1}{4\pi}\int_0^c w^2\,\delta K_1 ds = \frac{1}{4\pi}\int_0^c \beta(s)\delta K_1(s) ds$$

It worth comparing with a traditional way of doing this: introducing a weak thin lens (lumped focusing):



$$T = I\cos\mu + J\sin\mu = \begin{bmatrix}\cos\mu+\alpha\sin\mu & \beta\sin\mu \\ -\gamma\sin\mu & \cos\mu+\alpha\sin\mu\end{bmatrix}$$

$$T_1 = \begin{bmatrix} 1 & 0 \\ -\delta K_1 l & 1\end{bmatrix}\cdot T = \begin{bmatrix}\cos\mu+\alpha\sin\mu & \beta\sin\mu \\ ..... & \cos\mu+\alpha\sin\mu-\beta K_1 l\sin\mu\end{bmatrix}$$

$$TraceT^{'} = TraceT - \beta K_1 l\sin\mu;$$

$$\cos\mu' = \cos\mu - \frac{\beta\delta K_1 l}{2}\sin\mu; \beta K_1 l << 1 \to \mu' = \mu+\delta\mu$$

$$\cos(\mu+\delta\mu) \cong \cos\mu - \delta\mu\sin\mu \to \delta\mu = \frac{\beta\delta K_1 l}{2} \equiv \frac{\beta}{2f}; \delta Q = \frac{\beta}{4\pi f}.$$

Synchro-beatron coupling.

When we were discussing 3D motion and synchrotron oscillations we arrived to the following Hamiltonian

$$\tilde{\mathcal{H}} = \mathcal{H}_\beta + \mathcal{H}_\delta + \delta\mathcal{H}_\tau$$

$$\mathcal{H}_\beta = \frac{\pi_{x\beta}^{\ 2} + \pi_{y\beta}^{\ 2}}{2} + F\frac{x_\beta^{\ 2}}{2} + N x_\beta y_\beta + G\frac{y_\beta^{\ 2}}{2} + L\left(x_\beta \pi_{\beta y} - y_\beta \pi_{\beta x}\right);$$

$$\mathcal{H}_\delta = \left(\left(\frac{mc}{p_o}\right)^2 + g_x\eta_x + g_y\eta_y\right)\frac{\pi_\tau^{\ 2}}{2} = c_\tau \frac{\pi_\tau^{\ 2}}{2}$$

$$\delta\mathcal{H}_\tau = \frac{e}{p_o c}\sum_n \frac{|E_n|\cos\left(h_n k_o\left(\tilde{\tau} + \tau_{add}\right) + \phi_n\right)}{h_n k_o};$$

$$\tau = \tilde{\tau} + \tau_{add};\ \tau_{add} = \eta_x \pi_{x\beta} - \eta_{px} x_\beta + \eta_y \tilde{\pi}_y - \eta_{py} y_\beta = \eta^T \mathbf{S} X$$

$$\frac{d\pi_\tau}{ds} = -\frac{\partial(\delta H)}{\partial\tilde{\tau}} = \frac{e}{p_o c}\sum_n \frac{|E_n|\sin\left(h_n k_o\left(\tilde{\tau} + \tau_{add}\right) + \phi_n\right)}{h_n k_o};$$

To make it solvable we superficially removed τ_{add}

$$\mathcal{H}_s = \left(\left(\frac{mc}{p_o}\right)^2 + g_x\eta_x + g_y\eta_y\right)\frac{\pi_\tau^{\ 2}}{2} + \frac{e}{p_o c}\frac{\mathbf{E_o}(s)\cos\left(hk_o\tau + \varphi_o\right)}{hk_o}$$

or in linear case

$$\mathcal{H}_{sL} = \left(\left(\frac{mc}{p_o}\right)^2 + g_x\eta_x + g_y\eta_y\right)\frac{\pi_\tau^{\ 2}}{2} - hk_o\frac{\tau^2}{2}\frac{e\mathbf{E_o}(s)}{p_o c}\cos\varphi_o$$

We identify that such "removal" is valid is RF system is located at dispersion-free place, η =0. We further simplified the situation and replaces the RF cavity with the energy kick:

$$\mathbf{E_o}(s) = V_{RF}\delta(s - s_{RF})$$

Than we found that stability $0 < u\eta_c < 2;\ u = -\frac{eV_{rf}hk_0}{p_o c}\cos\varphi_0;\ \ \cos\varphi_0 \pm 1$

$$\mu_s = \sin^{-1}\sqrt{u\eta_c - \frac{(u\eta_c)^2}{4}};\beta_\tau = abs\left(\frac{\eta_c}{\sin\mu_s}\right);\ \alpha_\tau = \frac{|u\eta_c|}{2\sin\mu_s}$$

or in the case of slow synchrotron oscillations

$$\mu_s \cong \sqrt{u\eta_c};\beta_\tau = \sqrt{\left|\frac{\eta_c}{u}\right|};\ \alpha_\tau = 0;$$

In this approximation (weak longitudinal focusing) we can estimate effect on the transverse betatron motion if RF system is installed where $\eta \neq 0$.

$$\delta\mathcal{H} = u\delta(s - s_{rf})\left(\frac{\tau^2 - \tilde{\tau}^2}{2}\right) = u\delta(s - s_{rf})\cdot\left(\frac{\tau^2{}_{add}}{2} - \tilde{\tau}\tau_{add}\right);$$

$$\tau_{add} = \eta^T\mathbf{S}X; X = \operatorname{Re}\left(a_1Y_1e^{i\psi_1} + a_2Y_2e^{i\psi_2}\right);\ \tilde{\tau} == \operatorname{Re} w_\tau a_\tau e^{i\psi_s};\psi_s = \frac{\mu_s}{C}.$$

First, let's notice that term

$$\left\langle \tilde{\tau}\tau_{add} \right\rangle = \left\langle \mathrm{Re}\left(a_1 Y_1 e^{i\psi_1} + a_2 Y_2 e^{i\psi_2}\right) \mathrm{Re}\, w_\tau a_\tau e^{i\psi_s} \right\rangle$$

contains only oscillating terms like $\psi_{1,2} \pm \psi_s$ and averages to zero. While the second term

$$\left\langle \tau^2{}_{add} \right\rangle = \left\langle \left(\frac{a_1 \eta^T \mathbf{S} Y_1 e^{i\psi_1} + a_2 \eta^T \mathbf{S} Y_2 e^{i\psi_2} + c.c.}{2} \right)^2 \right\rangle =$$

$$\frac{|a_1|^2 \left|\eta^T \mathbf{S} Y_1\right|^2 + |a_2|^2 \left|\eta^T \mathbf{S} Y_2\right|^2}{2} = I_1 \left|\eta^T \mathbf{S} Y_1\right|^2 + I_2 \left|\eta^T \mathbf{S} Y_2\right|^2$$

$$\left\langle \delta \mathcal{H} \right\rangle == u\delta\left(s - s_{rf}\right) \cdot \frac{I_1 \left|\eta^T \mathbf{S} Y_1\right|^2 + I_2 \left|\eta^T \mathbf{S} Y_2\right|^2}{2};$$

$$\varphi_k^{/} = \frac{\partial \mathcal{H}}{\partial I_2} = \frac{u}{2} \cdot \left|\eta^T \mathbf{S} Y_k\right|^2 \delta\left(s - s_{rf}\right);$$

$$\Delta\mu_k = \frac{u}{2} \cdot \left|\eta\left(s_{rf}\right)^T \mathbf{S} Y_k\left(s_{rf}\right)\right|^2.$$

Finally in combination with $\mu_s{}^2 = u\eta_c$ we can show that betatron tunes shift is indeed

$$\Delta\mu_k = \frac{u}{2}\cdot\left|\eta\left(s_{rf}\right)^T \mathbf{S}Y_k\left(s_{rf}\right)\right|^2 = \frac{\mu_s{}^2}{2}\frac{\left|\eta\left(s_{rf}\right)^T \mathbf{S}Y_k\left(s_{rf}\right)\right|^2}{\eta_c}$$

proportional to $\mu_s{}^2$ and can be positive or negative depending on the "longitudinal mass" sign, e.g. the sign of η_c. We will see expression $\left|\eta\left(s_{rf}\right)^T \mathbf{S}Y_k\left(s_{rf}\right)\right|^2$ responsible for coupling between betatron and synchrony degrees of motion in many occasions. It is worth mentioning that it has dimension of length, L:

$$\eta^T \mathbf{S}Y_k \rightarrow w_{kx}\eta_{px}, w'_{kx}\eta_x, \eta_x / w_{kx};$$

$$\dim\left(\eta_x / w_{kx}\right)^2 = \dim\left(\frac{\eta_x{}^2}{\beta_{kx}}\right)^2 = \frac{L^2}{L} = L,$$

which just proves that previous equation is indeed has right dimensionally….

Sample V: Going beyond Hamiltonian system – taking dissipation into account

Let's consider that an additional linear term is no longer a Hamiltonian

$$\frac{dX}{ds} = (\mathbf{D}(s) + \varepsilon \mathbf{d}(s)) \cdot X;\ \mathbf{D} = \mathbf{SH}; Trace[\mathbf{D}] = 0; Trace[\mathbf{d}] \neq 0 \tag{25}$$

e.g. the overall motion is no longer symplectic

$$X(s) = \mathbf{R}(s) X_o \to \frac{d\mathbf{R}}{ds} = (\mathbf{D} + \varepsilon \mathbf{d})\mathbf{R} \to \frac{d \det[\mathbf{R}(s)]}{ds} = Trace[\mathbf{d}(s)]$$
$$\det[\mathbf{R}(s)] = \varepsilon \int_o^s Trace[\mathbf{d}(\xi)] d\xi; \tag{26}$$

Such contributions can come form natural dissipative (or anti-dissipative) processes such as radiation reaction (synchrotron radiation damping), ionization cooling or from special accelerator systems, such as electron or stochastic cooling. Here we are not specifying what is the source of the non-Hamiltonian force and only assume that it is linear.

Similarly to regular parameterization, we can assume that motion can be expanded as a set of eigen modes

$$X(s) = \tilde{V}(s)\chi(s) \cdot B = \sum_{k=1}^{2n} \tilde{V}_k(s) e^{\chi_k(s)} b_k; \det \tilde{V}(s) = 1;\ \det\left(\tilde{V}(0)\chi(0)\right) = 1;$$

$$\tilde{V}(s)\chi(s) = \mathbf{R}(s)\tilde{V}(0)\chi(0);\ \ \det \chi(s) = \prod_{k=1}^{2n} e^{\chi_k(s)} = \exp\left(\sum_{k=1}^{2n} \chi_k(s)\right);$$

$$\frac{d}{ds}\det\left(\tilde{V}(s)\chi(s)\right) = \frac{d}{ds}\left(\det \tilde{V}(s) \det \chi(s)\right) = \frac{d}{ds}\det\left(\mathbf{R}(s)\tilde{V}(0)\chi(0)\right) = \frac{d}{ds}\left(\det \mathbf{R}(s)\right) = Tr\mathbf{D};$$

$$\frac{d}{ds}\det \chi(s) = \sum_{k=1}^{2n} \chi_k'(s) = Tr\left((\mathbf{D}(s) + \varepsilon \mathbf{d}(s))\right) = \varepsilon Tr[\mathbf{d}(s)].$$

than (26) became

$$\frac{d}{ds}\sum_{k=1}^{2n} \chi_k(s) = \varepsilon Trace[\mathbf{d}(s)]$$
$$\sum_{k=1}^{2n} \chi_k(s) = \varepsilon \int_o^s Trace[\mathbf{d}(\xi)] d\xi; \tag{27}$$

which is commonly known as the sum of decrements theorem: sum of the decrements (or increments!) of all eigen modes is equal to the integral of the trace of the dissipative matrix. This is to a degree the most trivial and well-known relation for ordinary differential equation.

What is more interesting is to find decrements (increments) of the amplitudes of individual modes. Rewriting already established expansion (1)

$$X(s) = \frac{1}{2}\tilde{\mathbf{U}}(s)\cdot A(s) = \mathrm{Re}\sum_{k=1}^{n} Y_k(s)e^{i(\psi_k(s)+\varphi_k)}a_k(s); \frac{d}{ds}\tilde{\mathbf{U}}(s) = \mathbf{D}(s)\cdot\tilde{\mathbf{U}}(s)$$
$$\mathrm{Re}\sum_{k=1}^{n} Y_k(s)e^{i(\psi_k+\varphi_k)}\frac{da_k}{ds} = \varepsilon\mathbf{d}\cdot\mathrm{Re}\sum_{m=1}^{n} Y_m e^{i(\psi_m+\varphi_m)}a_m; \tag{28}$$

Using symplectic orthogonality of the eigen vectors we get equations of the evolution for individual amplitudes:

$$\frac{da_k}{ds} = \frac{\varepsilon}{2i}\cdot e^{-i(\psi_k+\varphi_k)}\left(\sum_{m=1}^{n} Y_k^{*T}(\mathbf{Sd})Y_m e^{i(\psi_m+\varphi_m)}a_m + Y_k^{*T}(\mathbf{Sd})Y_m^* e^{-i(\psi_m+\varphi_m)}a_m^*\right); \tag{29}$$

Hence, the perturbation can slightly change the eigen modes (as we discussed above in *ala quantum* perturbation) and phase of oscillations – the right side is not necessarily a real number.

But the main effect of-interest is a change of the oscillations amplitudes, which comes from a simple averaging of (29). Since

$$\Delta\psi_k = \psi_k(s+C) - \psi_k(s) = \mu_k$$
$$\Delta(\psi_k \pm \psi_m) = \mu_k \pm \mu_m ;$$

the only non-oscillating term in (14-29) is $Y_k^{*T}(\mathbf{Sd})Y_k$ and averaging yields

$$\left\langle\frac{da_k}{ds}\right\rangle = \frac{\varepsilon}{2i}Y^{*T}{}_k(s)(\mathbf{Sd}(s))Y_k(s)\langle a_k\rangle;$$
$$\langle a_k\rangle(s) = \langle a_k\rangle_o \exp\left[-\frac{\varepsilon}{2i}\int_0^s Y^{*T}{}_k(\xi)\cdot\mathbf{S}\cdot\mathbf{d}(\xi)\cdot Y_k(\xi)d\xi\right]; \tag{30}$$

At no surprise, we arrived to equation nearly identical to (23) with only exception that we did not assume that motion is Hamiltonian. Indeed, if

$$if \quad \varepsilon \mathbf{d}(s) = \mathbf{S}\delta\mathbf{H}_1$$

$$\langle a_k \rangle(s) = \langle a_k \rangle_o \exp\left[\frac{1}{2i}\int_0^s Y^{*T}{}_k\ (\xi)\delta\mathbf{H}_1 Y_m(\xi)d\xi\right];$$

$$\Delta\varphi = \frac{1}{2}\int_0^s Y^{*T}{}_k\ (\xi)\delta\mathbf{H}_1 Y_k(\xi)d\xi$$

It should not be surprising – we are solving similar problem using - more or less - the same method of varying constants.

The most useful form of (30) is calculation of dumping (or anti-damping) coefficients

$$|a_k| \cong |a_{k0}|\, e^{-\frac{\xi_k s}{C}}$$
$$\xi_k = -\frac{\varepsilon}{2}\int_0^C \mathrm{Im}\left(Y^{*T}{}_k\ (s)\left(\mathbf{Sd}(s)\right)Y_k(s)\right)ds; \tag{31}$$

Naturally, the sum of the decrements is determined by the trace of the matrix. What is non-trivial is that we can re-distribute some (if not all) decrements between various modes of oscillations using coupling between them.

As indicated above, we combine the real and imaginary parts:

$$a_k e^{i\varphi} \cong a_{k0}\cdot e^{\frac{s}{C}(i\Delta\mu - \xi_k)}$$
$$i\Delta\mu - \xi_k = \frac{\varepsilon}{2}\int_0^C \left(Y^{*T}{}_k\ (s)\left(\mathbf{Sd}(s)\right)Y_k(s)\right)ds; \tag{32}$$

We will use this expression now and again.

Again 1D case

It gives us know fact that damping of the amplitude of the oscillation is ½ of the dissipative term in $x'' - \xi_o x' + K_1(s)x = 0$:

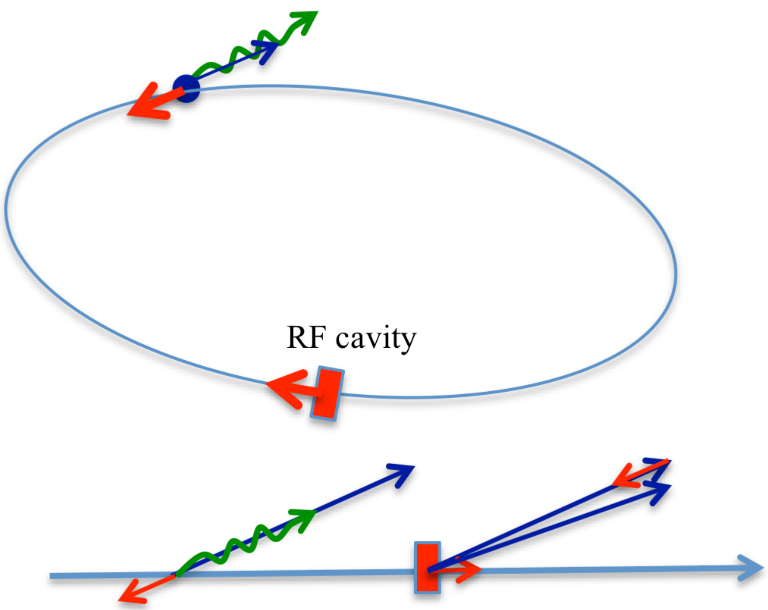
$$\varepsilon\mathbf{d} = \begin{bmatrix} 0 & 0 \\ 0 & -\xi_o \end{bmatrix}$$

$$\xi_x = -\frac{1}{2}\mathrm{Im}\begin{bmatrix} \mathrm{w} & \mathrm{w}' - \frac{i}{\mathrm{w}} \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & -\xi_o \end{bmatrix}\begin{bmatrix} \mathrm{w} \\ \mathrm{w}' + \frac{i}{\mathrm{w}} \end{bmatrix} = \frac{\xi_o}{2}\mathrm{Im}\begin{bmatrix} -\mathrm{w}' + \frac{i}{\mathrm{w}} & w \end{bmatrix}\begin{bmatrix} 0 \\ \mathrm{w}' + \frac{i}{\mathrm{w}} \end{bmatrix} = \frac{\xi_o}{2}.$$

By the way, the real part of the expression gives

$$\varphi_x' = \frac{1}{2}\mathrm{w}_x'\mathrm{w}_x\frac{\xi_o}{2}$$

while being interesting academically, it does not play too much role in the accelerators. We will return to damping when considering synchrotron radiation effects in accelerators.



How radiation cools beam in a storage ring: vertical motion: Particle radiate in the direction of the motion and RF cavity restores only longitudinal part of the momentum.

Sample VI: Going beyond Hamiltonian system – random kicks

Particle in accelerators frequently experience a sudden event, which change their momenta essentially in instance. Naturally, there are no sudden changes of position – it would require not infinite force, but also a finite time to change position.

Examples of such processes include adiation of a photon (so called quantum fluctuation of radiation), scattering on residual gas or on other particles inside the beam. The later is called intra-beam scattering and is one of limiting factors in attaining small beam emittances.

Again, let's just add an additional term in our equation of motion (1):

$$\frac{dX}{ds} = \mathbf{D}(s)\cdot X + DP(s); DP(s) = \sum_a \delta P_a \cdot \delta(s - s_a) \tag{33}$$

which has similar appearance as (2) but has very different nature – it represents a random process, not a regular continuous force. Nevertheless, we can find directly the change of the oscillation amplitude and phase at each random kick:

$$\begin{aligned} &\sum_{k=1}^{n} e^{i\psi_k(s_a)} Y_k(s_a)\delta\left(a_k e^{i\varphi}\right)_{s_a} = \delta P_a \rightarrow \delta\left(a_k e^{i\varphi}\right)_{s_a} = e^{-i\psi_k(s_a)} \frac{1}{2i} Y_k^{T*}(s_a)\mathbf{S}\delta P_a; \\ &a_k(s) e^{i\varphi} = a_{ok} + \sum_{s_a<s} e^{-i\psi_k(s_a)} \frac{1}{2i} Y_k^{T*}(s_a)\mathbf{S}\delta P_a; \end{aligned} \tag{34}$$

Naturally, the exact result depends on realization of the random process. But statistically we can write the average change if the actions:

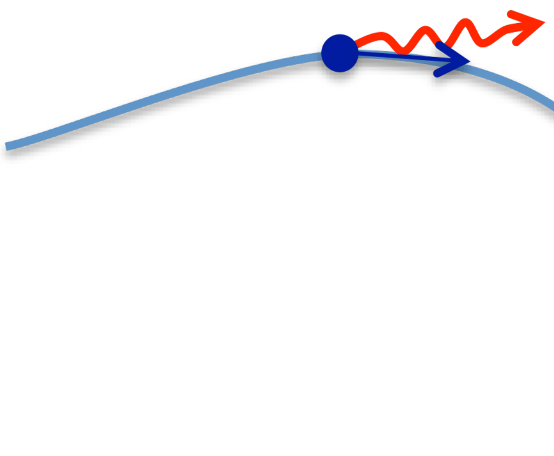
$$J_k = \frac{a_k^2}{2} \to \delta J_k = \frac{(a_k + \delta a_k)^2 - a_k^2}{2} = 2a_k \delta a_k + (\delta a_k)^2$$

Now we need to look on the average picture again:

$$\tilde{a}_k = a_k e^{i\varphi}; \delta\tilde{a}_k = e^{-i\psi_k(s_a)} \frac{1}{2i} Y_k^{T*}(s_a) \mathbf{S} \delta P_a$$

$$\delta|\tilde{a}_k|^2 \to (\tilde{a}_k + \delta\tilde{a}_k)(\tilde{a}_k^* + \delta\tilde{a}_k^*) - \tilde{a}_k\tilde{a}_k^* = |\delta\tilde{a}_k|^2 + 2\,\mathrm{Re}\,\tilde{a}_k^* \delta\tilde{a}_k$$

$$\tilde{a}_k^* \delta\tilde{a}_k = a_k e^{-i\varphi} e^{-i\psi_k(s_a)} \frac{1}{2i} Y_k^{T*}(s_a) \mathbf{S} \delta P_a$$



Since the kicks occur at random locations and the phase of the oscillation is randomized.

Hence,

$$\left\langle \tilde{a}_k^* \delta \tilde{a}_k \right\rangle = \left\langle a_k e^{-i\varphi} e^{-i\psi_k(s_a)} \frac{1}{2i} Y_k^{T*}(s_a) \mathbf{S} \delta P_a \right\rangle = 0 \tag{35}$$

and

$$\left\langle \delta J_k \right\rangle = \left\langle \frac{\delta a_k^2}{2} \right\rangle = \frac{1}{2} \left\langle \left| \delta \tilde{a}_k \right|^2 \right\rangle = \frac{1}{8} \left| Y_k^{T*}(s_a) \mathbf{S} \delta P_a \right|^2 \tag{36}$$

Now we need to introduce probability of the random kick δP at azimuth s to write an statistical average growth of the oscillation amplitude:

$$\left\langle \frac{dJ_k}{ds} \right\rangle = \frac{1}{8} \left\langle \left| Y_k^{T*}(s) \mathbf{S} \delta P \right|^2 \cdot \phi(s, \delta P) \right\rangle = D_k(s) \tag{37}$$

This growth is called diffusion (or random walk). It has interesting characteristic that amplitude of oscillations growth proportionally to the square root of time – e.g. the action grows linearly.

Again, we will discuss values for specific processes later. What is interesting now is to combine damping and diffusion. To do this we need to tone that without diffusion

$$\frac{dJ_k}{ds} = \frac{1}{2} \frac{da_k^2}{ds} = a_k \frac{da_k}{ds} = -2\xi_k J_k \tag{38}$$

and adding diffusion we get to

$$\frac{d\left\langle J_k \right\rangle}{ds} = -2\xi_k(s) \left\langle J_k \right\rangle + D_k(s);$$

$$\left\langle J_k(s) \right\rangle = J_{ok} e^{-2\int_o^s \xi_k(z)dz} + \int_0^s e^{-2\int_z^s \xi_k(u)du} D_k(z)dz; \tag{39}$$

In storage rings it is frequently that the processes are very slow and you can average the damping and the diffusion over the circumference

$$\langle D_k \rangle = \langle D_k(s) \rangle_C ; \langle \xi_k \rangle = \langle \xi_k(s) \rangle_C$$

$$\langle J_k(s) \rangle = e^{-2\langle \xi_k \rangle s} \left(J_{ok} + \langle D_k \rangle \int_0^s e^{2\langle \xi_k \rangle z} dz \right) = J_{ok} e^{-2\langle \xi_k \rangle s} + \frac{\langle D_k \rangle}{2\langle \xi_k \rangle} \left(1 - e^{-2\langle \xi_k \rangle s} \right); \tag{40}$$

and stationary action at large s (many turns) being

$$\langle J_k(s) \rangle \to \frac{\langle D_k \rangle}{2\langle \xi_k \rangle} \tag{41}$$

This formula is very useful for both calculating and estimating the beam emittances in presence of diffusion and dimpling.

Note, that an anti-damping $\langle \xi_k \rangle < 0$ will cause exponential growth of the oscillating amplitude and is almost as bad and instability of periodic Hamiltonian motion. Hence, this is important for accelerators where damping plays significant role in the beam dynamics, e.g. damping (anti-damping) time is much smaller or compatible with the beam lifetime in the accelerator.

Remarkably, I know about one storage ring (VEPP-4 in Novosibirsk), which was initially built for proton-antiproton collisions but then had been turned into electron-position collider. Since protons do not radiate any significant part of radiation, synchrotron radiation decrements were not important and neglected during design. When the switch to electrons and positrons, which have damping times of millisecond, it turned out that synchrotron radiation will damp one degree of freedom and anti-damp the other… It was required to add an additional radiation device into the lattice (a strong wiggler) to solve this important problem.

Let's repeat: We discuss several ways how the parameterization of the motion in linear Hamiltonian system can be used to solve variety of standard problems arising in accelerator physics. Some of them were exact solutions (like orbit distortions or dispersion function), but some of them were clearly perturbative and relied on averaging over fast oscillations. The later, while intuitively understandable, requires some more discussions – and this is what we start doing today. Let's consider an additional (not necessarily a simple, constant or linear, but weak) term in our equations of motion

$$\frac{dX}{ds} = \mathbf{D}(s) \cdot X + \varepsilon F(X,s); \tag{42}$$

Using our already well-established parameterization, we can always write:

$$X = \frac{1}{2}\tilde{\mathbf{U}}(s)A(s) = \mathrm{Re}\sum_{k=1}^{n} a_k(s)Y_k(s)e^{i(\psi_k(s)+\varphi_k(s))}; A(s) = \begin{bmatrix} \dots \\ a_k e^{i\varphi_k} \\ a_k e^{-i\varphi_k} \\ \\ \end{bmatrix};$$

$$\tilde{\mathbf{U}}(s)\frac{d}{ds}A = \varepsilon F\left(\tilde{\mathbf{U}}(s)A(s),s\right) \Leftrightarrow \frac{d}{ds}a_k e^{i\varphi_k} = \varepsilon \frac{e^{-i\psi_k(s)}}{i} Y_k^{*T}(s)\mathbf{S}F\left(\tilde{\mathbf{U}}(s)A(s),s\right) \tag{43}$$

If one likes real form of the equations, it can be written as

$$\begin{aligned} &\frac{d}{ds}a_k e^{i\varphi_k} = \left(a_k' + i\varphi_k' a_k\right)e^{i\varphi_k} = \varepsilon \frac{e^{-i\psi_k}}{i} Y_k^{*T}\mathbf{S}F; \\ &\frac{d}{ds}a_k e^{-i\varphi_k} = \left(a_k' - i\varphi_k' a_k\right)e^{-i\varphi_k} = -\varepsilon \frac{e^{i\psi_k}}{i} Y_k^{T}\mathbf{S}F; \\ &\frac{da_k}{ds} = \varepsilon\,\mathrm{Im}\left[e^{i(\psi_k+\varphi_k)}Y_k^T\mathbf{S}F\right]; a_k \cdot \frac{d\varphi_k}{ds} = \varepsilon\,\mathrm{Re}\left[e^{i(\psi_k+\varphi_k)}Y_k^T\mathbf{S}F\right]; \end{aligned} \tag{44}$$

In analytical mechanics, these equations for constant of motion in linear system are called "reduced" or "slow" equations when ε is so small that it significantly affect the motion only when right side of equation has constant terms, e.g. either the phase or amplitude of oscillations can grow in time, not just simply oscillate.

As an example, let's consider a 1D motion with write side having a power of x:

$$F=\begin{bmatrix} 0 \\ f(s)x^m \end{bmatrix}; x = a\mathrm{w}\cos(\psi+\varphi)$$

$$\frac{da}{ds}=\varepsilon f a^m \mathrm{w}^m \sin(\psi+\varphi)\cdot\cos^m(\psi+\varphi); \frac{d\varphi}{ds}=\varepsilon f \mathrm{w}^m a^{m-1}\cos^{m+1}(\psi+\varphi); \tag{45}$$

The equations (45) are non-linear and do not have explicit analytical solution in general case (we know that it can be parameterized for n=1). Let's now consider a periodical system:

$$\psi(s)+\mu \to \psi(s)=\chi(s)+\frac{\mu s}{C};$$

$$\chi(s+C)=\chi(s)\mathrm{w}(s+C)=\mathrm{w}(s); f(s+C)=f(s); \psi(s+C);$$

$$\frac{da}{ds}=\varepsilon f a^m \mathrm{w}^m \sin\left(\frac{\mu s}{C}+\chi+\varphi\right)\cdot\cos^m\left(\chi(s)+\frac{\mu s}{C}+\varphi\right); \frac{d\varphi}{ds}=\varepsilon f \mathrm{w}^m a^{m-1}\cos^{m+1}\left(\frac{\mu s}{C}+\chi+\varphi\right);$$

Considering that slow variable are nearly constant, we have on the right side terms oscillating with phase advancing as $(k\mu \pm 2\pi j)\frac{s}{C} = 2\pi\frac{s}{C}(kQ \pm j); -m \le k \le m; j - \text{integer}$. Only when $kQ \pm j = 0$ (or close to zero – see next) we have a stationary growth. Otherwise, the oscillating terms will average.

One can intuitively expand the variation of constants a power of the infinitesimal ε

$$a = a_o + \sum_{k=1} a_k \varepsilon^k; \varphi = \varphi_o + \sum_{k=1} \varphi_k \varepsilon^k$$

$$\frac{da}{ds} = \varepsilon a^m \mathrm{w}^m \sin(\psi + \varphi) \cdot \cos^m(\psi + \varphi); \frac{d\varphi}{ds} = \varepsilon \mathrm{w}^m a^{m-1} \cos^{m+1}(\psi + \varphi); \qquad (46)$$

and explore it further. But this will bring us to a method developed by Bogolyubov and Metropolsky (N. Bogolubov N. (1961). *Asymptotic Methods in the Theory of Non-Linear Oscillations*. Paris: Gordon & Breach. ISBN 978-0-677-20050-7.) in analytical mechanics. You can find a straightforward, but rather long derivation in the book – here we will only discuss the results.

Let's start from an equation of motion with a small (infinitesimally) perturbation for a linear system with deduced equation of

$$\frac{dA}{ds} = \varepsilon F(X,s); \tag{47}$$

than the first order perturbation can be written as

$$\begin{gathered} A = \xi(s) + \varepsilon \tilde{F}(\xi,s); \frac{d}{ds}\xi(s) = \langle F(\xi,s) \rangle; \\ \langle F(A,s) \rangle = \frac{1}{S}\int_s^{s+S} \langle F(A = const, s) \rangle ds; \;\; \tilde{F} = \int \left(F - \langle F \rangle \right) ds; \end{gathered} \tag{48}$$

What is quite remarkable, that they also derived a second order perturbation:

$$\begin{gathered} A = \xi(s) + \varepsilon \tilde{F}(\xi,s) + \varepsilon^2 \widetilde{\left\{ \left(\tilde{F}\frac{\partial}{\partial \xi} \right) F \right\}} - \varepsilon^2 \frac{\partial \tilde{\tilde{F}}}{\partial \xi} \langle F(\xi,s) \rangle; \\ \frac{d}{ds}\xi(s) = \varepsilon \left\langle F(\xi,s) + \varepsilon \tilde{F} \right\rangle \approx \varepsilon \left\langle \left(1 + \varepsilon \left(\tilde{F}\frac{\partial}{\partial \xi} \right) \right) F(\xi,s) \right\rangle. \end{gathered} \tag{49}$$

These equations were used and still used to derive several analytical expressions for nonlinear resonances and tune's dependences on the oscillation amplitudes (actions).

Let's consider a case we already studies during last class: **small variation of the quadrupole gradient**. It can come from errors in quadrupoles or from a deviation of the energy from the reference value. In 1D case (reduced) it is simple addition to the Hamiltonian: (including sextupole term!)

$$\delta H = \delta K_1(s)\frac{x^2}{2} = I \cdot \delta K_1(s)\beta(s)\cos^2\left(\psi(s)+\varphi\right)$$

$$\frac{d\varphi}{ds} = \frac{\partial \delta H}{\partial I} = \delta K_1 \beta \cos^2(\psi+\varphi) = \delta K_1 \beta \frac{1+\cos 2(\psi+\varphi)}{2}; \qquad (50)$$

$$\frac{dI}{ds} = -\frac{\partial \delta H}{\partial \varphi} = I \cdot \delta K_1(s)\beta(s)\sin 2(\psi+\varphi);$$

Using first order approximation we get:

$$\frac{d\langle\varphi\rangle}{ds} = \frac{1}{S}\int_s^{s+S} \delta K_1(s)\beta(s)\frac{1+\cos 2(\psi+\varphi)}{2} =$$

$$\frac{\langle \delta K_1(s)\beta(s)\rangle}{2} + \frac{\langle \delta K_1(s)\beta(s)\cos 2(\psi+\varphi_o)\rangle}{2} \qquad (51)$$

$$\frac{d\langle I\rangle}{ds} = I_o \cdot \langle \delta K_1(s)\beta(s)\sin 2(\psi+\varphi_o)\rangle;$$

We already got the first term average term

$$\frac{\langle \delta K_1(s)\beta(s)\rangle}{2} = \frac{1}{2C}\int_0^C \delta K_1(s)\beta(s) \tag{52}$$

while the amplitude does not have obvious non-oscillating term. Oscillating terms are also of some interest - let's explore them:

$$\begin{aligned}
\frac{d\tilde{\varphi}}{ds} &= \mathrm{Re}\frac{\delta K_1(s)\beta(s)}{2} e^{i\chi(s)} e^{i\frac{4\pi Q}{C}s} e^{2i\varphi_o} = \mathrm{Re}\sum_{k=-\infty}^{\infty} c_k e^{2\pi i\frac{2Q+k}{C}s} e^{2i\varphi_o} \\
\frac{d\tilde{I}}{ds} &= I_o \cdot \mathrm{Im}\frac{\delta K_1(s)\beta(s)}{2} e^{i\chi(s)} e^{i\frac{4\pi Q}{C}s} e^{2i\varphi_o} = I_o \cdot \mathrm{Im}\sum_{k=-\infty}^{\infty} c_k e^{2\pi i\frac{2Q+k}{C}s} e^{2i\varphi_o} \\
&\frac{\delta K_1(s)\beta(s)}{2} e^{i\chi(s)} = \sum_{k=-\infty}^{\infty} c_k e^{2\pi i\frac{k}{C}s}.
\end{aligned} \tag{53}$$

where we simply expanded periodic complex function into a Fourier series.

(53) is easy to integrate

$$\begin{aligned}
\tilde{\varphi} &= -\frac{C}{2\pi}\mathrm{Im}\sum_{k=-\infty}^{\infty}\frac{c_k}{2Q+k} e^{2\pi i\frac{2Q+k}{C}s} e^{2i\varphi_o} = \mathrm{Re}\,\phi(s)e^{2i(\psi+\varphi_o)};\;\; \phi(s+C)=\phi(s) \\
\tilde{I} &= -I_o \cdot \frac{C}{2\pi}\mathrm{Re}\sum_{k=-\infty}^{\infty}\frac{c_k}{2Q+k} e^{2\pi i\frac{2Q+k}{C}s} e^{2i\varphi_o} = I_o \cdot \mathrm{Re}\,\upsilon(s)e^{2i(\psi+\varphi_o)};\;\; \upsilon(s+C)=\upsilon(s)
\end{aligned} \tag{54}$$

Unless the accelerator is "sitting" at a parametric resonance $2Q = \pm k$, there oscillating term simply oscillating with double betatron frequency. Otherwise, at the parametric resonance $2Q = \pm k$ both the amplitude and the phase can grow – e.g. it is an instability we must stay away from. Parametric resonance is one you are using to increase amplitude of oscillation of a swing by periodically changing the "oscillation frequency" with your legs and body.

One more example: Octupole term in horizontal motion

Let's consider a 4th order term (non-linear) in our 1D Hamiltonian:

$$H = \frac{p_x^2}{2} + K_1(s)\frac{x^2}{2} + o(s)\frac{x^4}{4} \tag{55}$$

with transformation to action angle variables

$$x = \sqrt{2I} \cdot \mathrm{w}(s) \cdot \cos(\psi(s) + \varphi); \;\; p_x = x'; \qquad \{x, p_x\} \to \{\varphi, I\}. \tag{56}$$

which removes the linear part of the Hamiltonian leaving only nonlinear term, which we need to express using action-angle variables:

$$H_1(\varphi, I, s) = o(s)\mathrm{w}^4(s) I^2 \cos^4(\psi(s) + \varphi);$$
$$2\cos^2\theta = 1 + \cos 2\theta; \;\; 2(1+\cos 2\theta)^2 = 3 + 4\cos 2\theta + \cos 4\theta; \tag{57}$$
$$H_1 = o\mathrm{w}^4 \frac{I^2}{2}\left(\frac{3}{4} + \cos(2\psi + 2\varphi) + \frac{\cos(4\psi + 4\varphi)}{4}\right);$$

with equations of motion being

$$\varphi' = \frac{\partial H_1}{\partial I} = I o(s)\mathrm{w}^4(s)\left(\frac{3}{4} + \cos(2\psi + 2\varphi) + \frac{\cos(4\psi + 4\varphi)}{4}\right);$$
$$I' = -\frac{\partial H_1}{\partial \varphi'} = I^2 o(s)\mathrm{w}^4(s)\left(\sin(2\psi + 2\varphi) + \frac{\sin(4\psi + 4\varphi)}{2}\right); \tag{58}$$

Staying away from the parametric (second order) and 4th order resonances $4Q \neq \pm k$ we can first average (56) noting that oscillating term yelled zero in this approximation

$$\langle I' \rangle = 0 \to \bar{I} = const; \; \bar{\varphi}' = \frac{3}{4}\bar{I}\left\langle o(s)\mathrm{w}^4(s)\right\rangle \equiv \frac{3}{4}\bar{I}\left\langle o(s)\beta^2(s)\right\rangle;$$
$$\bar{\varphi} = \varphi_o + \frac{\partial\mu}{\partial I}\bar{I}\frac{s}{C} \equiv \varphi_o + \frac{\partial\mu}{\partial I}\frac{a^2}{2}; \; \frac{\partial\mu}{\partial I} = \frac{3}{4}\int_C o(s)\beta^3(s)ds; \tag{57}$$

e.g. while amplitude of oscillations remains constant, phase advance per turn (e.g. oscillation frequency) start depends on the square of amplitude of oscillations

$$Q = Q_o + \frac{1}{2\pi}\frac{\partial\mu}{\partial I}\frac{a^2}{2}. \tag{58}$$

Hence, octupole term in the Hamiltonian makes oscillations inharmonic. We always add oscillating terms

$$I = \bar{I} + \tilde{I}; \; \varphi = \bar{\varphi} + \tilde{\varphi};$$
$$\tilde{I}(s) = \bar{I}^2\int^s o(\xi)\mathrm{w}^4(\xi)\left(\sin\left(2\psi(\xi)+2\bar{\varphi}\right) + \frac{\sin\left(4\psi(\xi)+4\bar{\varphi}\right)}{2}\right)d\xi; \tag{59}$$
$$\tilde{\varphi} = \bar{I}\int^s o(\xi)\mathrm{w}^4(\xi)\left(\cos\left(2\psi(\xi)+2\bar{\varphi}\right) + \frac{\cos\left(4\psi(\xi)+4\bar{\varphi}\right)}{4}\right)d\xi;$$

which can be evaluated and expressed in terms oscillating with double and quadruple betatron frequency. Since we are considering quadrupole term being a perturbation, away from the resonance these oscillations are small. Naturally, one can go one extra step and write second order perturbation terms, which will be proportional to second order of quadrupole strength and higher order of action and harmonics of betatron frequencies. While it is possible, expression become rather long and are not very “educational.

It is not all… But already, not too shabby for a single parameterization

$$X(s)=\frac{1}{2}\tilde{\mathbf{U}}(s)\cdot A(s)=\mathrm{Re}\sum_{k=1}^{n}Y_k(s)e^{i(\psi_k(s)+\varphi_k)}a_k(s);$$

$$\frac{d}{ds}\tilde{\mathbf{U}}(s)=\mathbf{D}(s)\cdot\tilde{\mathbf{U}}(s);\tilde{\mathbf{U}}=\left[...Y_k e^{i\psi_k},Y_k^* e^{-i\psi_k},...\right];k=1,..,n$$

$$\tilde{\mathbf{U}}^T\mathbf{S}\tilde{\mathbf{U}}=2i\mathbf{S}.$$

Next step – solving nonlinear problems and finding solutions for particle distributions

What we learned today

- Using parameterization and – to a less degree action-angle variables – we wrote solutions for most common accelerator problems in most general form
- We avoided necessity to write a specific set of equations, finding way of solving it and than, finally, expressing it though lattice parameters
- We also confirmed that slow synchrotron oscillations indeed barely (e.g. in second order) change frequencies of betatron oscillations
- We did go beyond linearized motion to explore a classical perturbation theory, which is valid up to second order of approximation. More consistent approach of maps and Lie algebras will be topic later in the course
- We went beyond traditional Hamiltonian system and found damping decrements for each oscillation mode caused by weak dissipative force. We derived analytically so called "sum of the decrements" theorem
- We also calculated diffusion coefficients for each eigen mode (oscillator) caused by random kicks