

Homework 12. Due October 21

Problem 1. 10 points. Matrix gymnastics.

For a one-dimensional motion consider parameterization eigen vectors and their propagation along s :

$$Y(s) = \begin{bmatrix} w(s) \\ w'(s) + \frac{i}{w(s)} \end{bmatrix}; Y^*(s) = \begin{bmatrix} w(s) \\ w'(s) - \frac{i}{w(s)} \end{bmatrix};$$

$$\beta(s) \equiv w^2(s); \quad \alpha(s) \equiv -\frac{\beta'(s)}{2} \equiv -w(s)w'(s); \quad \frac{d\psi}{ds} = \frac{1}{w^2(s)} \equiv \frac{1}{\beta(s)};$$

$$Y(s_2)e^{i\Delta\psi} = M(s_1|s_2)Y(s_1) \Leftrightarrow Y^*(s_2)e^{-i\Delta\psi} = M(s_1|s_2)Y^*(s_1); \Delta\psi = \psi(s_2) - \psi(s_1).$$

e.g.

$$\begin{aligned} [Y(s_2)e^{i\Delta\psi}, Y^*(s_2)e^{-i\Delta\psi}] &= M(s_1|s_2)[Y(s_1), Y^*(s_1)]; \\ \tilde{W}(s) &= [Y(s)e^{i\psi(s)}, Y^*(s)e^{-i\psi(s)}]; \tilde{W}(s_2) = M(s_1|s_2)\tilde{W}(s_1); \end{aligned}$$

Show that then matrix can be

$$M(s_1|s_2) = \begin{bmatrix} \frac{w_2}{w_1} \cos \Delta\psi - w_1' w_2 \sin \Delta\psi & w_1 w_2 \sin \Delta\psi \\ -\sin \Delta\psi \left(\frac{1}{w_1 w_2} + w_1' w_2' \right) - \cos \Delta\psi \left(\frac{w_1'}{w_2} - \frac{w_2'}{w_1} \right) & \frac{w_1}{w_2} \cos \Delta\psi + w_2' w_1 \sin \Delta\psi \end{bmatrix}$$

or in traditional terms:

$$M(s_1|s_2) = \begin{bmatrix} \sqrt{\frac{\beta_2}{\beta_1}} (\cos \Delta\psi + \alpha_1 \sin \Delta\psi) & \sqrt{\beta_1 \beta_2} \sin \Delta\psi \\ -\frac{\sin \Delta\psi (1 + \alpha_1 \alpha_2) + \cos \Delta\psi (\alpha_2 - \alpha_1)}{\sqrt{\beta_1 \beta_2}} & \sqrt{\frac{\beta_1}{\beta_2}} (\cos \Delta\psi - \alpha_2 \sin \Delta\psi) \end{bmatrix}$$

Hint: if you are using $\tilde{W}(s)$, then use that $\tilde{W}^T S W = -2iS$. If you hate complex variables, then construct $\tilde{U}(s) = [\tilde{R}(s), \tilde{Q}(s)]$; $\tilde{R}(s) = \text{Re} \tilde{Y}(s)$; $\tilde{Q}(s) = \text{Im} \tilde{Y}(s)$; show that it is symplectic, and define how it transformed with s .