PHY 564
Advanced Accelerator Physics
Lecture 18

Beam emittance(s)

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Emittance of the beam. For quite a while we were saying words such as emittance or phase space volume occupied by a beam without a rigorous definition what it is? While intuitively we can understand this concept as well as get grip of Liouville theorem and Poincaré invariants. To no surprise, there is a number of definitions used for the beam emittances: RMS, core-, 95%, etc… Having something very rigorous would help you to navigate the topic without being lost…

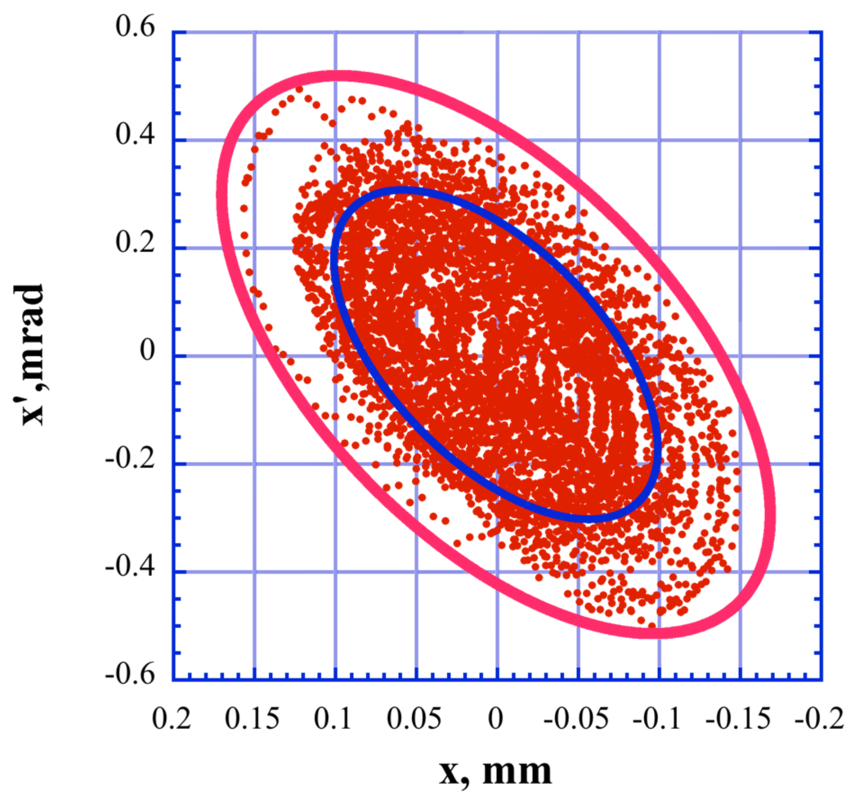
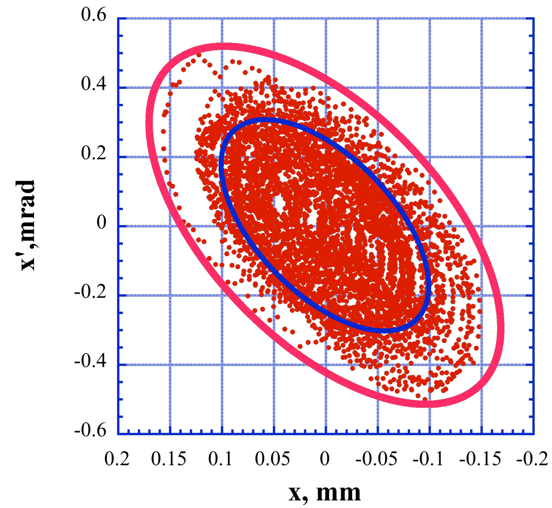


Fig. 1. 1D phase space distribution of particles with RMS-emittance ellipse and one containing all particles found in the plot.



Let's start from uncoupled 1D case. You can find RMS emittance definition in any text-book expressed via determinant of Σ matrix

$$\varepsilon_i^2 = \det\Sigma = \langle q_i^2\rangle\langle p_i^2\rangle - \langle q_i p_i\rangle^2, i = 1,2,...,n,$$

$$\Sigma = \langle X \cdot X^T \rangle = \left\langle \begin{bmatrix} x^2 & xp_x \\ xp_x & p_x^2 \end{bmatrix} \right\rangle; X^f = \mathbf{M}X^i \rightarrow \tag{1}$$

$$\Sigma^f = \mathbf{M}\Sigma^i\mathbf{M}^T \rightarrow \det\Sigma^f = (\det\mathbf{M})^2 \det\Sigma^i; \ \varepsilon^{2f} = \varepsilon^{2i} = inv,$$

which is invariant of 1D motion liner Hamiltonian motion - we used det$\mathbf{M}$=1.

In order to get to coupled case (e.g. a multi-dimensional linear Hamiltonian motion), let's start from equilibrium distribution in a storage ring. First, we should notice that in a stable ring without damping and diffusion, actions of eigen modes are preserved while phases, in general, are not. For example, nonlinearity of magnetic fields and RF curvature generate tune spread depending on 3 actions. It will spread phases randomly for all three oscillators. In this case one can assume that distribution functions depends only on action:

$$f = f\left(I_1, I_2, I_3\right) = f\left(\frac{\left|Y_1^T SX\right|^2}{2}, \frac{\left|Y_2^T SX\right|^2}{2}, \frac{\left|Y_3^T SX\right|^2}{2}\right). \tag{2}$$

It is even simpler in the case of stationary distribution established by synchrotron radiation damping and quantum fluctuations:

$$f\left(I,\varphi\right) = \prod_{k=1}^{3} \frac{1}{2\pi\varepsilon_k} \exp\left[-\frac{I_k}{\varepsilon_k}\right] = \left(\prod_{k=1}^{3} \frac{1}{2\pi\varepsilon_k}\right).\exp\left[-\sum_{k=1}^{3} \frac{I_k}{\varepsilon_k}\right]; \tag{3}$$

with natural substitutions

$$X = \frac{1}{2}\sum_{k=1}^{3}\left(\tilde{a}_k Y_k + \tilde{a}_k^* Y_k^*\right) \rightarrow i\tilde{a}_k = Y_k^{T*}\mathbf{S}X;\ \left| a_k\right|^2 = \left|Y_k^T \mathbf{S}X\right|^2;$$

$$f\left(X\right) = \prod_{k=1}^{3} \frac{1}{2\pi\varepsilon_k} \cdot \exp\left[-\sum_{k=1}^{3} \frac{\left|Y_k^T \mathbf{S}X\right|^2}{2\varepsilon_k}\right]; \tag{4}$$

The term in the exponent

$$q\left(X\right) = \sum_{k=1}^{3} \frac{\left|Y_k^T \mathbf{S}X\right|^2}{2\varepsilon_k} = \sum_{i,j=1}^{2n} q_{ij} x_i x_j;\ q_{ij} = q_{ji} \tag{5}$$

is a positively defined quadratic form of X components.

Now we should try to find the matrix of quadratic form and we will start from obvious complex form of (4)

$$\sum_{k=1}^{3}\frac{\left|a_k\right|^2}{\varepsilon_k}=\frac{1}{2}A^{*T}\Xi^{-1}A=\frac{1}{2}A^T\mathrm{E}^{-1}A^*;$$

$$A^T=\left(..,a_k,a_k^*,...\right);\Xi=\begin{bmatrix} ... & [0] & [0] \\ [0] & \begin{bmatrix} \varepsilon_k & 0 \\ 0 & \varepsilon_k \end{bmatrix} & [0] \\ [0] & [0] & ... \end{bmatrix}=\begin{bmatrix} ... & 0 & 0 \\ 0 & \varepsilon_k[I] & 0 \\ 0 & 0 & ... \end{bmatrix};[I]=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}; \quad (6)$$

$$\mathbf{S}=\begin{bmatrix} ... & 0 & 0 \\ 0 & [\sigma] & 0 \\ 0 & 0 & ... \end{bmatrix};[\sigma]=\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix};\Rightarrow \mathbf{S\Xi}=\mathbf{\Xi S}$$

with detailed structure

$$X=\frac{1}{2}\mathbf{U}\tilde{A}\Rightarrow\tilde{A}=2\mathbf{U}^{-1}X;\tilde{A}^*=2\mathbf{U}^{*-1}X;\ \mathbf{U}^T\mathbf{SU}=-2i\mathbf{S};$$

$$2\mathbf{U}^{-1}=-i\mathbf{SU}^T\mathbf{S};2\mathbf{U}^{*-1}=i\mathbf{SU}^{*T}\mathbf{S};$$

$$q(X)=\frac{1}{2}A^{*T}\Xi^{-1}A=2X^T\left[\left(\mathbf{U}^{*-1}\right)^T\Xi^{-1}\mathbf{U}^{-1}\right]X=\frac{1}{2}X^T\Omega X \quad (7)$$

$$\Omega=4\left[\left(\mathbf{U}^{*-1}\right)^T\Xi^{-1}\mathbf{U}^{-1}\right]=\mathbf{SU}^*\mathbf{S}\Xi^{-1}\mathbf{SU}^T\mathbf{S}=-\mathbf{SU}^*\Xi^{-1}\mathbf{US}=\mathbf{V}^*\Xi^{-1}\mathbf{V}^T;$$

$$\mathbf{V}=\mathbf{SU}=\left[..\mathbf{S}Y_k,\mathbf{S}Y^*{}_k...\right]$$

While it is OK to have this in complex form, it would be very nice to express it in real notations. Using the fact that X is real:

$$Y_k = R_k + iQ_k;$$

$$\left|X^T\mathbf{S}Y_k\right|^2 = \left|X^T\mathbf{S}\left(R_k + iQ_k\right)\right|^2 = \left|X^T\mathbf{S}R_k\right|^2 + \left|X^T\mathbf{S}Q_k\right|^2; \mathbf{O} = \left[...R_k, Q_k..\right]$$

$$X = \sum_{k=1}^{n}\left|a_k\right|\left(R_k\cos\psi - Q_k\sin\psi\right) = \mathbf{O}\tilde{B};\ \tilde{B}^T = \left[...\left|a_k\right|\cos\psi, -\left|a_k\right|\sin\psi\right];$$

$$\tilde{B} = \mathbf{O}^{-1}X;\ ;\ \tilde{B}^T\Xi^{-1}\tilde{B} = \sum_{k=1}^{n}\frac{\left|a_k\right|^2}{\varepsilon_k}\left(\cos^2\psi + \sin^2\psi\right) = \sum_{k=1}^{n}\frac{\left|a_k\right|^2}{\varepsilon_k}; \quad , \tag{8}$$

$$\sum_{k=1}^{n}\frac{\left|a_k\right|^2}{\varepsilon_k} = X^T\cdot\left(\mathbf{O}^T\right)^{-1}\Xi^{-1}\mathbf{O}^{-1}\cdot X$$

$$\mathbf{O}^T\mathbf{S}\mathbf{O} = \mathbf{S} \rightarrow \mathbf{O}^{-1} = -\mathbf{S}\mathbf{O}^T\mathbf{S};\ \left(\mathbf{O}^T\right)^{-1} = -\mathbf{S}\mathbf{O}\mathbf{S}$$

we get desirable symmetric form of the stationary distribution:

$$f(X) = \prod_{k=1}^{3}\frac{1}{2\pi\varepsilon_k}\exp\left[-\frac{X^T\cdot\left(\mathbf{O}^T\right)^{-1}\Xi^{-1}\mathbf{O}^{-1}\cdot X}{2}\right].,\tag{9}$$

Look at 1D case for simplicity:

$$\Xi^{-1} = \varepsilon^{-1}\mathbf{I}; \quad \rightarrow \mathbf{O}^{-1T}\Xi^{-1}\mathbf{O}^{-1} = \varepsilon^{-1}\mathbf{O}^{-1T}\mathbf{O}^{-1} = -\varepsilon^{-1}\mathbf{S}\mathbf{O}\mathbf{O}^{T}\mathbf{S}$$

$$\mathbf{O} = \begin{bmatrix} \mathrm{w} & 0 \\ \mathrm{w}' & 1/\mathrm{w} \end{bmatrix}; -\mathbf{S}\mathbf{O} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} \mathrm{w} & 0 \\ \mathrm{w}' & 1/\mathrm{w} \end{bmatrix} = \begin{bmatrix} -\mathrm{w}' & -1/\mathrm{w} \\ \mathrm{w} & 0 \end{bmatrix};$$

$$\mathbf{O}^{T}\mathbf{S} = \begin{bmatrix} \mathrm{w} & \mathrm{w}' \\ 0 & 1/\mathrm{w} \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -\mathrm{w}' & \mathrm{w} \\ -1/\mathrm{w} & 0 \end{bmatrix};$$

$$-\mathbf{S}\mathbf{O}\mathbf{O}^{T}\mathbf{S} = \begin{bmatrix} -\mathrm{w}' & -1/\mathrm{w} \\ \mathrm{w} & 0 \end{bmatrix}\begin{bmatrix} -\mathrm{w}' & \mathrm{w} \\ -1/\mathrm{w} & 0 \end{bmatrix} = \begin{bmatrix} \mathrm{w}'^2 + 1/\mathrm{w}^2 & -\mathrm{w}'\mathrm{w} \\ -\mathrm{w}'\mathrm{w} & \mathrm{w}^2 \end{bmatrix}, \qquad (10)$$

$$\rightarrow x^2 \frac{1+\mathrm{w}'^2\mathrm{w}^2}{\mathrm{w}^2} - 2\mathrm{w}'\mathrm{w}\cdot xx' + x'^2\mathrm{w}^2; \alpha = -\mathrm{w}'\mathrm{w}; \beta = \frac{1+\mathrm{w}'^2\mathrm{w}^2}{\mathrm{w}^2}$$

$$f(x,x') = \frac{1}{2\pi\varepsilon}\exp\left[-\frac{x^2 + (\alpha x + \beta x')^2}{2\beta\varepsilon}\right];$$

After a simple manipulations – which we forgo because we will do this for an arbitrary dimensionality- one can easily prove that

$$\det\Sigma = \langle x^2\rangle\langle x'^2\rangle - \langle xx'\rangle^2 = \frac{\langle x^2 + (\alpha x + \beta x')^2\rangle}{\beta} = \varepsilon$$

determinant of Σ matrix indeed an RMS emittance for Gaussian distributions.

While this is rather "convenient" to stop here – as in many accelerator text-books – for advanced AP course we should expand our studies to find **general moment invariants for linear Hamiltonian systems*.**

$$X_f = \mathbf{M}X_i;\;\; \mathbf{M}^T\mathbf{S}\mathbf{M} = \mathbf{S}; \tag{11}$$

with f(X) distribution function and define moments as

$$\langle x_{i_1}x_{i_2}\cdots x_{i_k}\rangle = \int f(X)x_{i_1}x_{i_2}\cdots x_{i_k}\,dX \leftrightarrow \frac{1}{N}\sum_{i=1}^{N} x_{i_1}x_{i_2}\cdots x_{i_k}; \tag{12}$$

with

$$\begin{aligned}\langle x_{i_1}x_{i_2}\cdots x_{i_k}\rangle^i &= \int f^i(X)x_{i_1}x_{i_2}\cdots x_{i_k}\,dX \leftrightarrow \frac{1}{N}\sum_{i=1}^{N}\left(x_{i_1}x_{i_2}\cdots x_{i_k}\right)^i\\ \langle x_{i_1}x_{i_2}\cdots x_{i_k}\rangle^f &= \int f^f(X)x_{i_1}x_{i_2}\cdots x_{i_k}\,dX \leftrightarrow \frac{1}{N}\sum_{i=1}^{N}\left(x_{i_1}x_{i_2}\cdots x_{i_k}\right)^f;\end{aligned} \tag{13}$$

Liouville's theorem requires that phase space density is preserved:

$$f^f(X^f) = f^i(X^i) \Leftrightarrow f^f(X) = f^i(\mathbf{M}^{-1}X); \tag{14}$$

and

$$\begin{aligned}\langle x_{i_1}x_{i_2}\cdots x_{i_k}\rangle^f &= \int f^i(\mathbf{M}^{-1}X)x_{i_1}x_{i_2}\cdots x_{i_k}\,dX;\\ X = \mathbf{M}\breve{X} \rightarrow \langle x_{i_1}x_{i_2}\cdots x_{i_k}\rangle^f &= \int f^i(\breve{X})(\mathbf{M}\breve{x})_{i_1}(\mathbf{M}\breve{x})_{i_2}\cdots(\mathbf{M}\breve{x})_{i_k}\,d\breve{X}\\ \langle x_{i_1}x_{i_2}\cdots x_{i_k}\rangle^f &= m_{i_1j_1}m_{i_2j_2}\cdots m_{i_kj_k}\langle x_{j_1}x_{j_2}\cdots x_{j_k}\rangle^i\end{aligned} \tag{15}$$

Dragt suggest following compact form

$$\begin{aligned}\langle x_{i_1}x_{i_2}\cdots x_{i_k}\rangle^f &= \left(\overset{k}{\otimes}\mathbf{M}\right)\langle x_{j_1}x_{j_2}\cdots x_{j_k}\rangle^i\\ \overset{k}{\otimes}\mathbf{M} &\equiv \mathbf{M}\otimes\mathbf{M}\otimes\cdots\otimes\mathbf{M}\end{aligned} \tag{15}$$

We identified the k-th order moments which are elements of *k*-th order tensor:

$$\begin{aligned}\mathbf{X}^{(k)} \Leftrightarrow \mathbf{X}^{(k)}{}_{i_1i_2\ldots i_k} &= \langle x_{i_1}x_{i_2}\cdots x_{i_k}\rangle\\ \mathbf{X}^{(k)f} &= \left(\overset{k}{\otimes}\mathbf{M}\right)\mathbf{X}^{(k)i}\end{aligned} \tag{16}$$

** for most of derivations we are following Alex J Dragt et all, Phys. Rev. A, **45**, 2572, 1992.*

Kinematic invariants. *A. General concepts*

Motivation – as we had seen in uncoupled or 1D case (eq,(1))

$$\varepsilon_i^2 = \langle q_i^2 \rangle \langle p_i^2 \rangle - \langle q_i p_i \rangle^2 = inv, i = 1,2,...,n \quad (17)$$

determinant of Σ matrix is invariant of 1D motion. Symplecticity conditions for transport matrix 1D case gives one invariant – its unit determinant.

An n-dimensional linear Hamiltonian has symplectic transport matrix with n(2n-1) conditions on its coefficients and one expect to have n(2n-1) independent (but in general case not necessarily all non-zero!) invariants. The corresponding invariant of motion is 1D emittance defined as (1) or (17).

In 2D case symplecticity of transport matrix gives 6 condition and we should expect 6 invariants of motion. In 3D case we have 15 conditions and should expect corresponding number of invariants.

For decoupled motion in 3D case we would have 3 conserved emittances as three invariants. All other invariants, which could be non-zero for coupled motion, are simply zeros in this case – and can be ignored. This is why accelerator physicist trying as much as possible to stay away from coupling…

Let's now look for generalized invariants of linear Hamiltonian system. Suppose we have a kinematic invariant function:

$$I\left(\left(\overset{k}{\otimes}\mathbf{M}\right)\mathbf{X}^{(k)}\right) = I\left(\mathbf{X}^{(k)}\right) \;\; \forall\; \mathbf{M} \in \left\{\mathbf{M}^T\mathbf{S}\mathbf{M} = \mathbf{S}\right\} \quad (18)$$

Let's define equivalence classes of k-th order moments: two moments are equivalent if and of if they are connected by symplectic transformation:

$$\mathbf{X}^{(k)\prime} \sim \mathbf{X}^{(k)} \Leftrightarrow \mathbf{X}^{(k)\prime} = \left(\overset{k}{\otimes}\mathbf{M}\right)\mathbf{X}^{(k)} \;\&\; \mathbf{M} \in \mathbf{M}^T\mathbf{S}\mathbf{M} = \mathbf{S} \quad (19)$$

Define set of equivalent k-th order moments

$$\left[\mathbf{X}^{(k)}\right] \Leftrightarrow \mathbf{X}^{(k)\prime} \in \left[\mathbf{X}^{(k)}\right] \to \mathbf{X}^{(k)\prime} \sim \mathbf{X}^{(k)} \quad (20)$$

From (18-20) we conclude that the kinematic invariant function is a class function:

$$I\left(\mathbf{X}^{(k)\prime}\right) = I\left(\mathbf{X}^{(k)}\right) \; if \;\; \mathbf{X}^{(k)\prime} \sim \mathbf{X}^{(k)} \to I = I\left(\left[\mathbf{X}^{(k)\prime}\right]\right) \quad (21)$$

B. Quadratic moment invariants

Consider a quantity of

$$I_2^{(n)}\left(\left[\mathbf{X}^{(2)}\right]\right)=tr\left[\left(\mathbf{X}^{(2)}\mathbf{S}\right)^n\right];\ \mathbf{X}_{ij}^{(2)}=\left\langle x_i x_j\right\rangle. \tag{21}$$

Let's show that $I_2^{(n)}$ is indeed a kinematic invariant:

$$\begin{gathered}(\mathbf{M}\otimes\mathbf{M})\,\mathbf{X}^{(2)}=\mathbf{M}\mathbf{X}^{(2)}\mathbf{M}^T;\ tr[\mathbf{ABC}]=tr[\mathbf{BCA}];\\ \mathbf{X}^{(2)}=(\mathbf{M}\otimes\mathbf{M})\,\mathbf{X}^{(2)}\rightarrow I_2^{(n)}\left(\left[(\mathbf{M}\otimes\mathbf{M})\,\mathbf{X}^{(2)}\right]\right)=tr\left[\left(\mathbf{M}\mathbf{X}^{(2)}\mathbf{M}^T\mathbf{S}\right)^n\right]=\\ tr\left[\left(\mathbf{X}^{(2)}\mathbf{M}^T\mathbf{S}\mathbf{M}\right)^n\right]=tr\left[\left(\mathbf{X}^{(2)}\mathbf{S}\right)^n\right]=I_2^{(n)}\left(\left[\ \mathbf{X}^{(2)}\right]\right)\#\end{gathered} \tag{22}$$

Hence, there is infinite number of quadratic moment invariants, but all odd order invariants are simple zeros: odd number invariant contains odd number of **S**, which is asymmetric. In contrast, $\mathbf{X}^{(2)}$ is symmetric by definition. Hence:

$$tr\left[\left(\mathbf{X}^{(2)}\mathbf{S}\right)^n\right]=tr\left[\left(\mathbf{X}^{(2)}\right)^n\mathbf{S}^n\right]=tr\left[\left(\mathbf{X}^{(2)}\right)^n\mathbf{S}^n\right]^T=(-1)^n tr\left[\left(\mathbf{X}^{(2)}\right)^n\mathbf{S}^n\right] \tag{23}$$

We can calculate $I_2^{(2)}$ directly:

$$\begin{gathered}n=2\rightarrow tr\left[\left(\mathbf{X}^{(2)}\mathbf{S}\right)^2\right]=\\ -2\left(\sum_{i=1}^{3}\left(\left\langle q_i^2\right\rangle\left\langle p_i^2\right\rangle-\left\langle q_i p_i\right\rangle^2\right)+2\sum_{i\neq j}\left(\left\langle q_i q_j\right\rangle\left\langle p_i p_j\right\rangle-\left\langle q_i p_j\right\rangle\left\langle p_i q_j\right\rangle\right)\right)\end{gathered} \tag{24}$$

It is clearly generalization of the 1D emittance definition, but it is not eigen emittances! It just a single number out of 3! It is definitely possible to write expressions for $I_2^{(4)}$ and $I_2^{(6)}$: the first will cover one page, the second quite a few!

Much more natural step is to determine number of independent invariants is to study properties of the form. Let's classify $\mathbf{X}^{(2)}$ according to its equivalency class:

$$\Sigma\overset{def}{\equiv}\mathbf{X}^{(2)};\ \ \mathbf{X}^{(2)\prime}=\mathbf{M}\mathbf{X}^{(2)}\mathbf{M}^T\Leftrightarrow\Sigma'=\mathbf{M}\Sigma\mathbf{M}^T \tag{25}$$

and we claim that any give has its "normal" form.

Theorem: Given a set of quadratic forms $\Sigma = \mathbf{X}^{(2)}$ there exists a symplectic matrix transferring it to a special form with

$$\langle q_i q_i \rangle = \langle p_i p_i \rangle = \lambda_i > 0; \quad \langle x_i x_{j\neq i} \rangle = 0; \tag{25}$$

Our Σ matrix

$$\Sigma = \left[\Sigma_{ij} \right]; \; \Sigma_{ij} = \langle x_i x_j \rangle = \int x_i x_j f(X) dX; \tag{26}$$

is obviously symmetric matrix. We need to prove that it is also positively defined!

Lemma 1. Since $f(X)$ is density of particles in the phase space, it must be positively defined, e.g. $f(X) \geq 0$. Suppose that $f(X)$ is continuous at some point X_o and $f(X_o) > 0$ - then X_{ij} is positively defined. *Proof*: Since $f(X)$ is non-zero and continuous at X_o, there exists a ball

$$B_\varepsilon = \{ X, |X - X_o| \leq \varepsilon \} \rightarrow f(X) \geq \delta > 0; \tag{27}$$

Let Z be any non-zero vector and

$$(Z, \Sigma Z) = \sum_{i,j} z_i \, \Sigma_{ij} z_j = \int \sum_{i,j} z_i x_i x_j z_j f(X) dX;$$

$$\sum_{i,j} z_i x_i x_j z_j = \left(\sum_i z_i x_i \right)^2 \rightarrow (Z, \Sigma Z) = \int \left(\sum_i z_i x_i \right)^2 f(X) dX; \tag{28}$$

$$f(X) \geq \delta \rightarrow (Z, \Sigma Z) \geq \delta \int \left(\sum_i z_i x_i \right)^2 dX > 0 \;\; \#.$$

Note: it is even easier for individual particles:

$$\Sigma = \left[\Sigma_{ij} \right]; \; \Sigma_{ij} = \langle x_i x_j \rangle = \frac{1}{N} \sum_{k=1}^{N} x_i^k x_j^k;$$

$$Z^T \Sigma Z = \frac{1}{N} \sum_{k=1}^{N} \sum_{i,j} z_i x_i^k x_j^k z_j = \frac{1}{N} \sum_{k=1}^{N} \left(\sum_i z_i x_i^k \right)^2 > 0 \#. \tag{29}$$

This is exactly definition of positively defined matrix. Proven#.

Lemma 2. Consider a Hamiltonian defined as:

$$H(Z) = \frac{1}{2} Z^T \Sigma Z = \frac{1}{2} X_{ij} z_i z_j; \tag{30}$$

which is positively definite. Hence, there exists c>0

$$H(Z) = \geq c\|Z\|^2 \tag{31}$$

Set $\|N\|=1$ and find minimum of $H(N)$ - since sphere $\|N\|=1$ is compact it has to have a minimum, which is greater than zero. The rest is just scaling:

$$Z = \|Z\| \cdot \frac{Z}{\|Z\|} \to H(Z) = \|Z\|^w H\left(\frac{Z}{\|Z\|}\right); \left\|\frac{Z}{\|Z\|}\right\| = 1. \tag{32}$$

Consider two matrices

$$\begin{gathered} \mathbf{M}^{-1} = -\mathbf{S}\mathbf{M}^T\mathbf{S};\ \Sigma' = \mathbf{M}\Sigma\mathbf{M}^T;\ \mathbf{T} = \mathbf{S}\Sigma;\ \mathbf{T}' = \mathbf{S}\Sigma'; \\ \to \Sigma = -\mathbf{S}\mathbf{T};\ \mathbf{T}' = \mathbf{S}\mathbf{M}\Sigma\mathbf{M}^T = \left(\mathbf{M}^T\right)^{-1} \Sigma \mathbf{M}^T \end{gathered} \tag{33}$$

e.g. matrices **T** and **T′** are similar and have the same eigen values. None of them equal zero, otherwise determinant of **T** is equal zero – but it is not possible sine it equal to determinant of **X**, which is positively defined with not zero determinant!

Lemma 3. Spectrum of is **T** purely imaginary pairs. Its eigen vectors form a basis and bring **T** to diagonal form, even in case of non-distinct eigen values.

Proof: Consider a Hamiltonian equations

$$Z' = \{Z, H(Z)\} = \mathbf{S}\Sigma \cdot Z = \mathbf{T}\cdot Z \rightarrow Z(s) = e^{\mathbf{T}s} Z(0) \quad (34)$$

Let's consider that matrix **T** can be brought to Jordan normal form

$$\mathbf{T} = \mathbf{A}\mathbf{N}\mathbf{A}^{-1} \rightarrow e^{\mathbf{T}s} = \mathbf{A}e^{\mathbf{N}s}\mathbf{A}^{-1} \quad (35)$$

The matrix $\exp(\mathbf{N}s)$ has also normal form, which we studied in the Sylvester formulae class. The set of eigen values is a set of {λ, -λ} pairs. If one of the eigen values, λ_k , is not purely imaginary, than we should have either $\exp(\lambda_k s)$ or $\exp(-\lambda_k s)$ growing exponentially with

$$\|Z(s)\| = \|e^{\mathbf{T}s}\| \|Z(0)\| \rightarrow \infty \Rightarrow H(Z(s)) > c\|Z(s)\|^2 \rightarrow \infty \quad (36)$$

which is in contradiction with the simple fact that energy is conserved for s-independent Hamiltonian:

$$H(Z(s)) = H(Z(0)) = const \quad (37)$$

Similarly, if some of Jordan block has dimension >1 (e.g. matrix **N** is not diagonal!), we would have an elements proportional to *s* at least in first power:

$$\|Z(s)\| \propto \|s^n e^{\lambda s}\| \|Z(0)\| \rightarrow \infty \Rightarrow H(Z(s)) > c\|Z(s)\|^2 \rightarrow \infty \quad (38)$$

which again contradicts energy conservation. Proven#.

Thus, $\mathbf{T}$ can be diagonalized with all imaginary eigen values and linearly independent eigen vectors:

$$\begin{gathered}\{\lambda_i, -\lambda_i\}, i = 1,2\ldots; \operatorname{Im}\lambda_i = \varepsilon_i > 0;\\ \mathbf{T}\cdot\Upsilon_i = \lambda_i\Upsilon_i; \mathbf{T}\cdot\Upsilon_i^* = \lambda_i^*\Upsilon_i^*;\end{gathered} \tag{39}$$

Let's introduce a new angular inner product with matrix $\mathbf{K}$:

$$\begin{gathered}\mathbf{K} = -i\mathbf{S};\ \ \mathbf{K}^\dagger = \left(\mathbf{K}^T\right)^* = \mathbf{K};\ \mathbf{S}^T = \mathbf{S};\\ \langle A,B\rangle \equiv A^{*T}\mathbf{K}B;\ \langle A,B\rangle^* = -\langle A,B\rangle;\\ A^T\mathbf{S}B = \left(A^T\mathbf{S}B\right)^T = -B^T\mathbf{S}A \to \langle A,B\rangle^{T*} = \langle B,A\rangle;\end{gathered} \tag{40}$$

and use it for eigen vectors

$$\begin{gathered}\mathbf{S}\Sigma\cdot\Upsilon_i = \lambda_i\Upsilon_i; \lambda_i = i\varepsilon_i \to \Sigma\cdot\Upsilon_i = -\lambda_i\mathbf{S}\Upsilon_i = \varepsilon_i\mathbf{K}\Upsilon_i\\ \mathbf{K}\Upsilon_i = \frac{1}{\varepsilon_i}\Sigma\cdot\Upsilon_i \to \Upsilon_i^\dagger\mathbf{K}\Upsilon_i = \langle\Upsilon_i,\Upsilon_i\rangle = \frac{1}{\varepsilon_i}\Upsilon_i^\dagger\Sigma\cdot\Upsilon_i; \varepsilon_i > 0.\\ \Upsilon_i = \mathcal{R}_i + i\mathcal{Q}_i;\ \ \Upsilon_i^\dagger\Sigma\cdot\Upsilon_i = \mathcal{R}_i^T\Sigma\cdot\mathcal{R}_i + \mathcal{Q}_i^T\Sigma\cdot\mathcal{Q}_i > 0\\ \langle\Upsilon_i,\Upsilon_i\rangle > 0\end{gathered} \tag{41}$$

To prove that

$$\begin{gathered}\langle\Upsilon_i,\Upsilon_j\rangle = -i\Upsilon_i{}^{*T}\,\mathbf{S}\Upsilon_j = 0;\ \lambda_k \neq \lambda_k\\ \Sigma = -\mathbf{S}\mathbf{T} \to \Sigma^T - \Sigma = 0 \to \mathbf{T}^T\mathbf{S} + \mathbf{S}\mathbf{T} = 0;\\ \Upsilon_i^{*T}\left(\mathbf{T}^T\mathbf{S} + \mathbf{S}\mathbf{T}\right)\Upsilon_j = \left(\lambda_j - \lambda_i\right)\Upsilon_i{}^{*T}\,\mathbf{S}\Upsilon_j = 0\,\#\end{gathered} \tag{42}$$

is easy. Similarly

$$\begin{gathered}\langle\Upsilon^*{}_i,\Upsilon_j\rangle = -i\Upsilon_i{}^T\mathbf{S}\Upsilon_j - 0;\ \mathbf{T}^T\mathbf{S} + \mathbf{S}\mathbf{T} = 0;\\ \Upsilon_i{}^T\left(\mathbf{T}^T\mathbf{S} + \mathbf{S}\mathbf{T}\right)\Upsilon_j = \left(\lambda_j + \lambda_i\right)\Upsilon_i{}^{*T}\,\mathbf{S}\Upsilon_j = 0\,\#\end{gathered} \tag{43}$$

Lemma 5. Starting with vector Υ_i, one can construct vectors Υ_j such that

$$\begin{aligned}&(1)\quad \mathbf{T}\Upsilon_j = \lambda_j \Upsilon_j = i\varepsilon_j \Upsilon_j,\ \varepsilon_j > 0;\\ &(2)\quad \left\langle \Upsilon_j, \Upsilon_k \right\rangle = \delta_{ik};\\ &(3)\quad \left\langle \Upsilon_j, \Upsilon^*_k \right\rangle = 0.\end{aligned} \tag{44-46}$$

Proof. In simplest case of distinct eigen values, it is coming from previous lemma plus simple normalization of the vectors.

My proof for arbitrary Let's consider a degeneracy of λ_k of order h (in 3D case it is either 2 or 3). Since matrix is diagonalized, there is h linearly independent eigen vectors

$$\begin{gathered}\mathbf{T}\Upsilon_k^m = \lambda_k \Upsilon_k^m = i\varepsilon_k \Upsilon_k^m,\ \varepsilon_k > 0 \rightarrow \left\langle \Upsilon_k^m, \Upsilon_k^m \right\rangle > 0;\ k = 1,\ldots,h;\\ \breve{\Upsilon} = \sum_m \alpha_m \Upsilon_k^m \rightarrow \mathbf{T}\Upsilon = \lambda_k \breve{\Upsilon}.\end{gathered} \tag{47}$$

Let's construct first eigen vector perpendicular to the rest using (seen to be called Gram-Schmidt) following procedure:

$$\begin{aligned}&\breve{\Upsilon}_k^1 = \Upsilon_k^1;\\ &\breve{\Upsilon}_k^2 = \Upsilon_k^2 - \frac{\left\langle \breve{\Upsilon}_k^1, \Upsilon_k^2 \right\rangle}{\left\langle \breve{\Upsilon}_k^1, \breve{\Upsilon}_k^1 \right\rangle}\Upsilon_k^1;\quad \left\langle \breve{\Upsilon}_k^2, \breve{\Upsilon}_k^1 \right\rangle = 0;\\ &\breve{\Upsilon}_k^3 = \Upsilon_k^3 - \sum_{m=1}^{2} \frac{\left\langle \breve{\Upsilon}_k^m, \Upsilon_k^3 \right\rangle}{\left\langle \breve{\Upsilon}_k^m, \Upsilon_k^m \right\rangle}\Upsilon_k^m;\ \left\langle \breve{\Upsilon}_k^l, \breve{\Upsilon}_k^3 \right\rangle = 0;\ l = 1,2\\ &\ldots.\\ &\breve{\Upsilon}_k^h = \Upsilon_k^3 - \sum_{m=1}^{h-1} \frac{\left\langle \breve{\Upsilon}_k^m, \Upsilon_k^3 \right\rangle}{\left\langle \breve{\Upsilon}_k^m, \Upsilon_k^m \right\rangle}\Upsilon_k^m;\ \left\langle \breve{\Upsilon}_k^l, \breve{\Upsilon}_k^h \right\rangle = 0;\ l = 1,2,..,h-1\end{aligned} \tag{48}$$

$$\hat{\Upsilon}_k^m = \frac{\breve{\Upsilon}_k^m}{\left\langle \breve{\Upsilon}_k^m, \breve{\Upsilon}_k^m \right\rangle} \rightarrow \left\langle \hat{\Upsilon}_k^m, \hat{\Upsilon}_k^m \right\rangle = 1 \tag{49}$$

which makes complete set of symplectically normalized and mutually orthogonal eigen vectors. We then simply remunerate these vectors in continuous sequence to drop and extra index. This ends the proof #.

These eigen vectors are definitely complex with non-zero real and imaginary part

$$\langle \Upsilon_k, \Upsilon_k \rangle = -i\Upsilon_k^{*T}\mathbf{S}\Upsilon_k = 1; \sqrt{2}\Upsilon_k = \mathcal{R}_k + i\mathcal{Q}_k;\ \Upsilon_k^{*T} \equiv \Upsilon_k^{\dagger};$$
$$A^T\mathbf{S}A^T \equiv 0; \Rightarrow \mathcal{R}_k^T\mathbf{S}\mathcal{Q}_k \equiv \left(\mathcal{R}_k, \mathbf{S}\mathcal{Q}_k\right) = 1 = -\mathcal{Q}_k^T\mathbf{S}\mathcal{R}_k. \tag{50}$$

– otherwise their symplectic product would be equal zero! A simple observation is that definitions used above are simply by $\sqrt{2}$ differ from eigen vectors we defined for periodic systems – but beware it is not one to one relation with eigen vectors we had defined:

$$\Upsilon_k \rightarrow \sqrt{2}\Upsilon_k \tag{51}$$

Lemma 6. One can construct symplectic matrix Θ from $\mathcal{Q}_k, \mathcal{R}_k$ that bring the matrix $\mathbf{\Sigma}$ to diagonal form with all positive identical pairs of diagonal elements

$$\Theta^T\Sigma\Theta = diag\left\{\varepsilon_1, \varepsilon_1 \ldots \varepsilon_n, \varepsilon_n\right\} = \begin{bmatrix} \ldots & 0 & 0 \\ 0 & \begin{bmatrix} \varepsilon_i & 0 \\ 0 & \varepsilon_i \end{bmatrix} & 0 \\ 0 & 0 & \ldots \end{bmatrix} \tag{52}$$

Proof. Let's construct Θ in the following way:

$$\Theta = \left[\mathcal{R}_1, \mathcal{Q}_1 .. \mathcal{R}_k, \mathcal{Q}_k ... \mathcal{R}_n, \mathcal{Q}_n\right] \Rightarrow \Theta^T \mathbf{S} \Theta = \mathbf{S}. \tag{52}$$

From definition of matrix $\mathbf{T}$ we have:

$$\mathbf{\Sigma} = -\mathbf{ST} \to \mathbf{\Sigma}\Theta = -\mathbf{ST}\Theta;\ \mathbf{T}\Upsilon_{kk} = i\kappa_k \Upsilon_{kk};$$

$$\mathcal{R}_k = \sqrt{2}\left(\Upsilon_k + \Upsilon_k^*\right); i\mathcal{Q}_k = \sqrt{2}\left(\Upsilon_k - \Upsilon_k^*\right);$$

$$\mathbf{T}\mathcal{R}_k = i\varepsilon_k\sqrt{2}\left(\Upsilon_k - \Upsilon_k^*\right) = -\varepsilon_k \mathcal{Q}_k; \mathbf{T}\mathcal{Q}_k = \sqrt{2}\varepsilon_k\left(\Upsilon_k + \Upsilon_k^*\right) = \varepsilon_k \mathcal{R}_k;$$

$$\mathbf{\Sigma}\Theta = \left[\varepsilon_1 \mathbf{S}\mathcal{Q}_1, -\varepsilon_1 \mathbf{S}\mathcal{R}_1 .... \varepsilon_n \mathbf{S}\mathcal{Q}_n, -\varepsilon_n \mathbf{S}\mathcal{R}_n\right]$$

$$\Xi = \Theta^T \mathbf{\Sigma}\Theta = \left[\mathcal{R}_1, \mathcal{Q}_1 ...., \mathcal{R}_n, \mathcal{Q}_n\right]^T \left[\varepsilon_1 \mathbf{S}\mathcal{Q}_1, -\varepsilon_1 \mathbf{S}\mathcal{R}_1 .... \varepsilon_n \mathbf{S}\mathcal{Q}_n, -\varepsilon_n \mathbf{S}\mathcal{R}_n\right]$$

$$\begin{bmatrix} ... \\ \mathcal{R}_k^T \\ \mathcal{Q}_k^T \\ ... \end{bmatrix} \left[....\varepsilon_j \mathbf{S}\mathcal{Q}_j, -\varepsilon_j \mathbf{S}\mathcal{R}_j ....\right] = \left[\Xi_{kj}\right]$$

$$\Xi_{kj} = \varepsilon_j \begin{bmatrix} \mathcal{R}_k^T \mathbf{S}\mathcal{Q}_j & -\mathcal{R}_k^T \mathbf{S}\mathcal{R}_j \\ \mathcal{Q}_k^T \mathbf{S}\mathcal{Q}_j & -\mathcal{Q}_k^T \mathbf{S}\mathcal{R}_j \end{bmatrix} = \varepsilon_k \delta_{ikj} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Xi = \begin{bmatrix} ... & 0 & 0 \\ 0 & \begin{bmatrix} \varepsilon_k & 0 \\ 0 & \varepsilon_k \end{bmatrix} & 0 \\ 0 & 0 & ... \end{bmatrix}$$

This ends the proof.

It also identifies how one define emittances of arbitrary particles distribution of particles in 6D phase space as well as initial values for eigen vectors.

$$\Sigma = \left[\langle x_i x_j \rangle\right]; \Theta^T \mathbf{S} \Theta = \mathbf{S}$$

$$\Xi = \Theta^T \Sigma \Theta = \begin{bmatrix} \ldots & 0 & 0 & 0 \\ 0 & \varepsilon_k & 0 & 0 \\ 0 & 0 & \varepsilon_k & 0 \\ 0 & 0 & 0 & \ldots \end{bmatrix} \to \Sigma = \left(\Theta^T\right)^{-1} \Xi \Theta^{-1} = \mathbf{S}\Theta\mathbf{S}\Xi\mathbf{S}\Theta^T\mathbf{S}; \quad (53)$$

Hence, we found the way to connect something we invented for periodic system – a parameterization using symplectically orthogonal eigen vectors of transport matrix for a single period. Now we found a way to connect this description with a beam having an arbitrary distribution in a phase space. To find a set of eigen vectors suitable to describe the actual 6D beam distribution in phase space for any transport channel we can use the correlation matrix

$$\Sigma = \left[\Sigma_{ij}\right]; \ \Sigma_{ij} = \langle x_i x_j \rangle = \frac{1}{N} \sum_{k=1}^{N} x_i^k x_j^k \quad (55)$$

than find eigen values of supporting matrix T by solving cubic equation on squares of its eigen values

$$\det[\mathbf{T} - \lambda \mathbf{I}] = \det[\mathbf{S}\mathbf{X} + \lambda \mathbf{I}] = \prod_{k=1}^{3} (\lambda - i\varepsilon_k)(\lambda + i\varepsilon_k) = \prod_{k=1}^{3} \left(\lambda^2 + \varepsilon_k^2\right); \ \varepsilon_k > 0; \quad (56)$$

and finding full set of eigen vectors of matrix **T** by picking them from columns of following matrices (beware of :

$$\left(\mathbf{T} - i\varepsilon_k \mathbf{I}\right) \prod_{\kappa_j \neq \kappa_k} \left(\mathbf{T}^2 - \varepsilon_j^2 \mathbf{I}\right) \quad (57)$$

and follow the Gram-Schmidt procedure to find the set of symplectically orthogonal eigen vectors. These eigen vectors will give the parameterization of the beam while ε_k with give 3 eigen emittance of the beam. As we proven, these eigen emittance can not be changed in any linear Hamiltonian transport (even though can be spoiled in non-linear one!) and are invariant of motion. There product is called 3D emittance of the beam

$$\varepsilon_{3D} = \varepsilon_1 \varepsilon_2 \varepsilon_3 = \sqrt{\det \Sigma} \quad (58)$$

One can change appearances of phase space projections into 1D phase space (frequently called emittance exchange between different degrees of freedom) but can not modify neither the values of the eigen emittance nor their product. In contrast with eigen emittances, eigen vectors can be multiplied by a complex exponent without modifying the result (50-31)

$$\Upsilon_k \to \breve{\Upsilon}_k = \Upsilon_k e^{i\varphi_k} \Leftrightarrow \left\langle \breve{\Upsilon}_k, \breve{\Upsilon}_j \right\rangle = \left\langle \Upsilon_k, \Upsilon_j \right\rangle \tag{59}$$

which is essentially flexibility of separating oscillation phase from phase of the oscillator.

One of the things we want to do is to connect set of eigen vectors (59) with our parameterization. When the parameterization eigen vectors are defined at s_o, we can propagate them according to already established rules using transport matrix:

$$\tilde{Y}_k(s) = \mathbf{M}(s_o|s) Y_k(s_o). \tag{60}$$

Making a dedicated transport channel to have a specific form (again, defined with flexibility of phase advance (59)), for example to fit it with one in a periodic lattice, injection into a storage ring or for a special device (a wiggler or interaction region for beam collisions), is called matching. Traditionally, when the energy if the beam is fixed, it is reduced to matching transverse eigen vectors using magnetic elements – e.g. 2D or 4D phase space problem. But it also involve matching transverse dispersion functions and bunch length.

But in modern accelerators, such as energy recovery linacs or sophisticated beam manipulation system with emittance exchange, matching can involve all six component in the phase space. Hence the most general treatment of the problem.

1D case it is rather simple for selecting phase in (59) to have zero imaginable part of Q:

$$Y=\begin{bmatrix} \mathrm{w} \\ \mathrm{w}'+i/\mathrm{w} \end{bmatrix};\ \Upsilon = Y/\sqrt{2};R=\begin{bmatrix} \mathrm{w} \\ \mathrm{w}' \end{bmatrix};Q=\begin{bmatrix} 0 \\ 1/\mathrm{w} \end{bmatrix};\mathbf{U}=\begin{bmatrix} \mathrm{w} & 0 \\ \mathrm{w}' & 1/\mathrm{w} \end{bmatrix}. \qquad (61)$$

In case of higher dimensions (two and above) this choice is not obvious, since any of eigen vector component in general can be zero.

In 2D case for x-y coupling

$$Y_k(s)=\begin{bmatrix} \mathrm{w}_{kx}e^{i\chi_{kx}} \\ \left(\mathrm{v}_{kx}+i\dfrac{q_{kx}}{\mathrm{w}_{kx}}\right)e^{i\chi_{kx}} \\ \mathrm{w}_{ky}e^{i\chi_{ky}} \\ \left(\mathrm{v}_{ky}+i\dfrac{q_{ky}}{\mathrm{w}_{ky}}\right)e^{i\chi_{ky}} \end{bmatrix};\ \text{or for x-}\tau\text{coupling } Y_k(s)=\begin{bmatrix} \mathrm{w}_{kx}e^{i\chi_{kx}} \\ \left(\mathrm{v}_{kx}+i\dfrac{q_{kx}}{\mathrm{w}_{kx}}\right)e^{i\chi_{kx}} \\ \mathrm{w}_{k\tau}e^{i\chi_{k\tau}} \\ \left(\mathrm{v}_{k\tau}+i\dfrac{q_{k\tau}}{\mathrm{w}_{k\tau}}\right)e^{i\chi_{k\tau}} \end{bmatrix};k=1,2 \quad (62\text{-}63)$$

with conditions

$$Y_k^T S Y_j = 0;\quad Y_j^{*T} S Y_k = 2i\delta_{kj}; \qquad (64)$$

resulting in partial conditions

$$\begin{aligned} & q_{kx}+q_{ky}=1;\ k=1,2 \rightarrow q_{1x}=q_{2y}=q;\ q_{2x}=q_{1y}=1-q; \\ & or \\ & q_{kx}+q_{k\tau}=1;\ k=1,2 \rightarrow q_{1x}=q_{2\tau}=q;\ q_{2x}=q_{1\tau}=1-q. \end{aligned} \qquad (65)$$

The last equation is nothing else but conservation of phase space projection (including sign! – q can be negative or larger than 1!) on two 1D phase spaces for each oscillator – you may still remember one of Poincaré invariants:

$$\begin{aligned} & \sum_{i=1}^{n}\iint dq_i dP^i = \iint dx dP^x + \iint dy dP^y = inv \\ & or \\ & \sum_{i=1}^{n}\iint dq_i dP^i = \iint dx dP^x + \iint d\tau dP^\tau = inv \end{aligned} \qquad (66)$$

In 3D case

$$Y_k(s) = \begin{bmatrix} \mathrm{w}_{kx} e^{i\chi_{kx}} \\ \left(\mathrm{v}_{kx} + i\dfrac{q_{kx}}{\mathrm{w}_{kx}}\right) e^{i\chi_{kx}} \\ \mathrm{w}_{ky} e^{i\chi_{ky}} \\ \left(\mathrm{v}_{ky} + i\dfrac{q_{ky}}{\mathrm{w}_{ky}}\right) e^{i\chi_{ky}} \\ \mathrm{w}_{k\tau} e^{i\chi_{k\tau}} \\ \left(\mathrm{v}_{k\tau} + i\dfrac{q_{k\tau}}{\mathrm{w}_{k\tau}}\right) e^{i\chi_{k\tau}} \end{bmatrix}; k = 1,2,3 \tag{67}$$

$$Y_k^{\ T} S Y_j = 0; \quad Y^{*}{}_j^{\ T} S Y_k = 2i\delta_{kj}; \tag{68}$$

$$q_{kx} + q_{ky} + q_{k\tau} = 1; \ k = 1,2,3 \tag{69}$$

In 3D case we have following well-know Poincaré invariants:

$$\sum_{i=1}^{n} \iint dq_i dP^i = \iint dx dP^x + \iint dy dP^y + \iint d\tau dP^\tau = inv$$

$$\sum_{i \neq j} \iint\iint dq_i\, dP^i\, dq_j\, dP^j = \tag{70}$$

$$\iint\iint dx\, dP^x\, dy\, dP^y + \iint\iint dx\, dP^x\, d\tau\, dP^\tau + \iint\iint d\tau\, dP^\tau\, dy\, dP^y = inv$$

[2]e.g. conservation is sum of projections.

How to calculate the Σ matrix and connect it with parameterization

In practice particle's displacements are taken from the position of reference particle (orbit) and if beam as a whole is displaced

$$\langle x_i \rangle \neq 0 \tag{81}$$

its center will execute oscillation (or at least collective motion) in the beam-line. If the position in the phase of the beam centroid can be corrected (or used as the reference!), we can remove the average displacement and use more traditional definition of the correlation matrix:

$$\begin{gathered}\Sigma = \left[\Sigma_{ij}\right];\ \langle x_i \rangle = \sum_{k=1}^{N} x_i^k; \\ \Sigma_{ij} = \left\langle \left(x_i - \langle x_i \rangle\right)\left(x_j - \langle x_j \rangle\right)\right\rangle = \frac{1}{N}\sum_{k=1}^{N}\left(x_i^k - \langle x_i \rangle\right)\left(x_j^k - \langle x_j \rangle\right).\end{gathered} \tag{82}$$

with the rest of treatment being identical to the above.

The only connection we need to make is with the Gaussian distribution in the storage ring

$$f(X)=\prod_{k=1}^{3}\frac{1}{2\pi\varepsilon_k}\exp\left[-\frac{X^T\left[\left(\mathbf{O}^T\right)^{-1}\Xi^{-1}\mathbf{O}^{-1}\right]X}{2}\right]., \tag{83}$$

to show that

$$\Sigma=\left[\Sigma_{ij}\right];\ \Sigma_{ij}=\int x_i x_j f(X)dX; f(X)=\prod_{k=1}^{3}\frac{1}{2\pi\varepsilon_k}\exp\left[-\frac{X^T\cdot\left(\mathbf{O}^T\right)^{-1}\Xi^{-1}\mathbf{O}^{-1}\cdot X}{2}\right]$$

$$\Sigma=\int\left[\Sigma_{ij}\right]f(X)dX \tag{84}$$

$$\Rightarrow\Sigma=\mathbf{O}\Xi\mathbf{O}^T$$

It can be done by changing variable under the integral to 3 oscillators and taking the integral

$$outer-product:\ \left[X\cdot X^T\right]_{ij}=x_i x_j;$$

$$X=\mathbf{O}\breve{A};\breve{A}^T=\left[..a_k\cos\varphi_k,-a_k\sin\varphi_k...\right]$$

$$\mathbf{O}=\left[...R_k,Q_k...\right];\quad \mathbf{O}^T\mathbf{S}\mathbf{O}=\mathbf{S};\ \left[X\cdot X^T\right]=\mathbf{O}\left[\breve{A}\cdot\breve{A}^T\right]\mathbf{O}^T \tag{85}$$

$$\Sigma=\int\left[X\cdot X^T\right]f(X)dX=\mathbf{O}\left(\int\left[\breve{A}\cdot\breve{A}^T\right]f(X)dX\right)\mathbf{O}^T$$

$$f(X)=\left(\prod_{k=1}^{3}\frac{1}{2\pi\varepsilon_k}\right)\exp\left[-\frac{X^T\left(\mathbf{O}^T\right)^{-1}\Xi^{-1}\mathbf{O}^{-1}X}{2}\right]=\left(\prod_{k=1}^{3}\frac{1}{2\pi\varepsilon_k}\right)\exp\left[-\frac{\breve{A}^T\Xi^{-1}\breve{A}}{2}\right]$$

with trivial follow up by making Canonical (unit determinant) transformation of variables:

$$\Sigma = \int \left[X \cdot X^T \right] f(X) dX = \mathbf{O} \left(\int \left[\breve{A} \cdot \breve{A}^T \right] f(X) dX \right) \mathbf{O}^T$$

$$X \to \breve{A} \to \int ...dX = \int ...\det \mathbf{O}\, d\breve{A} = \int ...\det \mathbf{O}\, d\breve{A} = \int ...\prod_{k=1}^{n} d\varphi_k d\frac{a_k^2}{2};$$

$$\breve{A}^T \Xi^{-1} \breve{A} = \sum_k \frac{a_k^2}{\varepsilon_k} \left(\cos^2 \varphi_k + \sin^2 \varphi_k \right) = \sum_k \frac{a_k^2}{\varepsilon_k} \tag{86}$$

$$\exp\left[-\frac{\breve{A}^T \Xi^{-1} \breve{A}}{2} \right] = \prod_{k=1}^{3} \exp\left[-\frac{a_k^2}{2\varepsilon_k} \right]$$

$$\Sigma = \mathbf{O} \left(\int \left[\breve{A} \cdot \breve{A}^T \right] \prod_{k=1}^{3} \left(\frac{1}{2\pi\varepsilon_k} \exp\left[-\frac{a_k^2}{2\varepsilon_k} \right] d\frac{a_k^2}{2} d\varphi_k \right) \right) \mathbf{O}^T$$

Now it is good time to look onto the inner product $\left[\breve{A}\cdot\breve{A}^T\right]$ under the integral

$$\left[\breve{A}\cdot\breve{A}^T\right]_{ij}=\breve{A}_i\breve{A}_j;\ \breve{A}^T=\left[...a_k\cos\varphi_k,-a_k\sin\varphi_k...\right];$$

$$\left[\breve{A}\cdot\breve{A}^T\right]=\begin{bmatrix}...\\ a_i\cos\varphi_i\\ -a_i\sin\varphi_i\\ ...\end{bmatrix}\bullet\left[...a_j\cos\varphi_j-a_j\sin\varphi_j..\right]$$

$$\breve{A}_{2i-1}\breve{A}_{2j-1}=a_ia_j\cos\varphi_i\cos\varphi_j;\breve{A}_{2i}\breve{A}_{2j}=a_ia_j\sin\varphi_i\sin\varphi_j;$$

$$\breve{A}_{2i-1}\breve{A}_{2j}=-a_ia_j\cos\varphi_i\sin\varphi_j;\ \breve{A}_{2i-1}\breve{A}_{2j};\breve{A}_{2i}\breve{A}_{2j-1}=-a_ia_j\sin\varphi_i\cos\varphi_j;$$

$$\int_0^{2\pi}...\iint\breve{A}_i\breve{A}_j\prod_{k=1}^{3}d\varphi_k=(2\pi)^n\delta_{ij}\frac{a_m^{\ 2}}{2};\ m=\mathrm{int}\left(\frac{i+1}{2}\right); \tag{87}$$

$$\Sigma=\mathbf{O}\left(\int\begin{bmatrix}... & 0 & 0 & 0\\ 0 & \frac{a_k^{\ 2}}{2} & & 0\\ 0 & 0 & \frac{a_k^{\ 2}}{2} & 0\\ 0 & 0 & 0 & ...\end{bmatrix}\prod_{k=1}^{3}\left(\frac{1}{\varepsilon_k}\exp\left[-\frac{a_k^2}{2\varepsilon_k}\right]d\frac{a_k^2}{2}\right)\right)\mathbf{O}^T$$

with finish line as:

$$\Sigma = \mathbf{O}\begin{bmatrix} ... & 0 & 0 & 0 \\ 0 & \alpha_i & & 0 \\ 0 & 0 & \alpha_i & 0 \\ 0 & 0 & 0 & ... \end{bmatrix}\mathbf{O}^T; \quad \alpha_i = \int \frac{a_i^{\,2}}{2}\prod_{k=1}^{3}\left(\frac{1}{2\pi\varepsilon_k}\exp\left[-\frac{a_k^2}{2\varepsilon_k}\right]d\frac{a_k^2}{2}d\varphi_k\right)$$

$$\alpha_i = \int \frac{a_i^{\,2}}{2}\prod_{k=1}^{3}\left(\frac{1}{\varepsilon_k}\exp\left[-\frac{a_k^2}{2\varepsilon_k}\right]d\frac{a_k^2}{2}\right) = \varepsilon_i \int \xi_i \prod_{k=1}^{3}\left(\exp\left[-\xi_k\right]d\xi_k\right); \; \xi_k = \frac{a_k^2}{2\varepsilon_k} \in \{0, +\infty\};$$

$$\alpha_i = \varepsilon_i\left(\int_0^\infty \xi_i e^{-\xi_i}\,d\xi_i\right)\prod_{k\neq i}\int_0^\infty e^{-\xi_k}\,d\xi_k; \int_0^\infty e^{-\xi_k}\,d\xi_k = 1; \left(\int_0^\infty \xi_i e^{-\xi_i}\,d\xi_i\right) = 1; \alpha_i = \varepsilon_i \tag{88}$$

$$\Sigma = \mathbf{O}\begin{bmatrix} ... & 0 & 0 & 0 \\ 0 & \varepsilon_i & & 0 \\ 0 & 0 & \varepsilon_i & 0 \\ 0 & 0 & 0 & ... \end{bmatrix}\mathbf{O}^T = \mathbf{O\Xi O}^T \, \# .$$

and the same time we have from (53)

$$\Sigma = \left(\Theta^T\right)^{-1} \Xi \Theta^{-1} \tag{89}$$

Comparing (89) with (88) finally give us relations between eigen vectors and Σ matrix and our parameterization:

$$\Sigma = \left(\Theta^T\right)^{-1} \Xi \Theta^{-1} = \mathbf{O}\Xi\mathbf{O}^T \rightarrow$$
$$\mathbf{O} = \left(\Theta^T\right)^{-1} = -\mathbf{S}\Theta\mathbf{S}; \tag{90}$$

Hence, we closed the circle: Any arbitrary **Σ** matrix can be brough to diagonal form

$$\Sigma = \mathbf{O}\Xi\mathbf{O}^T;\ \Xi = \begin{bmatrix} \ldots & 0 & 0 & 0 \\ 0 & \varepsilon_i & & 0 \\ 0 & 0 & \varepsilon_i & 0 \\ 0 & 0 & 0 & \ldots \end{bmatrix} \tag{91}$$

with real symplectic matrix **O** which can be used as definition of eigen vectors for any beam distribution. At the same time, Gaussian distribution in a storage ring (or a periodic system) using parameterization (in real notations)

$$f(X) = \prod_{k=1}^{3} \frac{1}{2\pi\varepsilon_k} \exp\left[-\frac{X^T\left[\left(\mathbf{O}^T\right)^{-1}\Xi^{-1}\mathbf{O}^{-1}\right]X}{2}\right], \tag{92}$$

will generate **Σ** matrix in eq. (90). Hence, we established one to one correspodnece between various defintions of emittance. Definition that we estanlish here

As the final note – we found thre out of 15 possible inependent invariant of motion. We will dicuss the others during next class.

What we learned today

- We studies some of best known kinematic invariants of motion in linear Hamiltonian systems – eigen "RMS" emittances
- We define classes of invariants, including those coming from quadratic form (Σ-matrix) of phase space particles positions
- We eigen "RMS" emittances them by transforming the quadratic form (Σ-matrix) using a symplectic transformation Θ of coordinates to positively defined double-degenerated diagonal matrix
- The diagonal terms are nothing else that eigen emittances which are invariants of motion
- We than compared our finding with parameterization we used for the describing particles motion – using a Gaussian distribution we got for a storage ring with synchrotron radiation - and found relation between the parameterization and the symplectic matrix Θ : $\mathbf{O} = \left[\ldots \mathrm{Re}\, Y_k, \mathrm{Im}\, Y \ldots \right] = \left(\Theta^T \right)^{-1} == -\mathbf{S}\Theta\mathbf{S}$
- This provided us with additional way of determining parameterization of particle's motion in any piece of accelerator, not only in period systems.