PHY 564
Advanced Accelerator Physics
Lecture 3

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Recap of lecture 2 - see in lecture 2

Before going to relativistic action -
let's discuss Lorentz group in matrix representation -
an exercise we will repeat quite a few times

Lorentz Group - Matrix representation

Jackson's Classical Electrodynamics, Section 11.7 [CED] has an excellent discussion of this topic. Here, we will review it briefly with some attention to the underlying mathematics. Generic Lorentz transformation involves a boost (a transformation from K to K' moving with some velocity $\vec{V}$) and an arbitrary rotation in 3D space. Matrix representation is well suited to describe 4-vectors transformations. The coordinate vector is defined as

$$X = \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}; \qquad \text{(B-29)}$$

and standard scalar product of 4-vectors is defined by $(a,b) = \tilde{a}b$, where $\tilde{a}$ is the transposed vector. The 4-scalar product involves the metric tensor (matrix):

$$a \cdot b \equiv a^i \cdot b_i = (a, gb) = (ga, b) = \tilde{a}gb\,; \qquad \text{(B-30)}$$

$$g = \tilde{g} = \{g^{ik}\} = \{g_{ik}\} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}. \qquad \text{(B-31)}$$

Lorentz transformations A (or the group of Lorentz transformations[1]) are linear transformations that preserve the interval, or scalar product (B-30):

$$X' = AX;\ \tilde{X}'gX' = \tilde{X}\tilde{A}gAX = \tilde{X}gX; \Rightarrow \tilde{A}gA = g. \qquad \text{(B-33)}$$

Using standard ratios for matrices

$$\det(\tilde{A}gA) = \det^2 A \det g = \det g \Rightarrow \det A = \pm 1; \qquad \text{(B-34)}$$

we find that the matrices of Lorentz transformation have det=±1. We will consider only *proper Lorentz transformations* with unit determinants $\det A = +1$. Improper Lorentz transformations, like space- and time-inversions, should be considered as special transformations and added to the proper ones.

A 4x4 matrix has 16 elements. Equation (B-33) limited number of independent elements in matrix A of Lorentz transformations. Matrices on both sides are symmetric. Thus, there are 10 independent conditions on matrix A, leaving six independent elements there. This is unsurprising since rotation in 3D space is represented by 3 angles and a boost is represented by 3 components of velocity. Intuitively, then there are six independent rotations: (xy), (yz), (zx), (t, x), (t, y), and (t, z). No other combinations of 4D coordinates are possible: $C_4^2 = \frac{4!}{2!2!} = 6$.

[1] Group G is defined as a set of elements , with a definition of a product of any two elements of the group; $P = A \bullet B \in G$; $A, B \in G$. The product must satisfy the associative law : $A \bullet (B \bullet C) = (A \bullet B) \bullet C$; there is an unit element in the group $E \in G; E \bullet A = A \bullet E = A: \forall A \in G;$ and inverse elements: $\forall A \in G; \exists B(called\ A^{-1}) \in G: A^{-1}A = AA^{-1} = E.$

Matrices NxN with non-zero determinants are examples of the group. Lorentz transformations are other examples : the product of two Lorentz is defined as two consequent Lorentz transformations. Therefore, the product also is a Lorentz transformation whose velocity is defined by rules discussed in previous lectures. The associative law is straightforward: unit Lorentz transformation is a transformation into the same system. Inverse Lorentz transformation is a transformation with reversed velocity. Add standard rotation s, to constitute the Lorentz Group

Major leap to matrix functions!

We next consider the properties of A in standard way, representing A through a generator L:

$$A = e^{L}; \tag{B-35}$$

where we use matrix exponent defined as the Taylor expansion:

$$e^{L} \downarrow_{def} \equiv \sum_{n=0}^{\infty} \frac{L^n}{n!};\ L^0 = I;\ I = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}; \tag{B-36}$$

where I is the unit matrix. Using (B-35) and $g^2 = I$ we find how to compose the inverse matrix for A:

$$\tilde{A} g A = g \Rightarrow A^{-1} = g \tilde{A} g; \tag{B-37}$$

which, in combination with

$$\tilde{A} = transpose(e^{L}) = \sum_{n=0}^{\infty} \frac{\tilde{L}^n}{n!} = e^{\tilde{L}};\ e^{gUg} = \sum_{n=0}^{\infty} \frac{(gUg)^n}{n!} = \sum_{n=0}^{\infty} g \frac{U^n}{n!} g; \tag{B-38}$$

gives

$$A^{-1} = g \tilde{A} g = e^{g \tilde{L} g}\ . \tag{B-39}$$

We can show that matrix exponent has similar properties as the regular exponent, i.e. $e^{U}e^{-U}=I$ by explicitly using Taylor expansion to collect the powers of U:

$$e^{U}e^{-U}=\left(\sum_{n=0}^{\infty}\frac{U^{n}}{n!}\right)\left(\sum_{k=0}^{\infty}(-1)^{k}\frac{U^{k}}{k!}\right)=\sum_{k=0,n=0}^{\infty}(-1)^{k}\frac{U^{n+k}}{n!k!}=I+\sum_{m=1}^{\infty}c_{m}U^{m}; \qquad \text{(B-40)}$$

and the well-known expansion of $(1\text{-}x)^{m}$. Our goal is to show that all c_m are zero:

$$(1-x)^{m}=\sum_{n=0}^{m}\frac{(-1)^{n}m!}{n!(m-n)!}x^{n}\Rightarrow m!c_{m}=\sum_{n=0}^{m}\frac{(-1)^{n}m!}{n!(m-n)!}=(1-1)^{m}=0\,. \qquad \text{(B-41)}$$

Now (B-39) can be rewritten

$$A^{-1}=g\tilde{A}g=e^{g\tilde{L}g}=e^{-L}\;\Rightarrow g\tilde{L}g=-L;\Rightarrow \widetilde{gL}=-gL \qquad \text{(B-42)}$$

Hence, gL is an asymmetric matrix and has six independent elements as expected:

$$gL=\begin{bmatrix}0 & L_{01} & L_{02} & L_{03}\\ -L_{01} & 0 & -L_{12} & -L_{13}\\ -L_{02} & L_{12} & 0 & -L_{23}\\ -L_{03} & L_{13} & L_{23} & 0\end{bmatrix};L=g(gL)=\begin{bmatrix}0 & L_{01} & L_{02} & L_{03}\\ L_{01} & 0 & L_{12} & L_{13}\\ L_{02} & -L_{12} & 0 & L_{23}\\ L_{03} & -L_{13} & -L_{23} & 0\end{bmatrix}. \qquad \text{(B-43)}$$

Each independent element represents an irreducible (fundamental) element of the Lorentz group or rotations and boosts, as discussed above. The six components of the L can be considered as six components of 3-vectors in the form ("-" is a convention):

$$L = -\vec{\omega}\vec{S} - \vec{\varsigma}\vec{K}; A = e^{-\vec{\omega}\vec{S}-\vec{\varsigma}\vec{K}}; \quad \text{(B-44)}$$

with

$$\vec{S} = \hat{e}_x \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} + \hat{e}_y \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} + \hat{e}_z \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}; \quad \text{(B-45)}$$

$$\vec{K} = \hat{e}_x \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \hat{e}_y \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \hat{e}_z \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}; \quad \text{(B-46)}$$

where $\vec{\omega}\vec{S}$ represents the orthogonal group of rotations in 3D space (O_3^+), and $\vec{\varsigma}\vec{K}$ represents the boosts caused by transformation into a moving system. It is easy to check that these matrices satisfy commutation rules of

$$[S_i, S_k] = e_{ikl}S_l; [S_i, K_k] = e_{ikl}K_l; [K_i, K_k] = -e_{ikl}S_l;\ [A,B] \equiv AB - BA; \quad \text{(B-47)}$$

where e_{ikl} is the totally asymmetric 3D-tensor. You should be familiar with 3D rotation $e^{-\vec{\omega}\vec{S}}$ by $\vec{\omega}$: the direction of $\vec{\omega}$ is the axis of rotation and the value of $\vec{\omega}$ is the angle of rotation.

For the arbitrary unit vector $\hat{e}$

$$(\hat{e}\vec{S})^3 = -\hat{e}\vec{S};\ (\hat{e}\vec{K})^3 = \hat{e}\vec{K}. \quad \text{(B-48)}$$

Therefore, $\vec{S}$ "behaves" as an imaginary "i" and we should expect *sin* and *cos* to be generated by exp(..$\vec{S}$.); exp(..$\vec{K}$.) should generate hyperbolic functions *sinh* and *cosh.* It is left for your homework to show, in particular, that boost transformation is:

$$A(\vec{\beta} = \vec{V} / c) = e^{-\vec{\beta}\vec{K}\tanh^{-1}\beta} \,. \tag{B-49}$$

Finally, all fully relativistic phenomena naturally have six independent parameters. For example, electromagnetic fields are described by two 3D vectors: the vector of the electric field and that of the magnetic field, or in equivalent form of an asymmetric 4-tensor of an electromagnetic field with six components. Furthermore, electric fields give charged particles energy boosts, while magnetic field rotates them without changing the energy….

Not surprisingly, the EM fields reflect the structure of the 4D space and its transformations.

Back to the relativistic mechanics…

Let's use Principle of Least Action for a relativistic particle. To determine *the action integral for a free particle* (which does not interact with the rest of the world), we must ensure that the action integral does not depend on our choice of the inertial system. Otherwise, the laws of the particle motion also will depend on the choice of the reference system, which contradicts the first principle of relativity. Therefore, the action must be invariant of Lorentz transformations and rotation in 3D space; i.e., it must depend on a 4D scalar. So far, *from Appendix A*, we know of one 4D scalar for a free particle: the interval. We can employ it as trial function for the action integral, and, by comparing the result with classical mechanics find a constant α connecting the action with the integral of the interval:

$$ds^2 = dx^i dx_i \equiv \sum_{i=1}^{4} dx^i dx_i = (cdt)^2 - (d\vec{r})^2$$

$$S = -\alpha \int_A^B ds = -\alpha \int_A^B \sqrt{(cdt)^2 - d\vec{r}^{\,2}}\,. \qquad (16)$$

The minus sign before the integral reflects a natural phenomenon: the law of inertia requires a resting free particle to stay at rest in inertial system. The interval $ds = cdt$ has a maximum possible value ($cdt \ge \sqrt{(cdt)^2 - d\vec{r}^{\,2}}$) and requires for the action to be minimal, that the sign is set to be "-".

The integral (16) is taken along the world line of the particle. The initial point A (event) determines the particle's start time and position, while the final point B (event) determines its final time and position. The action integral (16) can be represented as integral with respect to the time:

$$S=-\alpha\int_A^B\sqrt{(cdt)^2-d\vec{r}^{\,2}}=-\alpha c\int_A^B dt\sqrt{1-\vec{v}^{\,2}/c^2}=\int_A^B \mathbf{L}dt\,;\qquad \mathbf{L}=-\alpha c\sqrt{1-\frac{\vec{v}^{\,2}}{c^2}}\,;\ \vec{v}=\frac{d\vec{r}}{dt}\,;$$

where $\mathbf{L}$ signifies the Lagrangian function of the mechanical system. It is important to note that while the action is an invariant of the Lorentz transformation, the Lagrangian is not. It must depend on the reference system because time depends on it. To find coefficient α, we compare the relativistic form with the known classical form by expanding $\mathbf{L}$ by $\vec{v}^{\,2}/c^2$:

$$\mathbf{L}=-\alpha c\sqrt{1-\frac{\vec{v}^{\,2}}{c^2}}\approx-\alpha c+\alpha\frac{\vec{v}^{\,2}}{2c};\qquad \mathbf{L}_{classical}=m\frac{\vec{v}^{\,2}}{2}\,;$$

which confirms that α is positive and $\alpha=mc$, where m is the mass of the particle. Thus, we found the action and the Lagrangian for a relativistic particle:

$$S=-mc\int_A^B ds\,; \tag{17}$$

$$\mathbf{L}=-mc^2\sqrt{1-\frac{\vec{v}^{\,2}}{c^2}}\,; \tag{18}$$

The energy and momentum of the particles are defined by the standard relations eqs. (4) and (5):

$$\vec{p}=\frac{\partial\mathbf{L}}{\partial\vec{v}}=\frac{m\vec{v}}{\sqrt{1-\frac{\vec{v}^{\,2}}{c^2}}}=\gamma m\vec{v}; \tag{19}$$

$$E=\vec{p}\vec{v}-L=\gamma mc^2;\ \ \gamma=1/\sqrt{1-\vec{v}^{\,2}/c^2} \tag{20}$$

with ratio between them of

$$E^2=\vec{p}^{\,2}c^2+\left(mc^2\right)^2. \tag{21}$$

Four-momentum, conservation laws. The least-action principle gives us the equations of motion and an expression for the momentum of a system. Let us consider the total variation of an action for a single particle:

$$\delta S = -mc\delta\int_A^B ds = -mc\delta\int_A^B \sqrt{dx^i dx_i} = -mc\{\int_A^B \sqrt{d\delta x^i dx_i} + \sqrt{dx^i d\delta x_i}\} =$$

$$-mc\{\int_A^B \frac{d\delta x^i dx_i}{2\sqrt{dx^i dx_i}} + \frac{dx^i d\delta x_i}{2\sqrt{dx^i dx_i}}\} = -mc\int_A^B \frac{dx^i d\delta x_i}{ds} = -mc\int_A^B u^i d\delta x_i;$$

where $u^i \equiv dx^i / ds$ is 4-velocity. Integrating by parts,

$$\delta S = -mcu^i \delta x_i \Big|_A^B + mc\int_A^B \delta x_i \frac{du^i}{ds} ds; \tag{22}$$

we obtain the expression that can be used for all purposes. First, using the least-action principle with fixed A and B $\delta x_i(A) = \delta x_i(B) = 0$, to derive the conservation of 4-velocity for a free particle: $\frac{du^i}{ds} = 0;\ u^i = const$ or <u>**the inertia law.**</u>

Along a real trajectory $mc\int_A^B \delta x_i \frac{du^i}{ds} ds = 0$ the action is a function of the limits A and B (see eq. (12): $\delta S_{real\ traj} = (-E\delta t + \vec{P}\delta\vec{r})\Big|_A^B$, i.e., $dS_{real traj} = -Edt + \vec{P}d\vec{r}$ is the full differential of t and $\vec{r}$ with energy and momentum as the parameters. We note that this form of the action already is a Lorentz invariant:

$$\delta S_{real\ traj} = (-E\delta t + \vec{P}\delta\vec{r})\Big|_A^B = (-P^i\delta x_i)\Big|_A^B;$$

<u>**i.e. classical Hamiltonian mechanics always encompassed a relativistic form and a metric: a scalar δS is a 4-product of P^i and δx_i with the metric (1,-1,-1,-1).**</u> **Probably one of most remarkable things in physics is that its classic approach detected the metric of 4-D space and time at least a century before Einstein and Poincaré.**

To get 4-momentum, we consider a real trajectory $mc\int_A^B \delta x_i \frac{du^i}{ds} ds = 0$ and set $\delta x_i(B) = \delta x_i$:

$$p^i = -\frac{\partial S}{\partial x_i} = -\partial^i S = mcu^i = \left(\gamma mc, \gamma m\vec{v}\right) = \left(E / c, \vec{p}\right) \qquad (23)$$

with an obvious scalar product ($u^i u_i = 1$, see Appendix A. eq. (A.42))

$$p^i p_i = E^2 / c^2 - \vec{p}^{\,2} = m^2 c^2 u^i u_i = m^2 c^2 \ . \qquad (24)$$

Equivalent forms of presentation are

$$p^i = (E / c, \vec{p}) \equiv m\gamma_v (c, \vec{v}) \equiv \frac{(mc, m\vec{v})}{\sqrt{1 - v^2 / c^2}} \qquad (25)$$

and, Lorentz transformation (P^i is a 4-vector, K' moves with $\vec{V} = \hat{e}_x V$):

$$E = \gamma_V (E' + c\beta_V p'_x); p_x = \gamma_V (p'_x + \beta_V E' / c); p_{y,z} = p'_{y,z}; \gamma_V = 1 / \sqrt{1 - \beta_V^{\ 2}}; \beta_V = V / c; \qquad (26)$$

where subscripts are used for γ, β to define the velocity to which they are related. .

Equation (24) expresses energy, velocity, and the like in terms of momenta and allows us to calculate all differentials:

$$E = c\sqrt{\vec{p}^2 + m^2c^2};\, dE = cd\sqrt{\vec{p}^2 + m^2c^2} = \frac{d\vec{p}\cdot c\vec{p}}{\sqrt{\vec{p}^2 + m^2c^2}} = \frac{c^2\vec{p}\cdot d\vec{p}}{E} = \vec{v}\cdot d\vec{p}; \tag{27}$$

$$\vec{v} = \frac{c\vec{p}}{\sqrt{\vec{p}^2 + m^2c^2}};\, \vec{a}dt = d\vec{v} = d\frac{c\vec{p}}{\sqrt{\vec{p}^2 + m^2c^2}} =$$
$$\frac{c\left(d\vec{p}(\vec{p}^2 + m^2c^2) - \vec{p}(\vec{p}d\vec{p})\right)}{\left(\sqrt{\vec{p}^2 + m^2c^2}\right)^3} = c\frac{d\vec{p}\cdot m^2c^2 + \left[\vec{p}\times\left[d\vec{p}\times\vec{p}\right]\right]}{\left(\sqrt{\vec{p}^2 + m^2c^2}\right)^3}; \tag{28}$$

Coefficients $\gamma = E/mc^2;\ \vec{\beta} = \vec{v}/c$ differ from the above by constants, and satisfy similar relations.

The conservation laws reflect the homogeneity of space and time (see Mechanics): these natural laws do not change even if the origin of the coordinate system is shifted by δx . Then, $\delta x_i(A) = \delta x_i(B) = \delta x_i$. We can consider a closed system of particles (without continuous interaction, i.e., for most of the time they are free). Their action is sum of the individual actions, and

$$\sum_a \delta S_a = -(\sum_a m_a cu^i{}_a)\delta x_i\Big|_A^B = -(\sum_a m_a cu^i{}_a)\delta x_i\Big|_A^B = \{\sum_a p^i{}_a(A) - \sum_a p^i{}_a(B)\}\delta x_i = 0 \tag{29}$$

$$\sum_a p^i{}_a(A) = \sum_a p^i{}_a(B) = \left(\sum_a E_a / c, \sum_a \vec{p}\right) = const\,. \tag{30}$$

Relativism -> E&M

1.2 Particles in the 4-potential of the EM field.

The EM field propagates with the speed of light, i.e., it is a natural product of relativistic 4-D space-time; hence, the 4-potential is not an odd notion!

In contrast with the natural use of the interval for deriving the motion of the free relativistic particle, there is no clear guideline on what type of term should be added into action integral to describe a field. It is possible to consider some type of scalar function $\int A(x^i)ds$ to describe electromagnetic fields, but this would result in wrong equations of motion. Nevertheless, the next guess is to use a product of 4-vectors $A^i dx_i$, and surprisingly it does work, even though we do not know why? **Hence**, **the fact that electromagnetic fields are fully described by the 4-vector of potential $\boxed{A^i = (A^0, \vec{A})}$ must be considered as an experimental fact!**

Nevertheless, it looks natural that the interaction of a charge with electromagnetic field is represented by the scalar product of two 4-vectors with the $-e/c$ coefficient chosen by convention:

$$S_{\text{int}} = -\frac{e}{c}\int_A^B A^i dx_i\ ;\ A^i \equiv (A^o, \vec{A}) \equiv (\varphi, \vec{A}) \qquad (31)$$

where the integral is taken along the particle's world line. A charge e and speed of the light c are moved outside the integral because they are constant; hence, **we use the conservation of the charge** e **and constancy of the speed of the light !**

IT IS ESSENTIAL THAT FIELD IS GIVEN, SINCE WE ARE CONSIDERING A PARTICLE INTERACTING WITH A GIVEN FIELD.

Turning our attention back to the Least-Action Principle and Hamiltonian Mechanics

The standard presentation of 4-potential is

$$A^i \equiv (A^0, \vec{A}) \equiv (\varphi, \vec{A}) \; ; \tag{32}$$

where φ is called the scalar potential and $\vec{A}$ is termed the vector potential of electromagnetic field.

Gauge Invariance. As we discussed earlier the action integral is not uniquely defined; we can add to it an arbitrary function of coordinates and time without changing the motion: $S' = S + f(x_i)$. This corresponds to adding the full differential of f in the integral (31)

$$S' = \int_A^B \left(-mcds - \frac{e}{c} A^i dx_i + dx_i \partial^i f \right).$$

This signifies that the 4-potential is defined with sufficient flexibility to allow the addition of any 4-gradient to it (let us choose $f(x_i) = \frac{e}{c} g(x_i)$)

$$A'^i = A^i - \partial^i g(x_i) = A^i - \frac{\partial g}{\partial x_i}; \tag{33}$$

without affecting the motion of the charge, a fact called **THE GAUGE INVARIANCE .**

WE SHOULD BE AWARE THAT THE EVOLUTION OF THE SYSTEM DOES NOT CHANGE BUT APPEARANCE OF THE EQUATION OF THE MOTION FOR THE SYSTEM COULD CHANGE. FOR EXAMPLE, AS FOLLOWS FROM (33), THE CANONICAL MOMENTA WILL CHANGE:

$$P'^i = P^i - \partial^i f \; .$$

Nevertheless, only the appearance of the system is altered, not its evolution. Measurable values (such as fields, mechanical momentum) do not depend upon it. One might consider Gauge invariance as an inconvenience, but, in practice, it provides a great opportunity to find a gauge in which the problem becomes more comprehensible and solvable.

The action is an additive function: therefore, the action of a charge in electromagnetic field is simply the direct sum of a free particle's action and action of interaction: (remember $ds = ds^2 / ds = dx^i dx_i / ds = u^i dx_i$)

$$S = \int_A^B \left(-mcds - \frac{e}{c} A^i dx_i \right) = \int_A^B \left(-mcu^i - \frac{e}{c} A^i \right) dx_i \tag{34}$$

Then the total variation of the action is

$$\begin{aligned} \delta S = \delta \int_A^B \left(-mcds - \frac{e}{c} A^i dx_i \right) = \int_A^B \left(-mc \frac{dx^i d\delta x_i}{ds} - \frac{e}{c} A^i d\delta x_i - \frac{e}{c} \delta A^i dx_i \right) = \\ -\left[\left(mcu^i + \frac{e}{c} A^i \right) \delta x_i \right]_A^B + \int_A^B \left(mc \frac{du^i}{ds} \delta x_i ds + \frac{e}{c} \delta x_i dA^i - \frac{e}{c} \delta A^i dx_i \right) = 0. \end{aligned} \tag{35}$$

That gives us a 4-momentum

$$P^i = -\frac{\delta S}{\delta x_i} = \left(mcu^i + \frac{e}{c} A^i \right) = \left(H / c, \vec{P} \right) = p^i + \frac{e}{c} A^i; \tag{36}$$

with

$$\begin{aligned} H = E = c(mcu^0 + \frac{e}{c} A^0) = \gamma mc^2 + e\varphi = c\sqrt{m^2c^2 + \vec{p}^2} + e\varphi; \\ \vec{P} = \gamma m\vec{v} + \frac{e}{c} \vec{A} = \vec{p} + \frac{e}{c} \vec{A}; \Rightarrow \vec{p} = \vec{P} - \frac{e}{c} \vec{A}. \end{aligned} \tag{37}$$

The Hamiltonian must be expressed in terms of generalized 3-D momentum, $\vec{P} = \vec{p} + \frac{e}{c}\vec{A}$ and it is

$$H(\vec{r},\vec{P},t) = c\sqrt{m^2c^2 + \left(\vec{P} - \frac{e}{c}\vec{A}\right)^2} + e\varphi; \qquad (38)$$

with Hamiltonian equation following from it:

$$\vec{\mathrm{v}} = \frac{d\vec{r}}{dt} = \frac{\partial H}{\partial \vec{P}} = \frac{\vec{P}c - e\vec{A}}{\sqrt{m^2c^2 + \left(\vec{P} - \frac{e}{c}\vec{A}\right)^2}}$$

$$\frac{d\vec{P}}{dt} = \frac{d\vec{p}}{dt} + \frac{e}{c}\frac{d\vec{A}}{dt} = -\frac{\partial H}{\partial \vec{r}} = -e\vec{\nabla}\varphi - e\frac{\{(\vec{P} - \frac{e}{c}\vec{A})\cdot\vec{\nabla}\}\vec{A}}{\sqrt{m^2c^2 + \left(\vec{P} - \frac{e}{c}\vec{A}\right)^2}} = -e\vec{\nabla}\varphi - \frac{e}{c}(\vec{v}\cdot\vec{\nabla})\vec{A};$$

From this equation we can derive (without any elegance!) the equation for mechanical momentum $\vec{p} = \gamma m\vec{v}$. We will not do it here, but rather we will use easier way to obtain the 4D equation of motion via the least-action principle. We fix A and B to get from equation (35)

$$\delta S = \int_A^B \left(mcu^i \delta x_i ds + \frac{e}{c} \delta x_i dA^i - \frac{e}{c} \delta A^k dx_k \right) = \int_A^B \left(mc \frac{du^i}{ds} \delta x_i ds + \frac{e}{c} \frac{\partial A^i}{\partial x_k} \delta x_i dx_k - \frac{e}{c} \frac{\partial A^k}{\partial x_i} \delta x_i dx_k \right) =$$
$$\int_A^B \left(\frac{dp^i}{ds} + \frac{e}{c} \left\{ \frac{\partial A^i}{\partial x_k} - \frac{\partial A^k}{\partial x_i} \right\} u_k \right) \delta x_i ds = 0. \quad (39)$$

As usual, the expression inside the round brackets must be set at zero to satisfy (39); i.e., we have the equations of charge motion in an electromagnetic field:

$$mc \frac{du^i}{ds} \equiv \frac{dp^i}{ds} = \frac{e}{c} F^{ik} u_k; \quad (40)$$

wherein we introduce an anti-symmetric **electromagnetic field tensor**

$$F^{ik} = \frac{\partial A^k}{\partial x_i} - \frac{\partial A^i}{\partial x_k}. \quad (41)$$

Electromagnetic field tensor: The Gauge Invariance can be verified very easily:

$$F'^{ik} = \frac{\partial A'^k}{\partial x_i} - \frac{\partial A'^i}{\partial x_k} = F^{ik} - \frac{\partial^2 g}{\partial x_i \partial x_k} + \frac{\partial^2 g}{\partial x_k \partial x_i} = F^{ik};$$

which means that the equation of motion (40) is not affected by the choice of the gauge, and the **electromagnetic field tensor is defined uniquely!** Using the Landau convention, we can represent the asymmetric tensor by two 3-vectors (see Appendix A):

$$F^{ik} = (-\vec{E},\vec{B}); F_{ik} = (\vec{E},\vec{B});$$

$$F^{ik} = \begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{bmatrix}. \tag{42}$$

$\vec{E}$ is the so-called vector of the electric field and $\vec{B}$ is the vector of the magnetic field. Note the occurrence of the Lorentz group generator (see Appendix B) in (42). The 3D expressions of the field vectors can be obtained readily:

$$E^{\alpha} = F^{\alpha 0} = \frac{\partial A^0}{\partial x_\alpha} - \frac{\partial A^\alpha}{\partial x_0} - = -\frac{\partial \varphi}{\partial r_\alpha} - \frac{1}{c}\frac{\partial A^\alpha}{\partial t}; \ \alpha = 1,2,3; \vec{E} = -\frac{1}{c}\frac{\partial \vec{A}}{\partial t} - grad\varphi; \tag{43}$$

$$B^{\alpha} = -\frac{1}{2} e^{\alpha\kappa\lambda} F^{\kappa\lambda} = e^{\alpha\kappa\lambda}\left(\frac{\partial A^\lambda}{\partial x_\kappa} - \frac{\partial A^\kappa}{\partial x_\lambda}\right); \ \vec{B} = curl\vec{A}; \ F^{\kappa\lambda} = e^{\lambda\kappa\alpha} H_\alpha. \tag{44}$$

First pair of Maxwell equations – *free of charge*

A 3D asymmetric tensor $e^{\alpha\kappa\lambda}$ and the *curl* definition are used to derive last equation and use Greek symbols for the spatial 3D components. The electric and magnetic fields are also Gauge invariant being components of Gauge invariant tensor.

We have the first pair of Maxwell's equations without further calculation using the fact that differentiation is symmetric operator ($\partial^i\partial^k \equiv \partial^k\partial^i$):

$$e_{iklm}\partial^k F^{lm} = e_{iklm}\partial^k\left(\partial^l A^m - \partial^m A^l\right) = 2e_{iklm}(\partial^k\partial^l)A^m = 0; \qquad (45)$$

or explicitly:

$$\partial^k F^{lm} + \partial^l F^{mk} + \partial^m F^{kl} = 0. \qquad (46)$$

A simple exercise gives the 3D form of the first pair of Maxwell equations. They also can be attained using (43) and (44) and known 3D equivalencies: $div(curl\vec{A}) \equiv 0$; $curl(grad\varphi) \equiv 0$:

$$\begin{aligned} &\vec{E} = -grad\varphi - \frac{1}{c}\frac{\partial\vec{A}}{\partial t}; && curl\vec{E} = -curl(grad\varphi) - \frac{1}{c}curl\frac{\partial\vec{A}}{\partial t} = -\frac{1}{c}\frac{\partial\vec{B}}{\partial t}; \\ &\vec{B} = curl\vec{A}; && div\vec{B} = div(curl\vec{A}) \equiv 0; \end{aligned} \qquad (47)$$

I note that (47) is the exact 3D equivalent of invariant 4D Maxwell equations (45) that you may wish to verify yourself. There are 4 equations in (45): i=0,1,2,3. The *div* is one equation and *curl* gives three (vector components) equations. Even the 3D form looks very familiar; the beauty and relativistic invariance of the 4D form makes it easy to remember and to use.

EM Fields transformation, Invariants of the EM field. The 4-potential was defined as 4-vector and it transforms as 4-vector. The electric and magnetic fields, as components of the asymmetric tensor, follow its transformation rules (See Appendix A).

$$\begin{gathered}\varphi=\gamma(\varphi'+\beta A'_x);\ \ A_x=\gamma(A'_x+\beta\varphi');\\ E_y=\gamma(E'_y+\beta B'_z);\ E_z=\gamma(E'_z-\beta B'_y);\\ B_y=\gamma(B'_y-\beta E'_z);\ B_z=\gamma(B'_z+\beta E'_y).\end{gathered} \tag{48}$$

and the rest is unchanged. An important repercussion from these transformations is that the separation of the electromagnetic field in two components is an artificial one. They translate into each other when the system of observation changes and **MUST** be measured in the same units (Gaussian). The rationalized international system of units (SI) system measures them in V/m, Oe, A/m and T. Why not use also a horse power per square mile an hour, the old British thermal units as well? This makes about the same sense as using Tesla or A/m.

While the values and directions of 3D field components are frame-dependent, two 4-scalars can be build from the EM 4-tensor $F^{ik}=(-\vec{E},\vec{B})$

$$F^{ik}F_{ik}=inv;\qquad e^{iklm}F_{ik}F_{lm}=inv; \tag{49}$$

which in the 3D-form appear as

$$\vec{B}^2-\vec{E}^2=inv;\ (\vec{E}\cdot\vec{B})=inv. \tag{50}$$

This conveys a good sense what can and cannot be done with the 3D components of electromagnetic fields. Any reference frame can be chosen and both fields transferred in a minimal number of components limited by (50). For example; 1) if $|\vec{E}|>|\vec{B}|$ in one system it is true in all systems and vice versa; and (2) if fields are perpendicular in one frame, $(\vec{E}\cdot\vec{B})=0$, this is true in all frames. When $(\vec{E}\cdot\vec{B})=0$ a frame can always be found where E or B are equal to zero (locally!).

Lorentz form of equation of a charged particle's motion.

The equations of motion (40) can rewritten in the form:

$$\frac{d\mathrm{E}}{dt} = c\frac{dp^0}{dt} = eF^{0k}\mathrm{v}_k = e\vec{E}\cdot\vec{\mathrm{v}}; \qquad \mathrm{v}_k = \frac{dx_k}{dt} = (c, -\vec{\mathrm{v}})$$
$$\frac{d\vec{p}}{dt} = e\left(\hat{e}_\alpha F^{\alpha k}\frac{\mathrm{v}_k}{c}\right) = \frac{e}{c}\left(\hat{e}_\alpha \cdot cF^{\alpha 0} - \hat{e}_\alpha \cdot F^{\alpha\kappa}\mathrm{v}_k\right) = e\vec{E} + \hat{e}_\alpha e^{\alpha\kappa\lambda}B_\lambda\frac{\mathrm{v}_k}{c} = e\vec{E} + \frac{e}{c}\left[\vec{\mathrm{v}}\times\vec{B}\right]. \tag{51}$$

So, we have expressions for the generalized momentum and energy of the particle in an electromagnetic field. Generalized momentum is equal to the particle's mechanical momentum plus the vector potential scaled by *e/c*. The total energy of the charged particle is its mechanical energy, γmc^2, plus its potential energy , $e\varphi$, in an electromagnetic field. The Standard Lorentz (not Hamiltonian!) equations of motion for $\vec{p} = \gamma m\vec{v}$ are

$$\frac{d\vec{p}}{dt} = e\vec{E} + \frac{e}{c}\left[\vec{\mathrm{v}}\times\vec{B}\right]. \tag{52}$$

with the force caused by the electromagnetic field (Lorentz force) comprised of two terms: the electric force, which does not depend on particle's motion, and, the magnetic force that is proportional to the vector product of particle velocity and the magnetic field, i.e., it is perpendicular to the velocity. Accordingly, the magnetic field does not change the particle's energy. We derived it in Eq. (51):

$$mc^2\frac{d\gamma}{dt} = e\vec{E}\cdot\vec{\mathrm{v}}; \tag{53}$$

Eqs. (52) and (53) are generalized equations. Using directly standard Lorentz equations of motion in a 3D form is a poor option. The 4D form is much better (see below) and, from all points of view, the Hamiltonian method is much more powerful!

Prelude of things to come

$$X = \begin{bmatrix} x_1 \\ x_2 \\ .... \\ x_{n-1} \\ x_n \end{bmatrix}; \frac{dX}{dt} = D \cdot X; \quad D \begin{bmatrix} d_{11} & ... & ... & d_{n1} \\ ... & ... & ... & ... \\ ... & ... & ... & ... \\ d_{1n} & ... & ... & d_{nn} \end{bmatrix};$$

$$X(t) = e^{D \cdot t} \cdot X(0) \equiv M \cdot X(0);$$

$$\frac{dM}{dt} = D \cdot M; \quad M(t) = e^{D \cdot t}; \quad M(t_1 | t_2) = e^{D \cdot (t_2 - t_1)}$$

Unusual twist.

It is worth noting that the 4D form of the charge motion (40) and its matrix form is the most compact one,

$$u^i = \frac{dx^i}{ds};\ mc\frac{du^i}{ds} = \frac{e}{c}F^i{}_k u^k; \Rightarrow \frac{d}{ds}[x] = [I]\cdot[u];\ \frac{d}{ds}[u] = \frac{e}{mc^2}[F]\cdot[u] \tag{54}$$

and, in many cases, it is very useful. We treat the ***x, u*** as a vectors, and ***[F]*** as the 4x4 matrix. ***[I]*** is just the unit 4x4 matrix It has interesting formal solution in the matrix form:

$$[u] = e^{\int \frac{e}{mc^2}[F]ds}[u_0];\ [x] = [x_o] + \left[\int dse^{\int \frac{e}{mc^2}[F]ds}\right][u_0] \tag{55}$$

Its resolution is well defined when applied to the motion of a charged particle in uniform, constant EM field:

$$[u] = e^{\frac{e}{mc^2}[F](s-s_0)}[u_0];\ [x] = [x_o] + \left[\int e^{\frac{e}{mc^2}[F](s-s_0)}dse\right][u_0] \tag{56}$$

The Lorentz group of theoretical physics (see Appendix B) is fascinating, and the fact that EM field tensor has the same structure as the generator of Lorentz group is no coincidence – rather, it is indication that physicists have probably come very close to the roots of nature in this specific direction. This statement is far from truth for other fundamental forces and interactions.

To conclude this subsection, we will take one step further from (54) and write a totally linear evolution equation for a combination of 4D vectors

$$\frac{d}{ds}\begin{bmatrix} x \\ u \end{bmatrix} = [\Lambda]\cdot\begin{bmatrix} x \\ u \end{bmatrix};\ [\Lambda] = \begin{bmatrix} 0 & I \\ 0 & \frac{e}{mc^2}F \end{bmatrix} \tag{57}$$

where [Λ] is an 8x8 degenerated matrix. Similarly to (55) and (56)

$$\begin{bmatrix} x \\ u \end{bmatrix} = e^{\int [\Lambda]ds}\cdot\begin{bmatrix} x \\ u \end{bmatrix}_o;\ \begin{bmatrix} x \\ u \end{bmatrix} = e^{[\Lambda](s-s_o I}\cdot\begin{bmatrix} x \\ u \end{bmatrix}_o\ for\ [\Lambda] = const; \tag{58}$$

First pair of Maxwell's equations (a little more of juice)

We will derive full set of Maxwell equations using the least action principle. Nevertheless, you can consider the Maxwell equation as given - in any case they were derived originally from numerous experimental laws!

First pair of Maxwell's equations is the consequence of definitions of electric and magnetic field through the 4-potential:

$$\vec{E} = -grad\varphi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t}; \qquad \text{it is equivalent to} \qquad curl\vec{E} = -curl(grad\varphi) - \frac{1}{c}curl\frac{\partial \vec{A}}{\partial t} = \frac{1}{c}\frac{\partial \vec{H}}{\partial t};$$

$$\vec{H} = curl\vec{A}; \qquad div\vec{H} = div(curl\vec{A}) \equiv 0; \tag{59}$$

Nevertheless, it is very important to remember that they are actually originated from experiment. First Maxwell equation is the Faraday law and the second is nothing else that absence of magnetic charge! You should remember all time that inclusion of the term $S_{\text{int}} = -\frac{e}{c}\int_A^B A^i dx_i$ into action integral is consequence of experiment! Thus, the first pair of Maxwell equations governing the electromagnetic fields is:

$$curl\vec{E} = -\frac{1}{c}\frac{\partial \vec{H}}{\partial t}; \tag{60}$$

$$div\vec{H} = 0; \tag{61}$$

with well known integral ratios following it:

Gauss' theorem: $$\oint \vec{H}d\vec{a} = \int div\vec{H}dV = 0; \tag{62}$$

Stokes' theorem: $$\oint \vec{E}d\vec{l} = \int curl\vec{E}d\vec{a} = -\frac{1}{c}\frac{\partial}{\partial t}\int \vec{H}d\vec{a}; \tag{63}$$

where $d\vec{a}$ is vector of the element of the surface and $d\vec{l}$ is a vector of a contour length. Integral equations read: the

1) Flux of the of the magnetic field though the surface covering any volume V is equal zero;

2) The circulation of electric field around the contour (electromotive force) is equal to the derivative of the magnetic flux though the contour scaled down by "-c" - the Faraday law.