PHY 564
Advanced Accelerator Physics
Lectures 23

Vladimir N. Litvinenko
Yichao Jing
Gang Wang

Department of Physics & Astronomy, Stony Brook University
Collider-Accelerator Department, Brookhaven National Laboratory

Fokker-Plank equation. Distribution function of particles.

This probabilistic approach for Fokker-Plank equation follows §83 from *Thermodynamics, Statistical Physics and Kinetics* by Yu. Rumer and M. Ryvkin, Nauka, RAN, 2001 [TSPK]. English translation: Mir, Moscow, 1980

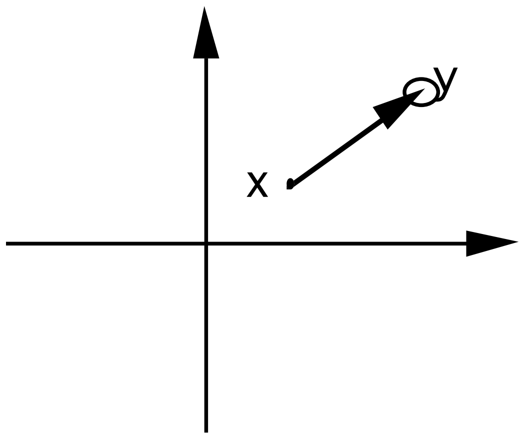
1. Particles described by a distribution function in the Phase Space $x = \{\vec{r}, \vec{P}\}$:

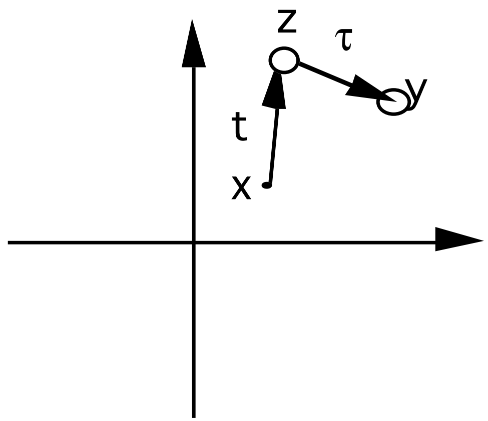
$$f\left(\vec{r}, \vec{P}, t\right) \equiv f(x,t); x = \{\vec{r}, \vec{P}\}: \quad \int f\left(\vec{r}, \vec{P}, t\right) d\vec{r}\, d\vec{P} = 1:$$

$$\Rightarrow \rho(\vec{r}, t) = \int f\left(\vec{r}, \vec{P}, t\right) d\vec{P}; \quad n\left(\vec{P}, t\right) = \int f\left(\vec{r}, \vec{P}, t\right) d\vec{r};$$

2. Markov's chain: no dependence on pre-history of the event=> ***hypothesis****: correlations exist only between two consequent events: the probability to "move" from point x in the phase space to y during time τ depends only on $\{x, y, t, \tau\}$:*

$$dw = W\left(y, x \middle| \tau, t_o\right) dy; \quad dy = d\vec{r}_y d\vec{P}_y.$$





There t is time for move $x \Rightarrow z$; τ is time for move $z \Rightarrow y$. Two events are independent and total probability is product of two probabilities: $x \Rightarrow z$ and then $z \Rightarrow y$:

$$W\left(y, z \middle| \tau, t_o + t\right) dy W\left(z, x \middle| t, t_o\right) dz$$

To find probability $W\left(y, x \middle| t + \tau, t_o\right)$ it is sufficient to integrate over all z :

$$W\left(y, x \middle| t + \tau, t_o\right) = \int dz W\left(y, z \middle| \tau, t_o + t\right) dy W\left(z, x \middle| t, t_o\right). \tag{23-1}$$

This is Smolukhovsky equation. **Fokker-Plank Equation** can be derived from (23-1) in following form ($t_o = 0$):

$$W\left(y, x \middle| t + \tau, 0\right) = \int dz W\left(y, z \middle| \tau, t\right) dy W\left(z, x \middle| t, 0\right). \tag{23-2}$$

Lets consider an analytical (integral-able) function $g(x)$, which is limited in all phase space and goes to zero with all it derivatives at the infinity (i.e. we use a finite system):

$$g(x) \quad \Rightarrow \quad g(x) \to 0; \quad \frac{\partial^n g(x)}{\prod_{k=1}^{n} \partial x_{i_k}} \to 0; \quad |x| \to \infty.$$

We should keep in mind that $g(x)$ can be a distribution function and these properties are natural for finite system with finite energy: $g(x)$ *when* $|\vec{r}| > r_{\max}$ *or* $|\vec{P}| > P_{\max}$ ($E_{\max}$). Multiplying (23-2) by $g(y)$ and integrating it give us:

$$\int g(y) W(y, x|t+\tau, 0) dy = \iint g(y) W(y, z|\tau, t) dy W(z, x|t, 0) dy dz. \tag{23-3}$$

$g(y)$ can be expanded into Taylor series:

$$g(y) = g(z) + \frac{\partial g}{\partial z_i}(y_i - z_i) + \frac{1}{2}\frac{\partial^2 g}{\partial z_i \partial z_k}(y_i - z_i)(y_k - z_k) + \ldots..$$

(summation is assumed on repeated indexes)

$$\int g(y) W(y, x|t+\tau, 0) dy =$$

$$\iint \left(g(z) + \frac{\partial g}{\partial z_i}(y_i - z_i) + \frac{1}{2}\frac{\partial^2 g}{\partial z_i \partial z_k}(y_i - z_i)(y_k - z_k) + \ldots.. \right) W(y, z|\tau, t) dy W(z, x|t, 0) dy dz. \tag{23-4}$$

Taking into account that:

$$\int g(z)W\left(y,x\middle|\tau,t\right)dy = g(z); \quad \int W\left(y,x\middle|\tau,t\right)dy \equiv 1;$$

$$\int g(y)W\left(y,x\middle|t+\tau,0\right)dy - \int g(z)W\left(z,x\middle|t,0\right)dz = \int g(y)\{W\left(y,x\middle|t+\tau,0\right) - W\left(y,x\middle|t,0\right)\}dy$$

we can rewrite (23-4) in following from:

$$\int g(y)\frac{W\left(y,x\middle|t+\tau,0\right) - W\left(y,x\middle|t,0\right)}{\tau}dy -$$

$$-\int a_i^{(\tau)}(y,t)\frac{\partial g}{\partial y_i}W\left(y,x\middle|t,0\right)dy$$

$$-\int b_{ik}^{(\tau)}(y,t)\frac{\partial^2 g}{\partial y_i \partial y_k}W\left(y,x\middle|t,0\right)dy - \ldots\ldots = 0.$$

where we introduce following notations:

$$a_i^{(\tau)}(y,t) = \frac{1}{\tau}\int (z_i - y_i)W\left(z,y\middle|\tau,t\right)dz;$$

$$b_{ik}^{(\tau)}(y,t) = \frac{1}{\tau}\int (z_i - y_i)(z_k - y_k)W\left(z,y\middle|\tau,t\right)dz;$$

....

2n-D vector $a=\left\{a_i^{(\tau)}\right\}$ is an “average speed” particles’ point on the in Poincaré plot in the phase space. Integration by parts gives us following: $b_{ik}^{(\tau)}$ is 2n-D tensor representing correlations between variations of i's and k’s components of $x=\{\vec{r},\vec{P}\}$ with the tensor’s trace giving RMS drift of the point $b_{ii}^{(\tau)}(y,t)=\dfrac{1}{\tau}\int\left(z_i-y_i\right)^2 W\left(z,y\middle|\tau,t\right)dz$.

Integrating by parts (here we use the boundary condition for finite system!):

$$\int a_i^{(\tau)}(y,t)\frac{\partial g}{\partial y_i}W\left(y,x\middle|t,0\right)dy=$$

$$\int\frac{\partial}{\partial y_i}\{a_i^{(\tau)}(y,t)W\left(y,x\middle|t,0\right)g(y)\}dy-\int g(y)\frac{\partial}{\partial y_i}\{a_i^{(\tau)}(y,t)W\left(y,x\middle|t,0\right)\}dy;$$

$$\int b_{ik}^{(\tau)}(y,t)\frac{\partial^2 g}{\partial y_i\partial y_k}W\left(y,x\middle|t,0\right)dy=\int\frac{\partial}{\partial y_i}\{b_{ik}^{(\tau)}(y,t)\frac{\partial g}{\partial y_k}W\left(y,x\middle|t,0\right)\}dy-$$

$$-\int\frac{\partial}{\partial y_i}\{g(y)\frac{\partial}{\partial y_k}[b_{ik}^{(\tau)}(y,t)W\left(y,x\middle|t,0\right)]\}dy+\int g(y)\frac{\partial^2}{\partial y_i\partial y_k}\{b_{ik}^{(\tau)}(y,t)W\left(y,x\middle|t,0\right)\}dy.$$

$$\int\frac{\partial}{\partial y_i}h(y)\prod_{k=1,\ldots 6}dy_k=\int\prod_{k\neq i}dy_k\int\frac{\partial}{\partial y_i}h(y)dy_k=\int\prod_{k\neq i}dy_k\{h(y_{k\neq i},y_i=+\infty)-h(y_{k\neq i},y_i=-\infty)\}=0.$$

$$\int g(y)\left\{\frac{\partial W(y,x|t,0)}{\partial t}+\frac{\partial}{\partial y_i}\left[a_i^{(\tau)}(y,t)W(y,x|t,0)\right]-\frac{1}{2}\frac{\partial^2}{\partial y_i\partial y_k}\left[b_{ik}^{(\tau)}(y,t)W(y,x|t,0)\right]\right\}dy=0.$$

$g(y)$ is arbitrary function which requires the expression in the brackets to be zero:

$$\frac{\partial W(y,x|t,0)}{\partial t}+\frac{\partial}{\partial y_i}\left[a_i^{(\tau)}(y,t)W(y,x|t,0)\right]-\frac{1}{2}\frac{\partial^2}{\partial y_i\partial y_k}\left[b_{ik}^{(\tau)}(y,t)W(y,x|t,0)\right]=0 \qquad (23\text{-}5)$$

This is called mono-molecular kinetic equation of Fokker and Plank. What about the distribution function? The Fokker Plank equation for the distribution function can be derived from this using connection between distribution function $f(x,t)\equiv f(\vec{r},\vec{p},t)$ and probability $W(y,x|\tau,t)$: deviation of the particles density in phase space volume *dx* during time t is equal to the difference between number of particles left this point and arrived into this point:

$$\left[f(x,t)-f(x,0)\right]dx=dx\int\left[W(x,z|t,0)f(z,0)-W(z,x|t,0)f(x,0)\right]dz \qquad (23\text{-}6)$$

Remembering that $\int W(z,x|t,0)dz=1$, we get

$$f(x,t)=\int W(x,z|t,0)f(z,0)dz \qquad (23\text{-}7)$$

which shows that multiplication on $W(y,x|t,0)$ and integrating over the phase space equivalent to a propagation in the phase space by (x-z) and in time by t.

Thus, multiplying (23-5) by $f(x,0)$ and integrating over x we obtaining Fokker-Plank equation for the distribution function:

$$\frac{\partial f(y,t)}{\partial t}+\frac{\partial}{\partial y_i}\left[a_i^{(\tau)}(y,t)f(y,t)\right]-\frac{1}{2}\frac{\partial^2}{\partial y_i \partial y_k}\left[b_{ik}^{(\tau)}(y,t)f(y,t)\right]=0 \qquad (23\text{-}8)$$

This equation also can be written as continuity equations in the phase space:

$$\frac{\partial f(y,t)}{\partial t}+\frac{\partial j_k}{\partial y_k}=0; \quad j_k=\left[a_k^{(\tau)}(y,t)f(y,t)\right]-\frac{1}{2}\frac{\partial^2}{\partial y_i}\left[b_{ik}^{(\tau)}(y,t)f(y,t)\right]. \qquad (23\text{-}9)$$

This is the final form of the of Fokker-Plank equation, where we just should recognize the terms such as motion of the particle and diffusion coefficients D:

$$\frac{\partial f(y,t)}{\partial t}+\frac{\partial}{\partial y_i}\left[\frac{dy_i(y,t)}{dt}f(y,t)\right]-\frac{1}{2}\frac{\partial^2}{\partial y_i \partial y_k}\left[D_{ik}(y,t)f(y,t)\right]=0$$

Finally, nobody told us to use time as independent variable, *s* is as good!

$$\frac{\partial f(y,s)}{\partial s}+\frac{\partial}{\partial y_i}\left[\frac{dy_i(y,s)}{ds}f(y,s)\right]-\frac{1}{2}\frac{\partial^2}{\partial y_i \partial y_k}\left[D_{ik}(y,s)f(y,s)\right]=0 \qquad (23\text{-}10)$$

Effects of synchrotron radiation on particle's distribution

I. Oscillator

Before we embark on detail studies of radiation effects on the beams in accelerators, let's look on a very simple model of harmonic oscillator:

$$H = \frac{P^2}{2m} + k\frac{x^2}{2} \quad or \quad h = \frac{p^2}{2} + \omega^2\frac{x^2}{2}\ ;\ \omega = \sqrt{\frac{k}{m}} \tag{23-11}$$

described by differential equations

$$\begin{gathered} x' = \frac{\partial h}{\partial p} = p;\ p' \equiv x'' = -\frac{\partial h}{\partial x} = -\omega^2 x;\ x = A\cdot\cos(\omega t + \varphi);\ p = -A\omega\cdot\sin(\omega t + \varphi) \\ X' = \begin{bmatrix} x' \\ p' \end{bmatrix} = S\frac{\partial h}{\partial X} = \begin{bmatrix} 0 & 1 \\ -\omega^2 & 0 \end{bmatrix}\cdot X;\ X = \operatorname{Re} aYe^{i(\omega t+\varphi)};\ Y = \begin{bmatrix} 1/\sqrt{\omega} \\ i\sqrt{\omega} \end{bmatrix};\ \left\{ I = \frac{a^2}{2}, \varphi \right\} \end{gathered} \tag{23-12}$$

Let's add a weak friction $\varepsilon << \omega$:

$$\begin{gathered} p = x';\ p' = -\omega^2 x - 2\alpha p;\ X' = D\cdot X = \begin{bmatrix} 0 & 1 \\ -\omega^2 & -2\alpha \end{bmatrix}\cdot X; \\ \det\begin{bmatrix} \lambda & -1 \\ \omega^2 & \lambda + 2\alpha \end{bmatrix} = \lambda(\lambda + 2\alpha) + \omega^2 = 0;\ \lambda = -\alpha \pm i\omega_1;\quad \omega_1 = \sqrt{\omega^2 - \alpha^2}; \\ x = A\cdot e^{-\alpha t}\cdot\cos(\omega_1 t);\ p = -A\cdot e^{-\alpha t}\left(\alpha\cdot\cos(\omega_1 t) + \sin(\omega_1 t)\right); \\ X = \operatorname{Re} aYe^{\lambda t + i\varphi} = a\cdot e^{-\alpha t}\operatorname{Re} Ye^{i\omega_1 t};\ Y = \begin{bmatrix} 1/\sqrt{\lambda} \\ i\sqrt{\lambda} \end{bmatrix}; \end{gathered} \tag{23-13}$$

which make <u>a very small change</u> to the frequency of the oscillations, but make free oscillations slowly decaying.

. Note that damping decrement α is only a half of that of simple decay:

$$p' = -2\alpha p \to p = p_o e^{-2\alpha t}.$$

This is the result of oscillations, where, time-averaged, only half energy is in the kinetic energy $p^2/2$, which decays. The potential energy decays only through its coupling to the kinetic energy via oscillations. The action of the oscillator, *I*, which represent the area of the phase space, decays with the simple decay rate towards zero:

$$I' = \left(\frac{a^2}{2}\right)' = -2\alpha I; I = I_o e^{-2\alpha t}, \qquad (23\text{-}14)$$

while the oscillator phase does not stationary point or any decay.

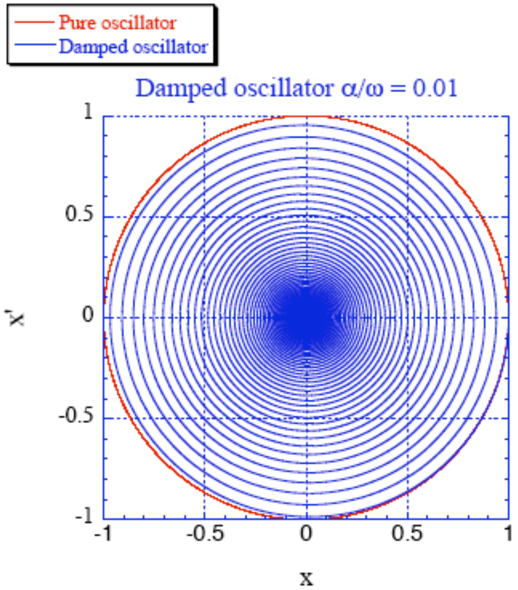


Fig. 23-1 Poincaré plot of trajectories of normal and damped oscillator in dimensionless coordinates x/a; x'/ωa.

Note the second fact, that trace of matrix $D=\begin{bmatrix}0 & 1\\ -\omega^2 & -2\alpha\end{bmatrix}$ gives the damping rate of the oscillator phase space volume. Let's add a random noise to the equations:

$$x' = p + \delta x(t);\ \ p' = -\omega^2 x - 2\alpha p + \delta x'(t);\ \ X' = \begin{bmatrix}0 & 1\\ -\omega^2 & -2\alpha\end{bmatrix}\cdot X + \begin{bmatrix}\delta x(t)\\ \delta x'(t)\end{bmatrix}; \tag{23-15}$$

$$\langle \delta x(t)\rangle = 0;\quad \langle \delta x(t)\rangle = 0.$$

where $\delta x(t), \delta x'(t)$ are "sudden" and randomly distributed in time and amplitude jumps.

One can easily calculate change in the amplitude and phase of the oscillator caused by a random kick:

$$\delta\left(ae^{i\varphi}\right) = -ie^{-i\omega_1 t}Y^{*T}\cdot S\cdot\begin{bmatrix}\delta x\\ \delta x'\end{bmatrix};$$

$$\delta a + ia\delta\varphi \cong -ie^{-i(\omega_1 t+\varphi)}Y^{*T}\cdot S\cdot\left(\delta x'/\sqrt{\omega} - i\delta x\sqrt{\omega}\right) \tag{23-16}$$

$$\langle \delta a\rangle = 0; \langle \delta\delta\varphi\rangle = 0;\ \ \delta I = a\delta a + \delta a^2/2; \langle \delta I\rangle = \left\langle \frac{\delta a^2}{2}\right\rangle = \frac{\langle \delta x'^2\rangle/\omega + \omega\langle \delta x^2\rangle}{2}$$

Thus, the only one thing is well determined – the average change of the action, *I*.

Adding damping term (23-14) to (23-16) we have:

$$\langle I \rangle' = -2\alpha \langle I \rangle + D/2; \quad D = \langle \delta x'^2 \rangle / \omega + \omega \langle \delta x^2 \rangle; \tag{23-17}$$

with stationary solution for average action (emittance) and RMS amplitude of the ensemble of oscillators:

$$\langle I \rangle = \frac{D}{4\alpha}, i.e.\ \varepsilon = \langle a^2 \rangle = \frac{D}{2\alpha} = \frac{\langle \delta x'^2 \rangle / \omega + \omega \langle \delta x^2 \rangle}{2\alpha}, \tag{23-18}$$

where ε is called emittance – phase space area occupied divided by π - of the ensemble of oscillators ($\varepsilon^2 = \langle x^2 \rangle \langle x'^2 \rangle - \langle xx' \rangle^2$).

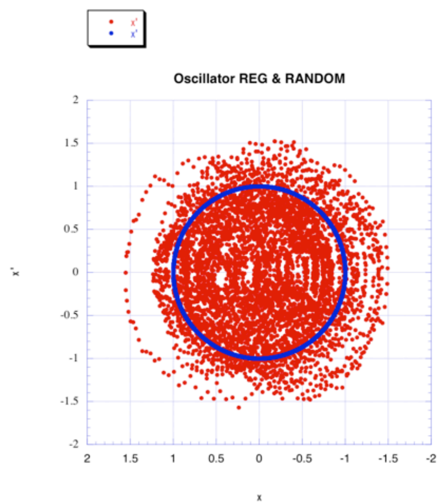


Fig. 23-2 Poincaré plot of trajectories of normal and damped oscillator with random kicks (in dimensionless coordinates x/a; x'/ωa).

Figure 2 shows Poincaré plot of few hundreds of such an oscillators starting from the same initial conditions (1,0) and going around for few damping times. Overall, a large ensemble of oscillators (or equivalently distribution of (x,p) for one oscillator in very long time – via Ergodic theorem, see **http://en.wikipedia.org/wiki/Ergodic_theory**) is described by distribution function.

Because *(I,φ)* is Canonical pair, it is natural to use them as independent variables for the distribution function, *f(I,φ,t)*. Few facts are apparent: the phases of oscillators walk randomly and because phase is cyclic function it is distributed evenly in the interval {-π, π}. Thus, there is no dependence on *φ*: $\frac{\partial f}{\partial \varphi}=0$. Finding distribution function of the action, *f(I,t),* requires solution of Fokker-Plank equation:

$$\frac{\partial f}{\partial t}+\frac{\partial}{\partial I}\left(f\frac{dI}{dt}\right)-\frac{1}{2}\frac{\partial}{\partial I}\left(\left\langle \delta I^2\right\rangle\frac{\partial f}{\partial I}\right)=0. \tag{23-19}$$

We are interested in stationary solution, $\frac{\partial f}{\partial t}=0$, $\frac{d}{dI}\left(f\frac{dI}{dt}-\frac{1}{2}\left\langle \delta I^2\right\rangle\frac{df}{dI}\right)=0$

$$\begin{gathered}\left\langle \delta I^2\right\rangle=2I\left\langle \delta I\right\rangle \Rightarrow \left(2\alpha f+I\left\langle \delta I\right\rangle f'\right)=const\rightarrow 0\\ \ln f'=-\frac{2\alpha}{\left\langle \delta I\right\rangle};\Rightarrow f=a\cdot e^{-I/I_0};\, I_0=\left\langle \delta I\right\rangle/2\alpha\end{gathered} \tag{23-20}$$

Remembering that $I=\frac{x'^2\omega+\omega\delta x^2}{2}$, it gives us just a trivial Gaussian distribution for the oscillators

$$f(x,x')=\frac{1}{2\pi\varepsilon}e^{-\frac{a^2}{2\varepsilon}}=\frac{1}{2\pi\varepsilon}\exp\left(\frac{x^2\omega+x'^2/\omega}{2\varepsilon}\right)=\frac{1}{\sqrt{2\pi}\sigma_x}e^{-\frac{x^2}{2\sigma_x^2}}\frac{1}{\sqrt{2\pi}\sigma_{x'}}e^{-\frac{x'^2}{2\sigma_{x'}^2}} \tag{23-21}$$

$$\sigma_x=\sqrt{\varepsilon/\omega};\sigma_{x'}=\sqrt{\varepsilon\cdot\omega}$$

where we normalize it as $\iint f(x,x')dxdx'=1$.

Conclusions are easy to remember: Position independent diffusion in the presence of linear damping results in stationary Gaussian distribution of the oscillator's amplitudes, positions and velocities. Phases of individual oscillators become random. Naturally, this process takes few damping times $T_d=1/\alpha$, if initial distribution deviates from the stationary.

Now we are fully equipped to write distribution for in 6D phase space. Using well-established

$$X = \mathrm{Re}\sum_{k=1}^{3} a_k Y_k e^{i(\psi_k+\varphi_k)} \to a_k = \frac{e^{i(\psi_k+\varphi_k)}}{i} Y_k^{*T} SX; \;\; I_k = \frac{a_k^{\,2}}{2} = \frac{\left|Y_k^{\,T} SX\right|^2}{2}; \tag{23-22}$$

we can write that particles distribution at any location s:

$$f(X) = \frac{1}{2\pi\varepsilon_1} e^{-\frac{a_1^{\,2}}{2\varepsilon_1}} \frac{1}{2\pi\varepsilon_2} e^{-\frac{a_2^{\,2}}{2\varepsilon_2}} \frac{1}{2\pi\varepsilon_3} e^{-\frac{a_3^{\,2}}{2\varepsilon_3}} = \frac{1}{\prod_{k=1}^{3} 2\pi\varepsilon_k} \exp\left[-\frac{1}{2}\sum_{k=1}^{3} \frac{\left|Y_k^{\,T}(s) SX\right|^2}{\varepsilon_k} \right] \tag{23-23}$$

This is one of the most useful applications of the eigen vectors and their components. It worth mentioning that this distribution is positively defined quadratic for of the particles postions (x,y, τ) and corresponding Canonical momenta. In accelerator physics $\left|Y_k^{\,T}(s) SX\right|^2$ are can be also known as Courant-Snyder invariants, which they derived in 1950s for 1D case.

For 1D case and slow synchrotron oscillation it is easy for write detailed distribution

$$\left|Y_x^T(s)SX\right|^2 = \left(\mathrm{w}_x x' - \mathrm{w}'_x x\right)^2 + \frac{x^2}{\mathrm{w}_x{}^2} = \frac{x_\beta{}^2 + \left(\beta_x x'_\beta + \alpha_x x_\beta\right)^2}{\beta_x}; x_\beta = x - \eta_x \pi_\tau;$$
$$f_{6D} = f_x\ f_y\ f_s; \tag{23-24}$$
$$f_x = \frac{1}{2\pi\varepsilon_x} e^{-\frac{x_\beta{}^2 + \left(\beta_x x'_\beta + \alpha_x x_\beta\right)^2}{2\varepsilon_x \beta_x}};\ \ f_y = \frac{1}{2\pi\varepsilon_y} e^{-\frac{y^2 + \left(\beta_y y' + \alpha_y y\right)^2}{2\varepsilon_y \beta_y}};\ \ f_s = \frac{1}{2\pi\sigma_\delta\sigma_s} e^{-\frac{\pi_\tau{}^2}{2\sigma_\delta{}^2}} e^{-\frac{\tau^2}{2\sigma_s{}^2}}.$$

that you would find in most of the accelerator books. But you are now capable of doing it for any arbitrary coupling using (23-23). You also can calculate RMS beam size in any direction by integrating (23-23) over the rest of coordinated and momenta. For example:

$$\left\langle x^2 \right\rangle = \int x^2 f(X) dX = \int x^2 dX \frac{1}{\prod_{k=1}^{3} 2\pi\varepsilon_k} \exp\left[-\frac{1}{2} \sum_{k=1}^{3} \frac{\left|Y_k^T(s)SX\right|^2}{\varepsilon_k} \right] \tag{23-25}$$

with 1D result being

$$\left\langle x^2 \right\rangle = \beta_x\varepsilon_x + \eta_x^2\sigma_\delta^2; \left\langle y^2 \right\rangle = \beta_y\varepsilon_y; \left\langle \tau^2 \right\rangle = \sigma_s^2;$$
$$\left\langle x^2 \right\rangle = \varepsilon_x \frac{1+\alpha_x^2}{\beta_x} + \eta_x'^2\sigma_\delta^2; \left\langle y^2 \right\rangle = \varepsilon_y \frac{1+\alpha_y^2}{\beta_y}; \left\langle \pi_\tau{}^2 \right\rangle = \sigma_\delta^2. \tag{23-26}$$

Vlasov equation and collective effects.

While Fokker-Plank equation is an important tool in many areas of physics, a reduced version of it, Vlasov equation, is one of the most important tools in accelerator and plasma physics. In contrast with the Fokker-Plank equation, Vlasov equation does not take into account random processes, such as quantum fluctuations of spontaneous radiation or particle’s scattering on each other. Nevertheless, it is one of most useful tools for studying instabilities of beams, including one called Free Electron Laser.

What we are planning to do for next three classes is to discuss collective effects, or in other words, the action of the beam’s particles on each other. It can be through repelling each other (the same charges), inducing EM wave in surrounding environment (wake-fields in vacuum chambers, RF cavities, FELs) or effect of collective radiation (such as coherent synchrotron radiation).

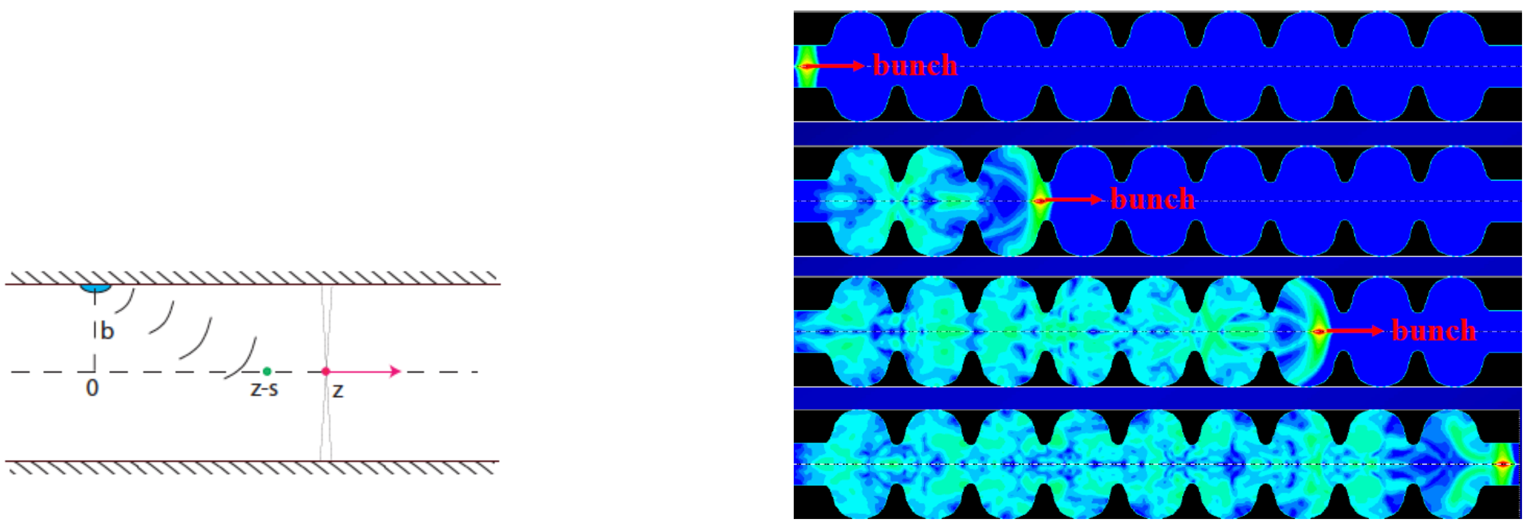


Fig. 23-3 A couple picture of wakefield generated by charged particles

In general, in addition to the describing the motion of particles in external (giving field), we need to find the fields induced by the particles moving (and accelerating) and include them into the equation of motion. This process is frequently complicated by necessity of including the boundary conditions and turns into very tough problem to crack. Still there is a number of approaches which allow to study some of the most important processes and instabilities analytically or semi-analytically. It usually involves solving self-consistently Vlasov equation and Maxwell equations:

$$div\vec{E} = 4\pi\rho;\ curl\vec{B} = \frac{4\pi}{c}\vec{j} + \frac{1}{c}\frac{\partial \vec{E}}{\partial t};\ curl\vec{E} = +\frac{1}{c}\frac{\partial \vec{B}}{\partial t};\ div\vec{B} = 0;$$
$$\rho = \sum_p e_p \delta\left(\vec{r} - \vec{r}_p(t)\right);\ \vec{j} = \rho\vec{\mathrm{v}}. \qquad (23\text{-}27)$$

Since you are well familiar with the later set of partial differential equations , you can guess that it is not a trivial problem and frequently requires significant simplification (assumptions) to be solved together with also not trivial partial differential equation.

Let's derive Vlasov equation for an ensemble of particles by considering a large number of them (in accelerator typical N ~ 10^{10}) evolving in the phase space. The microscopic distribution function

$$f = \sum_{part} \delta_{6D}\left(X - X_p(s)\right) \quad . \qquad (23\text{-}27)$$

while exact, results in N individual equations, which may be solvable exactly by computers in some distant-or-not-distant future. Meanwhile, so-called particle-in-cell (PIC) code are solving this equations typically using macro-particles and other simplifications.

Smoothing (23-27) out by using a space volume ΔV_{2n} containing large number of particles $\Delta N_p >> 1$ should allow us to introduce the distribution function:

$$f = f(X,s): \qquad dN_p = f(X,s)dX^{2n} \equiv f(X,s)dV_{2n}\,. \tag{23-28}$$

This immediately introduces the scale at which Vlasov equation is violated. While at the typical size σ scale ther is a huge number of particles, at a typical distance between particles $\delta l \sim \sigma / \sqrt[3]{N_p}$ they scatter on each other. Hence, according detailed (and non-trivial) studies there should be a scale L, when the scattering can be neglected,

$$\delta l << L << \sigma$$

and Vlasov equation can be used. We will assume the following:

1. The local interaction of the particles in small volume dV_{2n} is negligible compared with their interaction with the rest of the ensemble;
2. The system is Hamiltonian, i.e. dissipation is absent;
3. The consequence of 1) is that we neglect scattering processes between the particles! It is important – otherwise we could not say that number of particles in the phase space volume stays constant. We discussed such processes for Fokker-Plank equation.

Sub-ensemble of particles in small volume dV_{2n} satisfies the conditions we for derivation of Liouville's theorem. Let's draw the boundary of the infinitesimal phase-space volume around the particles. Because the phase space trajectories do not cross,

$$\left(X_1\left(s_o\right), s_o\right) = \left(X_2\left(s_o\right), s_o\right) \Leftrightarrow X_1\left(s\right) \equiv X_2\left(s\right).$$

the particles can not escape the volume. It means that phase density along the trajectory stays constant:

The number of the particles dN_p is constant;

The volume dV_{2n} is constant.

Thus,

$$f = f\left(X(s), s\right) = \frac{dN_p}{dV_{2n}} = const \ , \tag{23-29}$$

when $X(s)$ is the trajectory satisfying the equation of motion. The consequence of this equation is very powerful. If we follow the trajectory of the point in the phase space

$$X(s) = M : X_o(s_o) \Leftrightarrow X_o(s_o) = M^{-1} : X(s), \tag{23-30}$$

than particles density remains constant at that point and know initial particle's distribution $f_o\left(X, s_o\right)$ at s_o, then,

$$f\left(X, s\right) = f_o\left(M^{-1} : X, s_o\right). \tag{23-31}$$

It is called methods of trajectories and is used broadly from plasma physics to quantum field theory (famous Feynman's method of trajectories). While very interesting, finding (23-30) is equivalent to solving the pair of Maxwell-Vlasov equations. Hence, let's write Vlasov equation noting the total derivative of the distribution function along the (particle) trajectory is equal zero:

$$\frac{d}{ds} f\big(X(s),s\big) = \frac{\partial f}{\partial s} + \frac{\partial f}{\partial X}\frac{dX}{ds} = 0. \qquad (23\text{-}32)$$

This is famous Vlasov equation, which equivalent of the Liouville theorem. Do not forget that *s* is the independent variable, i.e. in most of the books *s*=*t*! Using the Hamiltonian equations to finish the job in matrix form:

$$\frac{\partial f}{\partial s} + \frac{\partial f}{\partial X} S \frac{\partial H}{\partial X} = 0 \qquad (23\text{-}33)$$

or in more traditional open form

$$\frac{\partial f}{\partial s} + \frac{\partial H}{\partial P_i}\frac{\partial f}{\partial Q_i} - \frac{\partial H}{\partial Q_i}\frac{\partial f}{\partial P_i} = 0. \qquad (23\text{-}34)$$

When time is used as independent variable (e.g. most of the books), the 3-D Vlasov equation reads:

$$\frac{\partial f}{\partial t} + \frac{\partial H}{\partial \vec{P}}\frac{\partial f}{\partial \vec{r}} - \frac{\partial H}{\partial \vec{r}}\frac{\partial f}{\partial \vec{P}} = 0. \qquad (23\text{-}35)$$