HW1 Monday, January 30, 2023, HWs solutions:

Problem 1: Reference particle and reference orbit. 6points
Using accelerator Hamiltonian (M1.19), corresponding differential equations (M1.20), expansion of the vector and scalar potentials (M1.21), show that for a reference particle that is following a reference "trajectory":

$$\vec{r} = \vec{r}_o(s); \quad t = t_o(s); \;\; H = H_o(s) = E_o(s) + \varphi_o(s, t_o(s)),$$

with $x \equiv 0;\; y \equiv 0;\; p_x \equiv 0; p_y \equiv 0$ and $h^*\big|_{ref} = -p_o(s)$ result in the following conditions:

$$K(s) \equiv \frac{1}{\rho(s)} = -\frac{e}{p_o c}\left(B_y\big|_{ref} + \frac{E_o}{p_o c} E_x\big|_{ref} \right); \tag{1}$$

$$B_x\big|_{ref} = \frac{E_o}{p_o c} E_y\big|_{ref}; \tag{2}$$

$$\frac{dt_o(s)}{ds} = \frac{1}{\mathrm{v}_o(s)} \tag{3}$$

$$\frac{dE_o(s)}{ds} = -e\frac{\partial \varphi}{\partial s}\bigg|_{ref} \equiv eE_2(s, t_o(s)). \tag{4}$$

Hints:

1. Use condition $\vec{A}\big|_{ref} = 0$ with

$$x\big|_{ref} = 0;\; y\big|_{ref} = 0;\; P_1\big|_{ref} = p_x\big|_{ref} + \frac{e}{c}A_1\big|_{ref} \equiv 0;\;\; P_3\big|_{ref} = p_y\big|_{ref} + \frac{e}{c}A_3\big|_{ref} \equiv 0;$$

or in the differential form

$$\frac{dx}{ds}\bigg|_{ref} = \frac{\partial h^*}{dP_1}\bigg|_{ref} = 0;\;\; \frac{dy}{ds}\bigg|_{ref} = \frac{\partial h^*}{dP_3}\bigg|_{ref} = 0;$$
$$\frac{dP_1}{ds}\bigg|_{ref} = -\frac{\partial h^*}{dx}\bigg|_{ref} = 0;\; \frac{dP_3}{ds}\bigg|_{ref} = -\frac{\partial h^*}{dy}\bigg|_{ref} = 0;$$

2. Keep only necessary (i.e. relatively low order) terms in expansion of vector potentials.

Solution: Start from the Hamiltonian

$$\begin{gathered} h^* = -\left(1 + Kx\right)\sqrt{G} - \frac{e}{c}A_2 + \kappa x\left(P_3 - \frac{e}{c}A_3 \right) - \kappa y\left(P_1 - \frac{e}{c}A_1 \right); \\ G = \frac{\left(H - e\varphi\right)^2}{c^2} - m^2c^2 - \left(P_1 - \frac{e}{c}A_1 \right)^2 - \left(P_3 - \frac{e}{c}A_3 \right)^2 \end{gathered} \tag{4}$$

and equations of motion:

$$x' = \frac{dx}{ds} = \frac{\partial h^*}{\partial P_1}; \quad \frac{dP_1}{ds} = -\frac{\partial h^*}{\partial x}; \qquad y' = \frac{dy}{ds} = \frac{\partial h^*}{\partial P_3}; \quad \frac{dP_3}{ds} = -\frac{\partial h^*}{\partial y}$$
$$t' = \frac{dt}{ds} = \frac{\partial h^*}{\partial P_t} \equiv -\frac{\partial h^*}{\partial H}; \quad \frac{dP_t}{ds} = -\frac{\partial h^*}{\partial t} \to \frac{dH}{ds} = \frac{\partial h^*}{\partial t} \tag{5}$$

we get rather trivial expression for coordinates derivatives:

$$x' = \frac{\partial h^*}{\partial P_1} = \frac{1+Kx}{\sqrt{G}}\left(P_1 - \frac{e}{c}A_1\right) - \kappa y \to x'\big|_{ref} = 0;$$
$$y' = \frac{\partial h^*}{\partial P_3} = \frac{1+Kx}{\sqrt{G}}\left(P_3 - \frac{e}{c}A_3\right) + \kappa x \to y'\big|_{ref} = 0; \tag{6}$$
$$t' = -\frac{\partial h^*}{\partial H} = \frac{1+Kx}{\sqrt{G}}\frac{H - e\varphi}{c^2} \to t'\big|_{ref} = \frac{E_o}{p_o c^2} = \frac{1}{\mathrm{v}_o(s)},$$

where we use $\vec{A}\big|_{ref} = 0$, $P_{1,3}\big|_{ref} = 0$ to arrive to first two equations, $E_o(s) = H - e\varphi\big|_{ref}\big|$ and $\sqrt{G}\big|_{ref} = p_o(s)$ to arrive to obvious $\mathrm{v}_o(s) = \dfrac{ds}{dt_o}$. Three other conditions require just a little bit more of work. Let's keep term(s) that do not vanish at the limit of the reference orbit and reference particle ins square brackets [...]. Let's start from $\dfrac{dP_1}{ds}$**:** differentiation on ***x*** (where most of the terms are turned into zero at the reference orbit, except $\partial_x\varphi$ and $\partial_x A_2$) we have

$$P_1' = -\frac{\partial h^*}{\partial x} = K\sqrt{G} - \frac{1+Kx}{\sqrt{G}}\left(\left[\frac{eE}{c^2}\frac{\partial\varphi}{\partial x}\right] + p_x\frac{e}{c}\frac{\partial A_1}{\partial x} + p_y\frac{e}{c}\frac{\partial A_3}{\partial x}\right) + \left[\frac{e}{c}\frac{\partial A_2}{\partial x}\right] + \kappa p_y + \kappa\left(\frac{e}{c}\frac{\partial A_1}{\partial x}y - \frac{e}{c}\frac{\partial A_3}{\partial x}x\right)$$

Let's first look at terms colored in red and prove that they are just a boring zero at the reference orbit:

$$p_{x,y}\big|_{ref} = 0; \; x,y\big|_{ref} = 0 \Rightarrow \left(p_x\frac{e}{c}\frac{\partial A_1}{\partial x} + p_y\frac{e}{c}\frac{\partial A_3}{\partial x}\right)_{ref} = \left(\kappa p_y\right) = \kappa\left(\frac{e}{c}\frac{\partial A_1}{\partial x}y - \frac{e}{c}\frac{\partial A_3}{\partial x}x\right)0$$

it means that

$$P_1'\big|_{ref} = p_o K - \frac{eE}{p_o c^2}\frac{\partial\varphi}{\partial x}\bigg|_{ref} + \frac{e}{c}\frac{\partial A_2}{\partial x}\bigg|_{ref} = p_o K - \frac{e}{\mathrm{v}_o}\frac{\partial\varphi}{\partial x}\bigg|_{ref} + \frac{e}{c}\frac{\partial A_2}{\partial x}\bigg|_{ref}$$

and we need expressions for potentials

$$A_1 = \frac{1}{2}\sum_{n.k=0}^{\infty} \partial_x^k \partial_y^n B_s\Big|_{ro} \frac{x^k}{k!}\frac{y^{n+1}}{(n+1)!};\ A_3 = -\frac{1}{2}\sum_{n.k=0}^{\infty} \partial_x^k \partial_y^n B_s\Big|_{ro} \frac{x^{k+1}}{(k+1)!}\frac{y^n}{n!}$$

$$A_2 = \sum_{n=1}^{\infty}\left\{\partial_x^{n-1}\big((1+Kx)B_y + \kappa x B_s\big)_{ro}\frac{x^n}{n!} - \partial_y^{n-1}\big((1+Kx)B_x - \kappa y B_s\big)_{ro}\frac{y^n}{n!}\right\} +$$
$$+\frac{1}{2}\sum_{n.k=1}^{\infty}\left\{\partial_x^{n-1}\partial_y^k\big((1+Kx)B_y + \kappa x B_s\big)_{ro}\frac{x^n}{n!}\frac{y^k}{k!} - \partial_x^n\partial_y^{k-1}\big((1+Kx)B_x - \kappa y B_s\big)_{ro}\frac{x^n}{n!}\frac{y^k}{k!}\right\};$$

$$\varphi = \varphi_o(s,t) - \sum_{n=1}^{\infty}\partial_x^{n-1}E_x\Big|_{ro}\frac{x^n}{n!} - \sum_{n=1}^{\infty}\partial_y^{n-1}E_y\Big|_{ro}\frac{y^n}{n!} - \frac{1}{2}\sum_{n.k=1}^{\infty}\left(\partial_x^{n-1}\partial_y^k E_x\Big|_{ro} + \partial_x^n\partial_y^{k-1}E_y\Big|_{ro}\right)\frac{x^n}{n!}\frac{y^k}{k!};$$

with a simple consideration that derivatives of potentials containing any power of x higher than one will be zero at reference orbitL

$$\left.\frac{\partial a_{nm}x^{n+1}y^m}{\partial x}\right|_{ref} = (n+1)a_{nm}\left(x^n y\right)^m{}_{x,y\to 0} = 0,\ n+m>0$$

hence

$$-\left.\frac{\partial\varphi}{\partial x}\right|_{ref} = E_x\Big|_{ref};\left.\frac{\partial A_2}{\partial x}\right|_{ref} = -B_y\Big|_{ref}$$

gives us connection between curvature of the reference orbit that components of electric and magnetic fields:

$$K(s) \equiv \frac{1}{\rho(s)} = -\frac{e}{p_o c}\left(B_y\Big|_{ref} + \frac{c}{\mathrm{v}_o}E_x\Big|_{ref}\right). \tag{7}$$

Second transverse coordinate:

$$P_3' = -\frac{\partial h^*}{\partial y} = -\frac{1+Kx}{\sqrt{G}}\left(\left[\frac{eE}{c^2}\frac{\partial\varphi}{\partial y}\right] + p_x\frac{e}{c}\frac{\partial A_1}{\partial y} + p_y\frac{e}{c}\frac{\partial A_3}{\partial y}\right) + \left[\frac{e}{c}\frac{\partial A_2}{\partial y}\right] - \kappa p_x + \kappa\left(\frac{e}{c}\frac{\partial A_1}{\partial y}y - \frac{e}{c}\frac{\partial A_3}{\partial y}x\right)$$
$$P_2'\Big|_{ref} = -\frac{eE}{p_o c^2}\left.\frac{\partial\varphi}{\partial y}\right|_{ref} + \frac{e}{c}\left.\frac{\partial A_2}{\partial y}\right|_{ref} = 0$$

where we eliminated terms in red as zeros at the reference orbit with the remaining need of two field components

$$-\left.\frac{\partial\varphi}{\partial y}\right|_{ref} = E_y\Big|_{ref};\left.\frac{\partial A_2}{\partial y}\right|_{ref} = B_x\Big|_{ref}$$

to arrive to “absence of curvature” in y direction:

$$B_x\Big|_{ref} = \frac{E_o}{p_o c} E_y\Big|_{ref}. \tag{8}$$

Finally, let's look at evolution of Hamiltonian of the reference particle:

$$\frac{dH_o(s)}{ds} = \frac{\partial h^*}{\partial t}\Bigg|_{ref} = (1+Kx)_{ref} \frac{\left(\left[\frac{eE}{c^2}\frac{\partial\varphi}{\partial t}\right] + p_x \frac{e}{c}\frac{\partial A_1}{\partial t} + p_y \frac{e}{c}\frac{\partial A_3}{\partial t}\right)_{ref}}{\sqrt{G}\Big|_{ref}}$$

$$-\left(\frac{e}{c}\frac{\partial A_2}{\partial t}\right)_{ref} + \kappa\left(\frac{e}{c}\frac{\partial A_1}{\partial x} y - \frac{e}{c}\frac{\partial A_3}{\partial x} x\right)_{ref} = \frac{eE_o}{p_o c^2}\frac{\partial\varphi}{\partial t}\Bigg|_{ref}$$

$$\frac{dH_o(s)}{ds} = \frac{\partial h^*}{dt}\Bigg|_{ref} = \frac{eE_o}{p_o c^2}\frac{\partial\varphi}{\partial t}\Bigg|_{ref};$$

which can be transferred using $H = E + e\varphi$ and $d\varphi_o(s,t_o(s)) = \frac{\partial\varphi_o}{\partial s}ds + \frac{\partial\varphi_o}{\partial t}\frac{ds}{\mathrm{v}_o(s)}$ into the energy gain of the reference particle along is trajectory:

$$\frac{dE_o(s)}{ds} = \frac{d\left(H_o(s) - \varphi_o(s,t_o(s))\right)}{ds} = -e\frac{\partial\varphi}{\partial s}\Bigg|_{ref} \equiv eE_2(s,t_o(s)). \tag{9}$$

Problem 2: Trace and determinant. 4 points

Solution of any linear n-dimensional differential equation

$$\frac{dX}{ds} = \mathbf{D}(s)X$$

can be expressed in a form of transport matrix

$$X(s) = \mathbf{M}(s)X_o;\ X_o = X(s=0)$$

with

$$\frac{d\mathbf{M}(s)}{ds} = \mathbf{D}(s)\mathbf{M}(s);\ \mathbf{M}(s=0) = \mathbf{I}; \tag{1}$$

where **I** is unit *nxn* matrix. Prove that

$$\det(\mathbf{M}(s)) = \exp\left(\int_0^s Trace(\mathbf{D}(\zeta))d\zeta\right).$$

Hints:

1. Prove first that

$$\frac{d}{ds}\det\mathbf{M} = Trace(\mathbf{D})\cdot\det\mathbf{M}$$

2. Use infinitesimally small step in eq. (1) to conclude that

$$d\mathbf{M}(s) = \mathbf{D}(s)\mathbf{M}(s)ds + O(ds^2) \Rightarrow \mathbf{M}(s+ds) = (\mathbf{I} + \mathbf{D}(s)ds)\cdot\mathbf{M}(s) + O(ds^2);$$

$$\det\mathbf{M}(s+ds) = \det(\mathbf{I} + \mathbf{D}(s)ds)\cdot\det\mathbf{M}(s) + O(ds^2) \rightarrow \tag{1}$$

$$\frac{1}{\det\mathbf{M}}\frac{d(\det\mathbf{M})}{ds} = \frac{\det(\mathbf{I} + \mathbf{D}(s)ds) - 1}{ds};$$

3. What remained is to prove us that

$$\det(\mathbf{I}+\varepsilon\mathbf{D}) = 1+\varepsilon\cdot Trace[\mathbf{D}]+O(\varepsilon^2)$$

where ε is infinitesimally small real number and term $O(\varepsilon^2)$ contains second and higher orders of ε.

4. First, fist look on the product of diagonal elements $\prod_{m=1}^{n}(1+\varepsilon a_{mm})$ in $\det[I+\varepsilon A]$ in the first order of ε. Then prove that contributions to determinant from non-diagonal terms $a_{km}; k\neq m$ is $O(\varepsilon^2)$ or higher order of ε. It is possible to do it directly for an arbitrary *nxn* matrix, or start from *n=1* and use induction from *n* to *n+1*.

Solution: Fist, let's assume that $\det(\mathbf{I}+\varepsilon\mathbf{D}) = 1+\varepsilon\cdot Trace[\mathbf{D}]+O(\varepsilon^2)$, then

$$\det(\mathbf{I}+\mathbf{D}ds) = 1+Trace\mathbf{D}ds;\;\; \mathbf{M}(0)=\mathbf{I}\rightarrow\det\mathbf{M}(0)=1;$$

$$\frac{1}{\det\mathbf{M}}\frac{d(\det\mathbf{M})}{ds} = \frac{d}{ds}\ln(\det\mathbf{M}) = Trace\mathbf{D}\rightarrow\ln(\det\mathbf{M}(s)) = \int_0^s Trace\mathbf{D}(\zeta)d\zeta \quad (3)$$

$$\det\mathbf{M}(s) = e^{\int_0^s Trace\mathbf{D}(\zeta)d\zeta}$$

Just for fun. let's use induction. For *n=1* we have

$$\det(1+\varepsilon d_{11}) = 1+\varepsilon d_{11};\; O(\varepsilon^2)=0 \quad 4)$$

For n=2

$$\det\begin{bmatrix} 1+\varepsilon d_{11} & d_{12} \\ d_{21} & 1+\varepsilon d_{22}\end{bmatrix} = (1+\varepsilon d_{11})(1+\varepsilon d_{22})+\varepsilon^2 d_{12}d_{21} = 1+\varepsilon(d_{11}+d_{22})+O(\varepsilon^2); \quad (5)$$

Let's assume that for *nxn* matrix our ration is correct

$$\det(\mathbf{I}_{nxn}+\varepsilon\mathbf{D}_{nxn}) = 1+\varepsilon Trace\mathbf{D}_{nxn}+O(\varepsilon^2) \quad (6)$$

and add element to expand to *(n+1)x(n+1)*:

$$\det\begin{bmatrix} 1+\varepsilon d_{1,1} & \ldots & \varepsilon d_{1,n} & \varepsilon d_{1,n+1} \\ \ldots & \ldots & \ldots & \ldots \\ \varepsilon d_{n,1} & \ldots & 1+\varepsilon d_{n,n} & \varepsilon d_{n,n+1} \\ \varepsilon d_{n+1,1} & \ldots & \varepsilon d_{n+1,} & 1+\varepsilon d_{n+1,n+1}\end{bmatrix} = \quad (7)$$

$$(1+\varepsilon d_{n+1,n+1})\det(I_{nxn}+\varepsilon D_{nxn})+\varepsilon\sum_{k=1}^{n}(d_{n+1,k}\cdot res_{n+1,k}+d_{k,n+1}\cdot res_{k,n+1})$$

The blue term is easy to evaluate

$$\begin{aligned}&(1+\varepsilon d_{n+1,n+1})\det(I_{nxn}+\varepsilon D_{nxn}) = (1+\varepsilon d_{n+1,n+1})(1+\varepsilon TraceD_{nxn}+O(\varepsilon^2)) = \\ &1+\varepsilon(TraceD_{nxn}+d_{n+1,n+1})+O(\varepsilon^2) = 1+\varepsilon TraceD_{n+1xn+1}O(\varepsilon^2)\end{aligned} \quad (8)$$

Part of the determinant containing $\varepsilon d_{n+1,k}$ or $\varepsilon d_{k,n+1}$ with k≠n+1 eliminated two noninfinite components $1+\varepsilon d_{k,k}$ and $1+\varepsilon d_{n+1,n+1}$. Since this part of the determinant contain the product contains *n+1* elements, and only maximum *n-1* of them are not infinitesimal, it means that its lowers power is ε^2. Hence, all this terms can be neglected when $\varepsilon -> 0$.

Munch more simple and straightforward is this prove:
The contribution to determinant from the diagonal elements is

$$\prod_{m=1}^{n}\left(1+\varepsilon a_{mm}\right)=1+\varepsilon\sum_{m=1}^{n}a_{mm}+O\left(\varepsilon^2\right)=1+\varepsilon\cdot Trace[A]+O\left(\varepsilon^2\right) \quad (1)$$

A generic term containing a non-diagonal element $a_{km}; k\neq m$, excludes from the product at least two diagonal elements $1+\varepsilon a_{mm}$ and $1+\varepsilon a_{kk}$.

$$\pm e_{m.....}e_{k.....}\varepsilon a_{m,k}\prod_{i\neq m;j\neq k}^{n}a_{i,j}\left(\delta_{ij}+\varepsilon a_{i,j}\right)$$

Since the total number of elements in the product is n, such term contains at least two non-diagonal elements, each of which contains ε. This proves that non-diagonal terms can contribute only second and higher order term into $O\left(\varepsilon^2\right)$.

Problem 3: Proving solutions of Vlasov and Fokker-Plank equation. 15 points

Part 1. **5 points**. Prove that for uncoupled vertical oscillations

$$\frac{dy}{ds}=y';\ \frac{dy'}{ds}\equiv y''=-K_1(s)y; \quad (1)$$

the phase space distribution

$$F(y,y',s)=f\left(\zeta(y,y',s)\right);\ \zeta(y,y',s)=\left(\mathrm{w}(s)y'-\mathrm{w}'(s)y\right)^2+\left(\frac{y}{\mathrm{w}(s)}\right)^2 \quad (2)$$

with an arbitrary differentiable $f(\zeta)$ and beam envelope

$$\mathrm{w}''(s)+K_1(s)\mathrm{w}(s)=\frac{1}{\mathrm{w}(s)^3} \quad (3)$$

satisfied Vlasov equation:

$$\frac{\partial F}{\partial s}+\frac{\partial F}{\partial y}y'+\frac{\partial F}{\partial y'}y''=0. \quad (4)$$

Hint: Use well-known $\frac{\partial_{y,y',s}f(\zeta)}{\partial y,y',s}=\frac{df(\zeta)}{d\zeta}\cdot\frac{\partial_{y,y',s}\zeta}{\partial y,y',s}$ and equations (1) and (3) to prove (4)

Part 2. **10 points**. Prove that phase space distribution

$$F(y,y',s)=f(\zeta)=c\cdot\exp\left(-\frac{\zeta}{2\varepsilon}\right); \quad (5)$$

satisfies **phase-averaged** Fokker Plank equation:

$$\frac{\partial F}{\partial s}+\frac{\partial F}{\partial y}y'-\frac{\partial}{\partial y'}F\left(K_1 y-\xi y'\right)=\frac{1}{2}\frac{\partial^2}{\partial y^2}\left(F\cdot D_{yy}\right)+\frac{1}{2}\frac{\partial^2}{\partial y\partial y'}\left(F\cdot D_{yy'}\right)+\frac{1}{2}\frac{\partial^2}{\partial y'^2}\left(F\cdot D_{y'y'}\right)=0 \tag{6}$$

for uncoupled vertical oscillations with additional damping terms and random noise (diffusion)

$$\frac{dy}{ds}=y';\ \frac{dy'}{ds}\equiv y''=-K_1(s)y-\xi(s)y'+\upsilon(s)\cdot\sum_{i=1}^{N}rnd_i\cdot\delta(s-s_i);s_i\in(0,C)$$
$$\langle rnd\rangle=0;\langle rnd^2\rangle=1 \tag{7}$$

with constant emittance $\varepsilon=\dfrac{\langle D_{y'y'}\mathrm{w}^2\rangle}{2\langle\xi\rangle}$.

Step 1: First, eliminate fast oscillating terms using eq. (4): $\dfrac{\partial F}{\partial s}=-\dfrac{\partial F}{\partial y}y'-\dfrac{\partial F}{\partial y'}y''$.

Step 2: Evaluate three diffusion coefficients

$$D_{uv}=\lim_{\tau\to 0}\frac{1}{\tau}\big(u(s+\tau)-u(s)\big)\big(v(s+\tau)-v(s)\big);$$

Show that $D_{yy}=0$ by finding that $\left(y(s+\tau)-y(s)\right)^2\sim\tau^2$, and that $\langle D_{yy'}\rangle=0$, when averaging is taken of the random kicks with $\langle g(y,y')\cdot rnd\rangle=g(y,y')\cdot\langle rnd\rangle=0$. Finally, calculate $\langle D_{y'y'}\rangle$ using following manipulations:

$$y'(s+\tau)=y'(s)+K(s^*)y(s^*)+\sum_{s_i\in\{s,s+\tau\}}\upsilon(s_i)\cdot rnd_i;\ s^*\in\{s,s+\tau\}$$

Show that after averaging over random kick strength, the only non-zero term originates only from square of the random kicks $\left\langle\left(\sum_{s_i\in\{s,s+\tau\}}\upsilon(s_i)\cdot rnd_i\right)^2\right\rangle\to\sum_{s_i\in\{s,s+\tau\}}\upsilon^2(s_i)\cdot\langle rnd_i^2\rangle$

Here you need to use the fact stand random kicks are not correlated:

$$\langle rnd_i\cdot rnd_{j\neq i}\rangle=0$$

to arrive to $\langle D_{y'y'}\rangle$ independent on *y* and *y'*, which allows you to take it out of $\dfrac{1}{2}\dfrac{\partial^2}{\partial y'^2}\left(F\cdot D_{y'y'}\right)$.

Step 3: after completing all differentiations, use expression for *y* and *y'*

$$y=a\mathrm{w}(s)\cdot\cos\varphi;\ y'=a\left(\mathrm{w}'(s)\cdot\cos\varphi-\frac{\sin\varphi}{\mathrm{w}(s)}\right)$$

and average over betatron phases φ arrive to equation in form of $F(y,y',s)\cdot g\left(\xi(s),\mathrm{w}(s)D_{y'y'}(s),a^2,\varepsilon\right)=0$, which means that *g*=0.

Step 3: Assuming that a^2,ε (i.e. practically are constants!) are slow function compared with $\xi(s),\mathrm{w}(s)D_{y'y'}(s)$, average over the ring circumference to arrive to conclusion that $\varepsilon=\dfrac{\left\langle D_{y'y'}\mathrm{w}^2\right\rangle_C}{2\langle\xi\rangle_C}$ satisfies the Fokker-Plank equation.

Solution:

Part one: Let's differentiate one by one, take $\dfrac{df}{d\zeta}$ outside of the bracket use, equation of motion $y''=ky$, combine terms and find that they cancel each other to get pure zero:

$$\frac{dF}{ds}=\frac{\partial F}{\partial s}+\frac{\partial F}{\partial y}y'+\frac{\partial F}{\partial y'}y''=0; F(y,y',s)=f\left(\left(\mathrm{w}y'-\mathrm{w}'y\right)^2+\left(\frac{y}{\mathrm{w}}\right)^2\right)$$

$$\frac{\partial F}{\partial s}=\frac{df}{d\zeta}\left(\left(\mathrm{w}y'-\mathrm{w}'y\right)\left(\mathrm{w}'y'-\mathrm{w}''y\right)-\frac{y^2}{\mathrm{w}^3}\mathrm{w}'\right)$$

$$y'\frac{\partial F}{\partial y}=-\frac{df}{d\zeta}\cdot y'\left(\mathrm{w}'\left(\mathrm{w}y'-\mathrm{w}'y\right)-\frac{y}{\mathrm{w}^2}\right)$$

$$y''\frac{\partial F}{\partial y'}=-\frac{df}{d\zeta}\left(K_1\mathrm{w}^2yy'-K_1\mathrm{w}\mathrm{w}'y^2\right);\ \ \mathrm{w}''+K_1\mathrm{w}=\frac{1}{\mathrm{w}^3};\ y''=ky;$$

$$\frac{df}{d\zeta}\left\{y^2\mathrm{w}'\left(\mathrm{w}''+K_1\mathrm{w}-\frac{1}{\mathrm{w}^3}\right)-y'^2\left(\mathrm{w}\mathrm{w}'-\mathrm{w}\mathrm{w}'\right)-y'y\left(\mathrm{w}'^2-\mathrm{w}'^2+\mathrm{w}\left(\mathrm{w}''+K_1\mathrm{w}-\frac{1}{\mathrm{w}^2}\right)^2\right)\right\}=0$$

$$\frac{\partial F}{\partial s}+\frac{\partial F}{\partial y}y'+\frac{\partial F}{\partial y'}Ky=0$$

It is quate natural, because $\zeta=\left(\mathrm{w}y'-\mathrm{w}'y\right)^2+\left(\dfrac{y}{\mathrm{w}}\right)^2=a^2=inv$, is invariant of motion, which means that

$$\text{Step }\frac{dF}{ds}=\frac{df}{d\zeta}\cdot\frac{d\zeta}{ds}=\frac{df}{d\zeta}\cdot\frac{da^2}{ds}=0.$$

Part two: by adding friction and random kicks :

$$\frac{dy}{ds}=y';\ \frac{dy'}{ds}\equiv y''=-K_1(s)y-\xi(s)y'+\upsilon(s)\cdot\sum_{i=1}^{N}rnd_i\cdot\delta(s-s_i);s_i\in(0,C)$$

$$\langle rnd\rangle=0;\langle rnd^2\rangle=1$$

we have some old and some new terms in the Fokker-plank equation:

$$\frac{\partial F}{\partial s}+\frac{\partial F}{\partial y}y'-\frac{\partial}{\partial y'}F\left(K_1 y-\xi y'\right)=DT;\ DT=\frac{1}{2}\frac{\partial^2}{\partial y^2}\left(D_{yy}F\right)+\frac{1}{2}\frac{\partial^2}{\partial y\partial y'}\left(D_{yy'}F\right)+\frac{1}{2}\frac{\partial^2}{\partial y'^2}\left(D_{y'y'}F\right)$$

$$\frac{\partial F}{\partial s}+\frac{\partial F}{\partial y}y'-\frac{\partial}{\partial y'}F\left(K_1 y-\xi y'\right)=\left(\frac{\partial F}{\partial s}+\frac{\partial F}{\partial y}y'-\frac{\partial F}{\partial y'}K_1 y\right)-\frac{\partial f}{\partial y'}\xi y'-\xi f=DT$$

where can use previously found $\frac{\partial F}{\partial s}+\frac{\partial F}{\partial y}y'-\frac{\partial F}{\partial y'}K_1 y=0$ to reduce it to

$$\xi\cdot\frac{\partial\left(y'F\right)}{\partial y'}+DT=0.$$

Now it is time to calculate diffusion coefficients. Let's start from easy one:

$$D_{yy}=\lim_{\tau\to 0}\frac{1}{\tau}\left(y(s+\tau)-y(s)\right)^2=\lim_{\tau\to 0}\left(y'(s^*)^2\,\tau\right)=0;$$

where we used well known formular from math analysis:

$$y(s+\tau)-y(s)=y'(s^*)\cdot\tau;s^*\in\left\{s,s+\tau\right\}.$$

Mixed term takes a bit more efforts because we need first find that

$$y'(s+\tau)-y'(s)=\int_s^{s+\tau}dz\left\{\upsilon(s)\cdot\sum_{i=1}^{N}rnd_i\cdot\delta(z-s_i)-K_1(z)y-\xi(z)y'\right\}=$$

$$\sum_{s_i\in\{s,s+\tau\}}\upsilon(s_i)\cdot rnd_i-\tau\left(K_1(s_1)y(s_1)-\xi(s_2)y'(s_2)\right);s_{1,2}\in\left\{s,s+\tau\right\}.$$

and recognizing that regular (non-random) term is ~ τ^2 and vanishes

$$D_{yy'}=\lim_{\tau\to 0}\frac{1}{\tau}\left(y(s+\tau)-y(s)\right)\left(y'(s+\tau)-y'(s)\right)=$$

$$-\lim_{\tau\to 0}\left(y'(s^*)\cdot\left(\left(K(s_1)y(s_1)+\xi(s_2)y'(s_2)\right)\tau-\sum_{s_i\in\{s,s+\tau\}}\upsilon(s_i)\cdot rnd_i\right)\right)=\lim_{\tau\to 0}\left(y'(s^*)\cdot\sum_{s_i\in\{s,s+\tau\}}\upsilon(s_i)\cdot rnd_i\right);$$

$$\left\langle D_{yy'}\right\rangle=\lim_{\tau\to 0}\left(y'(s^*)\cdot\sum_{s_i\in\{s,s+\tau\}}^{N}\upsilon(s_i)\cdot\left\langle rnd_i\right\rangle\right)=0;$$

while mixed product vanishes because zero average value of random kicks $\left\langle rnd_i\right\rangle$=0.
What is left is to calculate non-vanishing diffusion coeeficient

$$D_{y'y'}=\lim_{\tau\to 0}\frac{1}{\tau}\left(y'(s+\tau)-y'(s)\right)^2=\lim_{\tau\to 0}\frac{1}{\tau}\left(reg\cdot\tau-\sum_{s_i\in\{s,s+\tau\}}\upsilon(s_i)\cdot rnd_i\right)^2;reg=K(s_1)y(s_1)+\xi(s_2)y'(s_2)$$

$$D_{y'y'}=\lim_{\tau\to 0}\frac{1}{\tau}\left(reg^2\cdot\tau^2-2reg\cdot\tau\sum_{s_i\in\{s,s+\tau\}}\upsilon(s_i)\cdot rnd_i+\left(\sum_{s_i\in\{s,s+\tau\}}\upsilon(s_i)\cdot rnd_i\right)^2\right)$$

by eliminating regular term ~ τ^2, and product of the regular term with random kicks:

$$\lim_{\tau\to 0}\frac{1}{\tau}\left(reg\cdot\tau\right)^2 = 0;$$

$$\lim_{\tau\to 0}\frac{1}{\tau}\left(reg\cdot\tau\right)\sum_{s_i\in\{s,s+\tau\}}\upsilon\left(s_i\right)\cdot rnd_i = reg\cdot\sum_{s_i\in\{s,s+\tau\}}\upsilon\left(s_i\right)\cdot rnd_i$$

$$\left\langle reg\cdot\sum_{s_i\in\{s,s+\tau\}}\upsilon\left(s_i\right)\cdot rnd_i\right\rangle = reg\cdot\sum_{s_i\in\{s,s+\tau\}}\upsilon\left(s_i\right)\cdot\left\langle rnd_i\right\rangle = 0.$$

The remaining term requres a little bit of work by recognizing that randdom kicks occuring in different positions are uncorreltated and the only non-zero term comes from square of the kicks:

$$D_{y'y'} = \lim_{\tau\to 0}\frac{1}{\tau}\left(\sum_{s_i\in\{s,s+\tau\}}\upsilon\left(s_i\right)\cdot rnd_i\right)^2 = \lim_{\tau\to 0}\frac{\sum_i\sum_{j\neq i}\upsilon\left(s_i\right)\upsilon\left(s_j\right)\cdot rnd_i\cdot rnd_j}{\tau} + \lim_{\tau\to 0}\frac{\sum_{s_i\in\{s,s+\tau\}}\upsilon^2\left(s_i\right)\cdot rnd_i^2}{\tau};$$

$$\left\langle rnd_i\cdot rnd_{j\neq i}\right\rangle = 0;\ \left\langle D_{y'y'}\right\rangle = \lim_{\tau\to 0}\frac{\left\langle\sum_{s_i\in\{s,s+\tau\}}\upsilon^2\left(s_i\right)\cdot\left\langle rnd_i^2\right\rangle\right\rangle}{\tau} = \frac{N}{C}\left\langle\upsilon^2\left(s\right)\right\rangle$$

where we instriduce average frequency of kisk $\frac{N}{C}$ and their average RMS strenth $\left\langle\upsilon^2\left(s\right)\right\rangle$. In other words, this diffusion coefficient is completely defined by the frequency and strength of the random kicks and does not depend on y and y' – hence we can take is out from the differential:

$$\xi\cdot\frac{\partial\left(y'F\right)}{\partial y'} + \frac{D_{y'y'}}{2}\frac{\partial^2 F}{\partial y'^2} = 0.$$

Now we need to use specific expression for F:

$$F\left(y,y',s\right) = c\cdot\exp\left(-\frac{\left(\mathrm{w}\left(s\right)y' - \mathrm{w}'\left(s\right)y\right)^2 + \left(\frac{y}{\mathrm{w}\left(s\right)}\right)^2}{2\varepsilon}\right);$$

and calcuate the derivatives:

$$\frac{\partial F}{\partial y'} = -\frac{F\mathrm{w}}{\varepsilon}\left(\mathrm{w}y' - \mathrm{w}'y\right);\ \frac{\partial^2 F}{\partial y'^2} = \frac{F\mathrm{w}^2}{\varepsilon^2}\left\{\left(\mathrm{w}y' - \mathrm{w}'y\right)^2 - \varepsilon\mathrm{w}^2\right\};$$

$$\frac{\partial}{\partial y'}\left(F\xi y'\right) = F\xi\left(1 - \frac{\mathrm{w}}{\varepsilon}\left(\mathrm{w}y'^2 - \mathrm{w}'yy'\right)\right);$$

combining the terms we get to

$$F\left(\xi - D_{y'y'}\frac{\mathrm{w}^2}{2\varepsilon}\right) - \frac{F}{\varepsilon}\left(\xi\left(\mathrm{w}^2y'^2 - \mathrm{ww}'y^2\right) - D_{y'y'}\frac{\mathrm{w}^2\left(\mathrm{w}y' - \mathrm{w}'y\right)^2}{2\varepsilon}\right) = 0$$

where we need to introduce expressions for *y* and *y'*:

$$\mathrm{w}'y = a\cdot\mathrm{ww}'\cdot\cos\varphi;\quad \mathrm{w}y' = a\left(\mathrm{w}'\cdot\cos\varphi\text{-}\sin\varphi\right);\mathrm{w}y' - \mathrm{w}'y = -a\cdot\sin\varphi$$

and averrage ofve the betatron phases

$$\left\langle \mathrm{w}^2 y'^2 - \mathrm{w}\mathrm{w}' y^2 \right\rangle_\varphi = \frac{a^2}{2}; \mathrm{w}^2 \left\langle y'^2 \right\rangle_\varphi - \mathrm{w}\mathrm{w}' \left\langle y y' \right\rangle_\varphi = \frac{a^2}{2}$$

to get the desirable final product

$$F\left(1 - \frac{a^2}{2\varepsilon}\right)\left(\xi - D_{y'y'}\frac{\mathrm{w}^2}{2\varepsilon}\right) = 0.$$

With

$$\frac{a^2}{2\varepsilon} \cong const; F = f\left(a^2\right) \cong const.$$

being either constabt or very slow variable, we must concluded that

$$\left\langle \xi - D_{y'y'}\frac{\mathrm{w}^2}{2\varepsilon} \right\rangle_C = 0 \Rightarrow \varepsilon = \frac{1}{2}\frac{\left\langle D_{y'y'}(s)\mathrm{w}^2(s)\right\rangle_C}{\left\langle \xi(s)\right\rangle_C} \equiv \frac{1}{2}\frac{\left\langle D_{y'y'}(s)\beta_y(s)\right\rangle_C}{\left\langle \xi(s)\right\rangle_C}$$

where we average both the product of diffusion coefficient with vertical β-function and the decrement of the vertical oscillations over the circumference of the storage ring, *C*.