Homework 4, due September 21

Problem 1. 10 points. Long elements.

Prelude: Many elements of accelerators are straight – e.g. coordinate system is simply Cartesian (x,y,s=z). It allows you to forget about curvilinear coordinates and use simple *div* and *curl* and Laplacian… Many of them are DC - e.g. either with constant or nearly constant EM fields. Again, Maxwell equations without time derivatives – EM static. Furthermore, many of them are also long – e.g. have a constant cross-section with transverse size much smaller than the length of the element. It means that you can drop derivatives over z. Finally, all current and charges generating field are outside of the vacuum where particles propagate – e.g. Maxwell static equations are also homogeneous – charge and current densities are zero! It should come at no surprise – everybody like to have a solvable problems to rely upon.

(a) use electro-static equations for a long uniform electric element and show that

$$\vec{E} = \vec{\nabla}\,\mathrm{Re}\left[a_n (x+iy)^n\right] \qquad (1)$$

satisfy static Maxwell equations with a_n being a complex number. Electric elements with real a_n call regular elements (they have plane symmetry!), element with imaginary a_n are called skew .

(b) use magneto-static equations for a long uniform magnetic element

$$\vec{B} = \vec{\nabla}\,\mathrm{Re}\left[b_n (x+iy)^n\right] \qquad (2)$$

satisfy static Maxwell equations with *b* being a complex number. Magnetic elements with imaginary b_n call regular elements (they have plane symmetry!), element with real b_n are called skew.

(c) show that arbitrary combination of elements from (1) and (2) is also a solution of electrostatic equations.

Note: elements with various *n* have specific names: n=1 – dipole, n=2 – quadrupole, n=3 – sextupole, n=4 – octupole, …. Or 2n-pole element. Skew is added as needed. It also obvious that an arbitrary 2n-pole element can be constricted out combining a regular and a skew fields.

Problem 2. 5 points. Edge effects.

We continue with Cartesian (x,y,s=z) coordinates for a straight element. But now we will suggest that field in this element depends on *z*;

$$\vec{E},\vec{B} = \vec{\nabla}\,\mathrm{Re}\left[a_n(z)(x+iy)^n\right] \qquad (1)$$

Show that such elements will generate terms in the field which are not simply higher order multi-poles. Prove that a sum of higher order multi-poles with amplitudes dependent on z cannot be a solution for edge field.

Solutions.

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(a) use electro-static equations for a long uniform electric element and show that

$$\vec{E} = \vec{\nabla}\,\mathrm{Re}\left[a_n\left(x+iy\right)^n\right] \tag{1}$$

satisfy static Maxwell equations with a_n being a complex number. Electric elements with real a_n call regular elements (they have plane symmetry!), element with imaginary a_n are called skew.

(b) use magneto-static equations for a long uniform magnetic element

$$\vec{B} = \vec{\nabla}\,\mathrm{Re}\left[b_n\left(x+iy\right)^n\right] \tag{2}$$

satisfy static Maxwell equations with *b* being a complex number. Magnetic elements with imaginary b_n call regular elements (they have plane symmetry!), element with real b_n are called skew.

(c) show that arbitrary combination of elements from (1) and (2) is also a solution of electrostatic equations.

Solution: Most of Maxwell equations are satisfied automatically:

$$\vec{E} = \vec{\nabla}\phi_e = \vec{\nabla}\cdot\sum_{n=1}^{\infty}\phi_{en}; \quad \vec{B} = \vec{\nabla}\phi_b = \vec{\nabla}\cdot\sum_{n=1}^{\infty}\phi_{bn}$$

$$\text{(a)} \quad \phi_{en} = \mathrm{Re}\left[a_n\left(x+iy\right)^n\right]; \phi_{bn} = \mathrm{Re}\left[b_n\left(x+iy\right)^n\right];$$

$$curl\vec{E} = curl\left(\vec{\nabla}\phi_e\right) \equiv 0; \quad curl\,\vec{B} = curl\left(\vec{\nabla}\phi_b\right) \equiv 0;$$

the only non-trivial equations remain are:

$$\text{(b)} \quad \begin{aligned} &\phi_{en} = \mathrm{Re}\left[a_n\left(x+iy\right)^n\right]; \phi_{bn} = \mathrm{Re}\left[b_n\left(x+iy\right)^n\right]; \\ &div\vec{E} = \vec{\nabla}\cdot\left(\vec{\nabla}\phi_e\right) = \Delta\phi_e \equiv 0; \quad div\,\vec{B} = \vec{\nabla}\cdot\left(\vec{\nabla}\phi_b\right) = \Delta\phi_b \equiv 0; \end{aligned}$$

What we have to prove is trivial:

$$\Delta\,\mathrm{Re}\left[a_n\left(x+iy\right)^n\right] = \mathrm{Re}\left[a_n\cdot\Delta\left(x+iy\right)^n\right] = 0;$$

$$\text{(b)} \quad \Delta\left(x+iy\right)^n = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\left(x+iy\right)^n =$$

$$= n\left(n-1\right)\left(x+iy\right)^{n-2}\left(1+i^2\right) = 0$$

Needless to say that we discussed that the one of most important features of EM fields is principle of superposition: if two fields are satisfying Maxwell equations, than their linear combinations also does satisfy the equations.
What is really unusual is that we expressed magnetic field as a gradient of a scalar potential – it is only possible in the area where $curl\,\vec{B} = 0$, i.e. in the absence of currents and time dependent electric field! Do not try this for AC fields!

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We continue with Cartesian (x,y,s=z) coordinates for a straight element. But now we will suggest that field in this element depends on *z*;

$$\vec{E},\vec{B} = \vec{\nabla}\,\mathrm{Re}\left[a_n(z)\left(x+iy\right)^n\right] \qquad (1)$$

Show that such elements will generate terms in the field which are not simply higher order multi-poles.

(a) Prove that a sum of higher order multi-poles with amplitudes dependent on z can not be a solution for edge field.
(b) Would 2n-th other multi-pole generate lower order multi-poles?

Solution: Similar to problem 1, there is only one not-trivial equation for E or B:
;

$$\text{(b)} \quad \Delta\left\{a_n(z)\left(x+iy\right)^n\right\} = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\left\{a_n(z)\left(x+iy\right)^n\right\} =$$

$$= \frac{\partial^2 a_n(z)}{\partial z^2}\left(x+iy\right)^n \neq 0$$

Since uniform *x ,y* polynomials of n-th order can not be canceled by those of different order, this solution is invalid.