Homework PHY 554 #5.

HW 1 (3 points): A multi-cell accelerating RF linac operating at 500 MHz in a standing wave π-mode (e.g., each cell has opposite sign of the accelerating voltage from the neighboring cell) is used to accelerate non-relativistic heavy ion (Z=2, A=79) moving with velocity v=c/3 (β=1/3).

(a) find the length of the cell required for resonant acceleration in such a linac – 1 point.
(b) at what velocity (ies) (and energy(ies) of the ion), the energy gain in 5-cell cavity would vanish (became zero) – 2 points

Solution: Problem defines that we have a standing wave RF voltage operating in π-mode, e.g., electric field can be described as

$$E_n(z,t) = (-1)^{n-1} \cdot E_o(z) \cdot \cos(\omega t + \varphi) \equiv e^{i\pi(n-1)} \cdot E_o(z) \cdot \cos(\omega t + \varphi)$$

where $E_o(z)$ is the electric field pattern (envelope) of one cell.

(a) For resonant acceleration we need that acceleration is repeated in each and every cell, e.g., generally speaking RF phase advance should be equal odd number of π while particle traverse one cell:

$$\Delta t = \frac{l}{\mathrm{v}}; \Delta\varphi_{RF} = \omega\Delta t = \omega\frac{l}{\mathrm{v}} = (2m+1)\pi;$$

$$t_n = t_1 + (n-1)\frac{l}{\mathrm{v}}; \;\; \omega t_n = \omega t_1 + (n-1)(2m+1)\pi;$$

$$E_1(z,t) = E_o(z) \cdot \cos(\omega t_1 + \varphi);$$

$$E_n(z,t) = (-1)^{n-1} \cdot E_o(z) \cdot \cos(\omega t_1 + (n-1)(2m+1)\pi + \varphi);$$

$$\cos((n-1)(2m+1)\pi + \alpha) = \cos((n-1)\pi + \alpha) = (-1)^{n-1} \cos\alpha;$$

$$E_n(z,t) = (-1)^{2(n-)1} E_o(z) \cdot \cos(\omega t_1 + \varphi) = E_1(z).$$

e.g., as required, a particle moving with constant velocity see the same field in each cell when

$$l = (2m+1)\pi\frac{\mathrm{v}}{\omega} = \left(m + \frac{1}{2}\right)\lambda_{RF}\frac{\mathrm{v}}{c}$$

Naturally, m=0 is preferred case (e.g. particle sample accelerating field while propagating through each cell), which for this problem

$$\frac{\mathrm{v}}{c} = \frac{1}{2} \Rightarrow l = \frac{\lambda_{RF}}{6};$$

For *f=500 MHz* $\lambda_{RF} = 0.6\ m$ (rounded), resulting in *l=0.1 m*.

(b) The total energy gain/loss of the particle in a linac comprised on N identical cells with length *l* is given by:

$$\Delta E = q\int_0^{N\cdot l} E(z,t)dz = \sum_{n=1}^{N} e^{i\pi(n-1)}\cdot\int_o^l E_o(z)\cdot\cos\omega\left(t_o+\frac{z+(n-1)l}{\mathrm{v}}\right);\; T=\frac{l}{\mathrm{v}};\tau=\frac{z}{\mathrm{v}};\phi=\omega T;$$

$$\Delta E = q\frac{\mathrm{v}}{2}\int_o^T d\tau E_o(\mathrm{v}\tau)\cdot\sum_{n=1}^{N}\left(e^{i\omega(t_o+\tau)}e^{i(n-1)(\phi+\pi)}+e^{-i\omega(t_o+\tau)}e^{-i(n-1)(\phi-\pi)}\right)=$$

$$q\frac{\mathrm{v}}{2}\int_o^T d\tau E_o(\mathrm{v}\tau)\cdot\left(e^{i\omega(t_o+\tau)}\sum_{n=1}^{N}e^{i(n-1)(\phi+\pi)}+e^{-i\omega(t_o+\tau)}\sum_{n=1}^{N}e^{-i(n-1)(\phi-\pi)}\right);$$

$$e^{2\pi i(n-1)}=1\Rightarrow e^{i(n-1)(\phi+\pi)}=e^{i(n-1)(\phi+\pi)}e^{-2\pi i(n-1)}=e^{-i(n-1)(\phi-\pi)};\sum_{n=1}^{N}e^{i(n-1)\alpha}=\frac{1-e^{iN\alpha}}{1-e^{i\alpha}};$$

$$\Delta E = q\mathrm{v}\,\mathrm{Re}\left\{e^{i\omega t_o}\left(\frac{1-e^{iN(\phi-\pi)}}{1-e^{i(\phi-\pi)}}\cdot\int_o^T d\tau E_o(\mathrm{v}\tau)\cdot e^{i\omega\tau}\right)\right\};$$

e.g., energy gain/loss is zero independently of the time of the entering the linac (initial RF phase, ωt_o) when one of the multipliers inside the brackets is zero:

$$\frac{1-e^{iN(\phi+\pi)}}{1-e^{i(\phi+\pi)}}=0;\; or/and\; \int_o^T d\tau E_o(\mathrm{v}\tau)\cdot e^{i\omega\tau}=0;$$

While the second condition is possible, we cannot solve it without knowing the field pattern in the cell. Meanwhile the first condition is rather trivial to solve:

$$e^{iN(\phi-\pi)}=1\;\; when\;\; e^{i(\phi-\pi)}\neq 1\Rightarrow N(\phi-\pi)=2m\pi;\; \phi-\pi\neq 2k\pi;$$

$$\phi=\frac{2m\pi}{N}+\pi;\qquad m\neq k\cdot N;\quad (N,n,k \text{ integers})$$

In our case N=5 and we would have zero gain/loss is the cavity when

$$\phi=\omega\frac{l}{\mathrm{v}}=m\frac{2\pi}{5}+\pi;\quad m\neq 5k\Rightarrow\; \lambda_{RF}=\frac{2\pi c}{\omega};$$

$$\frac{\mathrm{v}}{c}=\frac{5\omega l}{\pi c(2m+5)}=\frac{10l}{\lambda_{RF}(2m+5)}\;\leq 1\Rightarrow m\geq\frac{5l}{\lambda_{RF}}-\frac{5}{2}$$

Again, in our case we know that $l=\frac{\lambda_{RF}}{6}$ and $m\geq\frac{5}{6}-\frac{5}{2}=-\frac{20}{12}\rightarrow m=-1,1,2,3,4,6....$;

It means that there are infinite number of velocities at each particles will not change energy (again, only if we assuming constant velocity!):

m=-1; v/c =0.555555556	m=0; v/c =0.333333333
m=1; v/c =0.238095238	m=2; v/c =0.185185185
m=3; v/c =0.151515152	m=4; v/c =0.128205128
m=6; v/c =0.098039216	….

It worth noting that at m=-2 velocity should exceed speed of the light and the above formulae would require velocity to be *5c/3,* so far unattainable. Second note –because the wave is standing, it is natural the same condition would be correct for particle moving in opposite direction – e.g., for the negative velocity with the same absolute value.

At m=0,5,10… we will have resonant interaction and energy gain/loos from each cell will simply add.

HW 2 (2 points): A N-cell standing wave cavity operates in π-mode with field on the axis describes as

$$E_z = E_o(z)\cdot \sin(\kappa z)\cdot \cos(\omega t+\varphi);\ \kappa = \omega/2c;$$

$$E_o(z) = \begin{pmatrix} E_o;\ 0 \le z \le \dfrac{n\pi}{\kappa} \\ 0;\ \ z<0 \\ 0;\ \ z>\dfrac{n\pi}{\kappa} \end{pmatrix}$$

Find the energy gain and transit time factor in such a linac for particle moving with the speed of light.

Extra points: what will be modification if v= βc; β ≠1.

Solution: Let's use result from our previous problem but with well defined *Eo(z):*

$$\Delta\mathrm{E} = q\mathrm{v}\,\mathrm{Re}\left\{ e^{i\omega t_o}\left(\frac{1-e^{iN(\phi-\pi)}}{1-e^{i(\phi-\pi)}}\cdot\int_o^T d\tau E_o\cdot e^{i\omega\tau}\right)\right\};\int_o^T d\tau E_o\cdot e^{i\omega\tau} = \frac{e^{i\omega T}-1}{i\omega} = Te^{i\frac{\omega T}{2}}\,\mathrm{sinc}\left(\frac{\omega T}{2}\right);$$

$$\mathrm{v}T = l;\ \ \phi = \omega T = \pi;\ \left.\frac{1-e^{iN(\phi-\pi)}}{1-e^{i(\phi-\pi)}}\right|_{\phi\to\pi} = N;\ \ \mathrm{sinc}\left(\frac{\omega T}{2}\right) = \frac{2}{\pi};$$

$$\Delta\mathrm{E} = qNE_o l\cdot \mathrm{sinc}\left(\frac{\omega T}{2}\right)\cos\left(\omega\left(t_o+\frac{T}{2}\right)\right) = V_{RF}\cos\varphi_o; V_{RF} = N\frac{2}{\pi}E_o l;\ \ \varphi_o = \omega\left(t_o+\frac{T}{2}\right);$$

Extra points: when v= βc; β ≠1, the phase advance ϕ is not equal to π and wee need to do a bit more

$$\Delta\mathrm{E} = q\mathrm{v}\,\mathrm{Re}\left\{ e^{i\omega t_o}\left(\frac{1-e^{iN(\phi-\pi)}}{1-e^{i(\phi-\pi)}}\cdot Te^{i\frac{\omega T}{2}}\,\mathrm{sinc}\left(\frac{\omega T}{2}\right)\right)\right\};\phi = \pi/\beta;\phi-\pi = \pi\cdot\frac{1-\beta}{\beta};$$

$$\frac{1-e^{iN(\phi-\pi)}}{1-e^{i(\phi-\pi)}} = e^{i(N-1)\pi\cdot\frac{1-\beta}{2\beta}}\frac{\sin\left(N\pi\cdot\dfrac{1-\beta}{2\beta}\right)}{\sin\left(\pi\cdot\dfrac{1-\beta}{2\beta}\right)} =;\ \ \mathrm{sinc}\left(\frac{\omega T}{2}\right) = \frac{\sin\left(\pi\cdot\dfrac{1-\beta}{2\beta}\right)}{\pi\cdot\dfrac{1-\beta}{2\beta}};$$

$$\Delta\mathrm{E} = qNE_o l\cdot \mathrm{sinc}\left(N\pi\cdot\frac{1-\beta}{2\beta}\right)\cos(\varphi_o) = V_{RF}\cos\varphi_o;$$

$$V_{RF} = \mathrm{sinc}\left(N\pi\cdot\frac{1-\beta}{2\beta}\right);\ \ \varphi_o = \omega\left(t_o+\frac{T}{2}\right)+(N-1)\pi\cdot\frac{1-\beta}{2\beta}.$$

HW 3 (5 points): A l=0.3 m long 500 MHz pillbox cavity operates in fundamental accelerating TM_{010} mode with peak accelerating electric field of 20 MV/m.

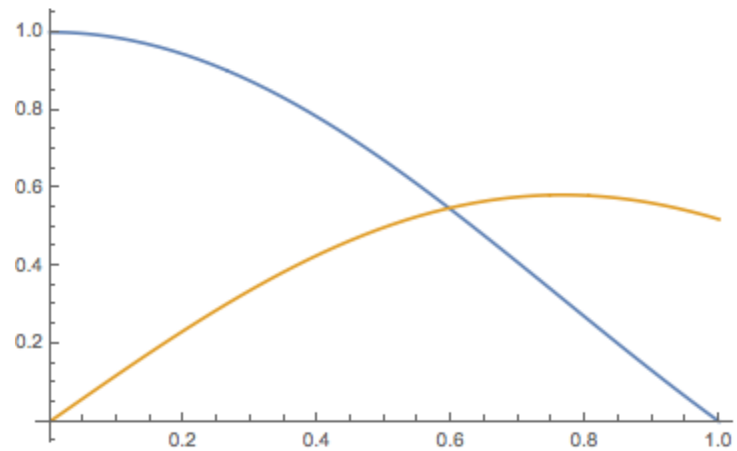
(a) Find the energy stored in electric and magnetic fields as function of time.
(b) What is the total energy of EM field in the cavity? Does it change with time?
(c) What will be losses of the energy for Q-factor of 30,000?

Solution: We first should write solutions for electric and magnetic fields for TM_{010} mode in pillbox cavity: (I am using here CGS units)

$$\mathbf{E}_z = E_o \cdot J_o\left(2.405\frac{r}{a}\right)\sin(\omega t);$$

$$\mathbf{B}_\theta = E_o \cdot J_1\left(2.405\frac{r}{a}\right)\cos(\omega t);$$

$$W_E = \frac{1}{8\pi}\int \vec{\mathbf{E}}^2\, dV; W_B = \frac{1}{8\pi}\int \vec{\mathbf{B}}^2\, dV;\ W = \frac{1}{8\pi}\int\left(\vec{\mathbf{E}}^2 + \vec{\mathbf{B}}^2\right)dV$$



r/a

and a is determined from the RF frequency

$$\omega = 2\pi f = \frac{2.405c}{a} \to a = \frac{2.405c}{2\pi f}$$

finding that a=0.2295 m. Than energy stored in electric field is:

$$W_E = \frac{1}{8\pi}\int \vec{\mathbf{E}}^2\, dV; dV = dz d\theta r dr; \int E_o^2 J_o\left(2.405\frac{r}{a}\right)^2 dV = 2\pi l \int_o^a J_o\left(2.405\frac{r}{a}\right)^2 r\, dr$$

$$W_E = \frac{E_o^2}{4} a^2 l \cdot \sin^2 \omega t \int_o^1 J_o(2.405x)^2\, x dx;\ W_B = \frac{E_o^2}{4} a^2 l \cdot \cos^2 \omega t \int_o^1 J_1(2.405x)^2\, x dx$$

It is feature of Bessel functions that two integrals taken to the root of Jo are equal. Indeed,

$$z_o = 2.404825557695773..; J_o(z_o) = 0;$$

$$I = \int_o^{z_o} J_o(x)^2\, r\, dr = \int_o^{z_o} J_1(x)^2\, r\, dr = 0.779325...$$

$$I_1 = \int_o^1 J_o(2.405x)^2\, x dx = \frac{I}{2.405^2} = 0.134757..;$$

$$W_E = W \cdot \sin^2 \omega t;\; W_B = W \cdot \cos^2 \omega t;\; W = \frac{E_o^{\ 2}}{4} a^2 l \cdot I_1.$$

As we discussed in class this is actually property of a cavity with ideally conducting walls fields that peak energy stored in magnetic field is equal that in electric field and, naturally – because of the energy conservation, total energy stored in EM field is constant. You can used SI formulae, but I am just transforming 20 MV/m field into Gauss: E_o= *666.7 Gs* and use the above formula to find stored energy being *W=2.37E8 erg*, or *W=23.7 J*.
By definition (lecture 11, slide 7) the losses are connected to the Q-factor s

$$P_{loss} = \frac{\omega W}{Q_o} \rightarrow P_{loss} = 2\pi \frac{5 \cdot 10^8\, Hz \cdot 23.7 J}{3 \cdot 10^4} = 2.48 \cdot 10^6 \;\; W$$

and such tiny cavity will endure about 2.5 MW losses in the wall. It makes it both "power hungry" and in practice, impossible to cool. Hence, Cu cavities usually operate at gradients ~ few MV/m with power dissipation measured in tens to few hundreds kW.

HW 4 (5 points): RF cavity beam loading/unloading.

A short ultra-relativistic (1-v/c <<1) bunch with charge of 5 nC is passing through a 0.3 meter long 500 MHz pillbox accelerating cavity operating at the fundamental TM_{010} with peak accelerating field of 5 MV/m.

(1) Find the change of the cavity voltage ΔV/V (accelerating field) after the beam passes through it as function of the phase of the beam passing the cavity. What are the maximum and minimum ΔV/V?

(2) How the beam loading ΔV/V depends on the accelerating field? At what level of accelerating it reaches ΔV/V 1%?

(a) Assume that beam does not change velocity in the cavity;
(b) Hint – use energy conservation law
(c) Assume that relative change of the voltage ΔV/V is small, e.g. the beam loading can be treated as a perturbation.

Solution:

First, we need to find the energy gain by each electron in the cavity operating at E=5 MV/m using RF phase in the center as the reference:

$$dz \cong ct;$$

$$\Delta E = ec\mathbf{E}\int_{-L/2c}^{L/2c}\cos(\omega t+\varphi)dt = eL\mathbf{E}\frac{\sin\left(\frac{\omega L}{2c}\right)}{\frac{\omega L}{2c}}\cos\varphi = eV_{RF}\cos\varphi$$

$$FF = \frac{\sin\left(\frac{\omega L}{2c}\right)}{\frac{\omega L}{2c}} = 0.636179.. \qquad V_{RF} = L\mathbf{E}\cdot FF = 0.9543\ MV$$

It means that that energy take/given by the beam is

$$\Delta U = qV_{RF}\cos\varphi = \Delta U_o\cos\varphi$$

$$q = 5nC = 5\cdot 10^{-9}C; V_{RF} = 0.9543\cdot 10^6 V$$

$$\Delta U_o = \text{sign}[q]\cdot 4.77\cdot 10^{-3} J$$

Naturally, when energy is taken by electron beam, $q\cos\varphi > 0$, RF voltage in the cavity drops (it is called beam loading) and with beam loses energy, $q\cos\varphi < 0$, RF voltage increases. To know the voltage change we need to know what EM energy is stored in the RF cavity. We should use your favorite units system (SI)

$$W = \int\left(\varepsilon_o\frac{\vec{\mathbf{E}}^2}{2} + \mu_o\frac{\vec{\mathbf{H}}^2}{2}\right)dV = \frac{\varepsilon_o}{2}\int\vec{\mathbf{E}}_o^{\,2}\,dV$$

or GSG

$$W = \frac{1}{8\pi}\int\left(\vec{\mathbf{E}}^2 + \vec{\mathbf{H}}^2\right)dV = \frac{1}{8\pi}\int\vec{\mathbf{E}}_o^{\,2}\,dV$$

and the field pattern we derived for TM_{010} mode.

$$\mathbf{E}_o = \hat{z}\cdot E_o\cdot J_o\left(2.405\frac{r}{a}\right);\ E_o = 5\cdot 10^6\frac{V}{m} \equiv 166.7\ Gs$$

where radius of the cavity, *a*, is defined by its frequency:

$$J_o(ka) \equiv J_o\left(\frac{\omega}{c}a\right) = 0 : TM_{010} \rightarrow \frac{\omega}{c}a = 2.405$$

$$a = \frac{2.405c}{\omega} = \frac{2.405}{2\pi}\frac{c}{f} = 0.2295\,m$$

and then integrate Bessel function over the radius of the cavity

$$\int J_o^2 r\, dr d\theta dz = 2\pi L \int_o^a J_o^2(kr) r dr = \frac{2\pi L}{k^2}\int_o^{x_o} J_o^2(x) x dx;$$

$$x_o = 2.404825557695773..: \quad \int_o^{x_o} J_o^2(x) x dx = 0.779325$$

$$\frac{2\pi L}{k^2}\int_o^{x_o} J_o^2(x) x dx = 1.337\cdot 10^{-2} m^3 = 1.337\cdot 10^4 cm^3$$

Then using your favorite units, we got identical

$$W = 1.48\, J = 1.48\cdot 10^7\, erg$$

Side note: *A smart RF engineer would use known value of R $_{sh}$/Q*

$$\frac{R_{sh}}{Q_0} = \frac{V_{RF}^2}{\omega_0 W} \rightarrow W = \frac{V_{RF}^2}{\omega_0} / \frac{R_{sh}}{Q_0}$$

for pillbox cavity of 196 Ohm (Slide 10, Lecture 11) to get the same:

$$\frac{R_{sh}}{Q_0} = 196; V_{RF} = 9.54E5\ V; \omega_0 = 3.141593E9\ Hz \Rightarrow W = 1.48 J$$

Finally, we should notice that

$$V_{RF} \sim \sqrt{W} \rightarrow \frac{\Delta V_{RF}}{V_{RF}} = \frac{1}{2}\frac{\Delta W}{W}; \Delta W = -\Delta U$$

and maximum voltage drop in our case is

$$\Delta V_{RF} = -V_{RF}\frac{1}{2}\frac{\Delta U}{W} = -1.54\ kV; \ \frac{\Delta V_{RF}}{V_{RF}} = -1.6\cdot 10^{-3} = 0.16\%$$

Beam loading dependence on the accelerating field (RF voltage) is very simple to find from following

$$\Delta U = qV_{RF} \sim E_o; W \sim V_{RF}^{\ 2} \sim E_o^{\ 2} \Rightarrow \frac{\Delta V_{RF}}{V_{RF}} \sim \frac{1}{V_{RF}} \sim \frac{1}{E_o}$$

e.g., beam loading is inverse proportional to the accelerating field. Thus, to increase beam loading from 0.16% to 1% we should make the accelerating voltage to be 0.16 Eo = 0.8 MV/m.