Homework 1.

Problem 1. 2 points. Lorentz transformations

Show explicitly that two successive Lorentz transformations in the same direction are equivalent to a single Lorentz transformation with $v = \dfrac{v_1 + v_2}{1 + (v_1 v_2 / c^2)}$.

Problem 2. 2 points. 4-invarints

Show that trace of a tensor is 4-invariant, i.e. $F_i^i \equiv \sum_{i=0}^{3} F_i^i = inv$.

Problem 3. Lorentz group

a) **5 points.** For the Lorentz boost and rotation matrices **K** and **S** show that

$$\left(\vec{\varepsilon}\vec{\mathbf{S}}\right)^3 = -\vec{\varepsilon}\vec{\mathbf{S}}; \left(\vec{\varepsilon}\vec{K}\right)^3 = \vec{\varepsilon}\vec{K}; \forall\, \vec{\varepsilon} = \vec{\varepsilon}^{\,*}; |\vec{\varepsilon}| = 1;$$

$$\text{or } \left(\vec{a}\vec{\mathbf{S}}\right)^3 = -\vec{a}\vec{\mathbf{S}} \cdot \vec{a}^2; \left(\vec{a}\vec{\mathbf{K}}\right)^3 = \vec{a}\vec{\mathbf{K}} \cdot \vec{a}^2; \forall\, \vec{a} = \vec{a}.$$

b) **5 points.** use this results to show that

$$e^{\vec{\omega}\vec{\mathbf{S}}} = I + \frac{\vec{\omega}\vec{\mathbf{S}}}{|\vec{\omega}|}\sin|\vec{\omega}| + \frac{\left(\vec{\omega}\vec{\mathbf{S}}\right)^2}{\vec{\omega}^2}(\cos|\vec{\omega}| - 1);$$

$$e^{\vec{\beta}\vec{K}} = I + \frac{\vec{\beta}\vec{\mathbf{K}}}{|\vec{\beta}|}\sinh|\vec{\beta}| + \frac{\left(\vec{\beta}\vec{\mathbf{K}}\right)^2}{\vec{\beta}^2}(\cosh|\vec{\beta}| - 1);$$

Draw connection to Lorentz transformations (e.g. boosts and rotations).

Note: in original problem there was typo “-“ instead of “+” before *sin* and *sinh.* Everybody who found correct sign had extra 10 points!

With solutions:

Problem 1. Show explicitly that two successive Lorentz transformations in the same direction are equivalent to a single Lorentz transformation with $v = \frac{v_1 + v_2}{1 + (v_1 v_2 / c^2)}$.

Solution: Each Lorentz transformations along x-axis corresponds to the block-diagonal matrix with parameterization of :

$$L_i = \begin{bmatrix} L_i & O \\ O & I \end{bmatrix}; L_i = \gamma_i \begin{bmatrix} 1 & \beta_i \\ \beta_i & 1 \end{bmatrix}; O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}; I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}; \ \det L_i = \gamma_i^2 (1 - \beta_i^2) = 1$$

and we should find parameters of L by brining it to the same form

$$L = \begin{bmatrix} L & O \\ O & I \end{bmatrix} = L_2 L_1 = \begin{bmatrix} L_2 & O \\ O & I \end{bmatrix} \begin{bmatrix} L_1 & O \\ O & I \end{bmatrix} = \begin{bmatrix} L_2 L_1 & O \\ O & I \end{bmatrix}; \ L = L_2 L_1.$$

The fact that $\det L = \gamma^2 (1 - \beta^2) = 1$ for any L is taking care of the rest:

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$$L = L_2 L_1 = \gamma_1 \gamma_2 \begin{bmatrix} 1 & \beta_2 \\ \beta_2 & 1 \end{bmatrix} \begin{bmatrix} 1 & \beta_1 \\ \beta_1 & 1 \end{bmatrix} = \gamma_1 \gamma_2 \begin{bmatrix} 1 + \beta_1 \beta_2 & \beta_1 + \beta_2 \\ \beta_1 + \beta_2 & 1 + \beta_1 \beta_2 \end{bmatrix} = \gamma_1 \gamma_2 (1 + \beta_1 \beta_2) \begin{bmatrix} 1 & \frac{\beta_1 + \beta_2}{1 + \beta_1 \beta_2} \\ \frac{\beta_1 + \beta_2}{1 + \beta_1 \beta_2} & 1 \end{bmatrix}$$

Problem 2. Show that trace of a tensor is 4-invariant, i.e. $F^i{}_i \equiv \sum_i F^i{}_i = inv$.

Solution: $$Trace(F') = F'^i{}_i = \frac{\partial x'^i}{\partial x^k}\frac{\partial x^j}{\partial x'^i}F^k{}_j = \frac{\partial x^j}{\partial x^k}F^k{}_j = \delta^j_k F^k{}_j = F^k{}_k = Trace(F) \ \#$$

Problem 3. Lorentz group

c) **5 points.** For the Lorentz boost and rotation matrices **K** and **S** show that

$$\left(\vec{\varepsilon}\vec{\mathbf{S}}\right)^3 = -\vec{\varepsilon}\vec{\mathbf{S}};\left(\vec{\varepsilon}\vec{K}\right)^3 = \vec{\varepsilon}\vec{K};\forall\,\vec{\varepsilon} = \vec{\varepsilon}^*;|\vec{\varepsilon}| = 1;$$

$$\text{or } \left(\vec{a}\vec{\mathbf{S}}\right)^3 = -\vec{a}\vec{\mathbf{S}}\cdot\vec{a}^2;\left(\vec{a}\vec{\mathbf{K}}\right)^3 = \vec{a}\vec{\mathbf{K}}\cdot\vec{a}^2;\forall\,\vec{a} = \vec{a}.$$

d) **5 points.** use this results to show that

$$e^{\vec{\omega}\vec{\mathbf{S}}} = I + \frac{\vec{\omega}\vec{\mathbf{S}}}{|\vec{\omega}|}\sin|\vec{\omega}| + \frac{\left(\vec{\omega}\vec{\mathbf{S}}\right)^2}{\vec{\omega}^2}(\cos|\vec{\omega}| - 1);$$

$$e^{\vec{\beta}\vec{K}} = I + \frac{\vec{\beta}\vec{\mathbf{K}}}{|\vec{\beta}|}\sinh|\vec{\beta}| + \frac{\left(\vec{\beta}\vec{\mathbf{K}}\right)^2}{\vec{\beta}^2}(\cosh|\vec{\beta}| - 1);$$

Draw connection to Lorentz transformations (e.g. boosts and rotations).

Solution: it is possible to do it by multiplying three matrices and getting confirmation. Otherwise, we can test that:

$$\left(\vec{a}\vec{K}\right)^3 = \left(\vec{a}\vec{K}\right)^2\cdot\vec{a}\vec{K};$$

$$K_\alpha K_\beta = \begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\delta_{\alpha\beta} + \begin{bmatrix}0 & 0\\ 0 & u_{\chi\varepsilon}\end{bmatrix}\delta_{\chi\alpha}\delta_{\varepsilon\beta};u = \begin{bmatrix}1 & 1 & 1\\ 1 & 1 & 1\\ 1 & 1 & 1\end{bmatrix};$$

and use it to calculate square of the matrix:

$$\left(\vec{a}\vec{K}\right)^2 \equiv \sum_{\alpha,\beta=1,2,3} a_\alpha a_\beta K_\alpha K_\beta = \begin{bmatrix}\vec{a}^2 & 0\\ 0 & 0\end{bmatrix} + \sum_{\alpha,\beta=1,2,3}\begin{bmatrix}0 & 0\\ 0 & a_\alpha a_\beta u_{\chi\varepsilon}\end{bmatrix}\delta_{\chi\alpha}\delta_{\varepsilon\beta} = \vec{a}^2 I + X;$$

$$X = \left(\sum_{\alpha,\beta=1,2,3}\begin{bmatrix}0 & 0\\ 0 & a_\alpha a_\beta u_{\chi\varepsilon}\end{bmatrix}\delta_{\chi\alpha}\delta_{\varepsilon\beta} - \vec{a}^2\begin{bmatrix}0 & 0\\ 0 & I\end{bmatrix}\right).$$

First term gives us desirable answer if product of matrix X and $\vec{a}\vec{K}$ is zero. It is easy to show:

$$\vec{a}\vec{K} = \begin{bmatrix}0 & \vec{a}\\ \tilde{\vec{a}} & 0_{3\times3}\end{bmatrix}; -\vec{a}^2\begin{bmatrix}0 & 0\\ 0 & I\end{bmatrix}\cdot\begin{bmatrix}0 & \vec{a}\\ \tilde{\vec{a}} & 0_{3\times3}\end{bmatrix} = -\vec{a}^2\begin{bmatrix}0 & 0\\ \tilde{\vec{a}} & 0_{3\times3}\end{bmatrix};$$

$$\sum_{\alpha,\beta=1,2,3} a_\alpha a_\beta\delta_{\chi\alpha}\delta_{\varepsilon\beta}\begin{bmatrix}0 & 0\\ 0 & u_{\chi\varepsilon}\end{bmatrix}\cdot\begin{bmatrix}0 & \vec{a}\\ \tilde{\vec{a}} & 0_{3\times3}\end{bmatrix} = \begin{bmatrix}0 & 0\\ \tilde{\vec{b}} & 0_{3\times3}\end{bmatrix}$$

$$b_\chi = \sum_{\alpha,\beta,\varepsilon=1,2,3} a_\alpha a_\beta\delta_{\chi\alpha}\delta_{\varepsilon\beta}a_\varepsilon = a_\chi\sum_{\alpha,\beta,\varepsilon=1,2,3} a_\beta\delta_{\varepsilon\beta}a_\varepsilon = a_\chi\cdot\vec{a}^2 \Rightarrow \vec{b} = \vec{a}\cdot\vec{a}^2 .\ \#K$$

For **S** it is even easier, noting that it is already block-diagonal matrix:

$$S_\alpha = e_{\alpha\beta\gamma}\begin{bmatrix} 0 & 0 \\ 0 & u_{\beta\gamma} \end{bmatrix}$$

and further we can drop all time components operating with 3x3 matrix:

$$\left[S_\alpha\right]_{\beta\gamma} = e_{\alpha\beta\gamma}; \left[\vec{a}\vec{S}\right]_{\beta\gamma} = a_\alpha e_{\alpha\beta\gamma};$$

$$\left(\vec{a}\vec{S}\right)^2{}_{\beta\eta} = \left[a_\alpha a_\varepsilon S_\alpha S_\varepsilon\right] = a_\alpha a_\varepsilon e_{\alpha\beta\gamma} e_{\varepsilon\gamma\eta}; \; e_{\alpha\beta\gamma} e_{\varepsilon\gamma\eta} = -e_{\alpha\beta\gamma} e_{\varepsilon\eta\gamma} = -\delta_{\alpha\varepsilon}\delta_{\beta\eta} + \delta_{\alpha\eta}\delta_{\beta\varepsilon};$$

$$\delta_{\alpha\varepsilon} a_\alpha a_\varepsilon = \vec{a}^2; a_\alpha a_\varepsilon \delta_{\alpha\eta}\delta_{\beta\varepsilon} = a_\eta a_\beta; \left(\vec{a}\vec{S}\right)^2{}_{\beta\eta} = I\vec{a}^2 + a_\beta a_\eta; \; a_\beta a_\eta a_\mu e_{\mu\eta\theta} \equiv 0!$$

which is equivalent to

$$\left(\vec{a}\vec{S}\right)^2\left(\vec{a}\vec{S}\right) = -\vec{a}^2\left(\vec{a}\vec{S}\right) \quad \#S$$

b) is trivial for any matrix

$$M^3 = (-1)^n x^2 M; \; n = 0,1$$

which also means that

$$M^4 = (-1)^n x^2 M^2;$$

Separating series into zero order, odd and even terms:

$$e^M = \sum_{k=0}^{\infty}\frac{M^k}{k!} = I + \sum_{k=0}^{\infty}\frac{M^{2k+1}}{(2k+1)!} + \sum_{k=1}^{\infty}\frac{M^{2k}}{(2k)!}$$

and then use induction principle to remove all powers higher then two:

$$\sum_{k=0}^{\infty}\frac{M^{2k+1}}{(2k+1)!} = \sum_{k=0}^{\infty}\frac{(M^2)^k}{(2k+1)!}M = M\sum_{k=0}^{\infty}\frac{\left\{(-1)^n x^2\right\}^k}{(2k+1)!};$$

$$\sum_{k=1}^{\infty}\frac{M^{2k}}{(2k)!} = \sum_{k=1}^{\infty}\frac{(M^2)^k}{(2k)!} = M^2\sum_{k=1}^{\infty}\frac{\left\{(-1)^n x^2\right\}^k}{(2k)!}$$

brining the rest of the problem to known exponents:

$$i^n x\sum_{k=0}^{\infty}\frac{\left\{(-1)^n x^2\right\}^k}{(2k+1)!} = \frac{1}{2}\left\{e^{i^n x} - e^{-i^n x}\right\};$$

$$1 + (-1)^n x^2\sum_{k=1}^{\infty}\frac{\left\{(-1)^n x^2\right\}^k}{(2k)!} = \frac{1}{2}\left\{e^{i^n x} + e^{-i^n x}\right\};$$

Therefore, both cases are identical with exception of the split between regular sin/cos and their hyperbolic twins.

In addition:

$$M = \vec{a}\vec{S} \Rightarrow x = |\vec{a}|; \Rightarrow \frac{M}{x} = \frac{\vec{a}\vec{S}}{|\vec{a}|} = \hat{e}\vec{S};$$

$$M = \vec{a}\vec{K} \Rightarrow x = |\vec{a}|; \Rightarrow \frac{M}{x} = \frac{\vec{a}\vec{K}}{|\vec{a}|} = \hat{e}\vec{K}; \#\#$$

$$\hat{e} \to \vec{\beta}; \; |\vec{a}| \to \zeta$$

What is left? Question about general expression for

$$e^{\vec{a}\vec{S}+\vec{b}\vec{K}} = \sum_{n=0} \frac{\left(\vec{a}\vec{S}+\vec{b}\vec{K}\right)^n}{n!}.$$